Int. J. Nonlinear Anal. Appl.
Volume 12, Special Issue, Winter and Spring 2021, 1731-1743
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.5861

A study on approximate and exact controllability of impulsive stochastic neutral integrodifferential evolution system in Hilbert spaces

R. Subalakshmi[a,*], B. Radhakrishnan[a]

[a]*Department of Mathematics, PSG College of Technology, Coimbatore 641004, TamilNadu, India*

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper, the authors establish the approximate and exact controllability of semilinear non-autonomous impulsive neutral stochastic evolution integrodifferential systems with variable delay in a real separable Hilbert space. The findings are determined by using the fixed point approach. Finally, an example is addressed in the proposed work.

Keywords: Impulsive neutral stochastic evolution equation, Fractional power operator, Approximate controllability, Exact controllability, Banach fixed point theorem.
2010 MSC: 39A10, 93B05.

1. Introduction

Impulsive differential equations are a type of important model that represents the behaviour of a system many evolution processes characterized by the fact that at certain moments of time they experience a change of state abruptly these processes are subject to short-term perturbations as impulsive effects also widely exist in stochastic and deterministic systems [2, 16, 29]. Stochastic impulsive mathematical models are studied in different research areas like population dynamics, biology, ecology and epidemic. Many authors [5, 13, 22, 24, 26, 25, 28] have investigated the fixed and random impulsive, delay differential equations in abstract spaces. The qualitative properties such as existence, stability, invariant measures, controllability and observability of deterministic impulsive systems have been established by some researches (see[4, 17] and references therein) for infinite

*Corresponding author
Email addresses: rsl.maths@psgtech.ac.in (R. Subalakshmi), brk.maths@psgtech.ac.in (B. Radhakrishnan)

Received: August 2021 *Accepted:* November 2021

dimensional linear and nonlinear impulsive systems. However, it should be stressed, that there has been few investigation on the controllability of impulsive stochastic systems in infinite dimensional spaces and from the last decade onwards the controllability of impulsive stochastic integrodifferential systems in infinite dimensional spaces begun to receive a significant amount of attention [28, 29]. The invertibility of a controllability operator is imposed in order to obtain controllability outcomes. In fact, it turns out that verifying this condition directly is difficult, as shown in [6], and it fails in infinite-dimensional spaces when the semigroup formed by A is compact. As a result, studying the weaker idea of controllability, especially approximation controllability for impulsive differential systems, is critical.

Klamka [14, 15] investigated constrained approximate controllability problems for linear abstract dynamical systems with linear unbounded control operator and piece-wise polynomial controls and also studied the stochastic controllability of linear systems with delay in controls and state variables. It should be noted that if $\mathcal{X}$ is infinite dimensional, the semigroup is compact and B is bounded, then the infinite-dimensional linear control system is not exactly controllable [6] and there by considering only the linear part for finding the approximate and exact controllability of the system. In [8, 18, 19], authors have studied the controllability for linear stochastic system of the following form

$$\begin{aligned} dx(t) &= [\mathcal{A}x(t) + \mathcal{B}u(t)dt + \tilde{\sigma}(t)dw(t) \\ x(0) &= x_0, \quad t \in J = [0, b] \end{aligned} \tag{1.1}$$

controllability of nonlinear stochastic systems with and without delays in finite and infinite dimensional spaces in Hilbert spaces [3, 20, 27]. [1, 11, 19] investigated the controllability of nonlinear stochastic evolution systems in infinite dimensional spaces. Furthermore, academics have shown a strong interest in neutral impulsive differential and integrodifferential equations [12, 23]. The approximate and perfect controllability of impulsive neutral stochastic functional integrodifferential evolutions systems with variable time delay is discussed in this study.

$$\begin{aligned} d[\mathcal{X}(t) + g(t, \mathcal{X}_t, \int_0^t h(t, s, \mathcal{X}_s)ds)] &= [-\mathcal{A}(t)\mathcal{X}(t) + \mathcal{B}u(t) + f(t, \mathcal{X}(t), \int_0^t h(t, s, \mathcal{X}_s)ds)]dt \\ &\quad + \sigma(t, \mathcal{X}_t, \int_0^t h(t, s, \mathcal{X}_s)ds)dW(t), \quad t \in J\backslash D \\ \mathcal{X}_0 &= \phi(t) \in L_p(\Omega, \mathcal{C}_\alpha), \\ \Delta\mathcal{X}(t_k) &= I_k(\mathcal{X}_{t_k}), \quad k = 1, ..., m \end{aligned} \tag{1.2}$$

where ϕ is $\mathcal{F}_0$-measurable and $\mathcal{A}(t)$ is a closed, densely defined operator generating a linear evolution operators $\{\mathcal{U}(t, s); t, s > 0\}$ on a Hilbert space $\mathcal{H}$ with inner product $\langle ., . \rangle$ and norm $\|.\|$. Define the Banach space $D(A^\alpha(t))$, with the norm $\|x\|_{\alpha,t} : \|A^\alpha(t)x\|$ for $x \in D(A^\alpha(t))$, where $D(A^\alpha(t))$ denotes the domain of the fractional power operator $A^\alpha(t) : D(A^\alpha(t)) \subset H \to H$ (Refer [21] for a detailed study on $A^\alpha(t)$).

In the sequel, we denote for brevity that $H_\alpha : D(A^\alpha(t_0))$ for some $t_0 > 0$, and $\mathcal{C}_\alpha = P\mathcal{C}([-r, 0], H_\alpha)$ be the space of all piecewise continuous functions from $[-r, 0]$ into $H_\alpha, 0 < r < \infty$. Let K, E be another separable Hilbert spaces, $\mathcal{B}$ is a bounded linear operator from U into $\mathcal{H}$. Suppose $W(t)$ is given K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$.

Assume $g : J \times \mathcal{C}_\alpha \times \mathcal{H} \to \mathcal{H}, f : J \times \mathcal{C}_\alpha \times \mathcal{H} \to \mathcal{H}, \sigma : J \times \mathcal{C}_\alpha \times \mathcal{H} \to L_2^0(K, \mathcal{H})$ and $h : J \times J \times \mathcal{C}_\alpha \to \mathcal{H}$ are given functions such that $f(t, 0, 0), g(t, 0, 0), h(t, s, 0)$ and $\sigma(t, 0, 0)$ are locally bounded in $\mathcal{H}$-norm and $L_2^0(K, \mathcal{H})$-norm, respectively. Here $L_2^0(K, \mathcal{H})$ denotes the space of all Q-Hilbert-Schmidt operators from K into $\mathcal{H}$. We also employ the same notation $\|.\|$ for the norm of $L(K,\mathcal{H})$, where $L(K,\mathcal{H})$ denotes the space of all bounded linear operators from K into $\mathcal{H}$. The nonlinear map $I_k : \mathcal{C}_\alpha \to \mathcal{C}_\alpha,\ k = 1,...,m$ is continuous.

2. Preliminaries

Here, the probability space $(\Omega,\mathcal{F},P)$ on which an increasing and right continuous family $\{\mathcal{F}_t : t \geq 0\}$ of complete sub-σ-algebras of $\mathcal{F}$ is defined. Suppose $\mathcal{X}(t) : \Omega \to H_\alpha, t \geq -r$, is a continuous $\mathcal{F}_t$-adapted, H_α-valued stochastic process we can associate with another process $X(t,\omega) : [0,T]\times\Omega \to \mathcal{C}_\alpha, t \leq 0$, by setting $\mathcal{X}(t,\omega) = \{\mathcal{X}(t+s)(\omega) : s \in [-r,0]\}$. This is regarded as a $\mathcal{C}_\alpha$-valued stochastic process. Let $\beta_n(t)(n = 1,2,...)$ be the sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega,\mathcal{F},P)$. Let $\psi \in \mathcal{L}(K,\mathcal{H})$ and define $\|\psi\|^2_Q = tr[\psi Q\psi^*] = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\psi e_n\|^2$.

If $\|\psi\|_Q < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Recall that f and g are said to be $\mathcal{F}_t$-adapted if $f(t,.,.) : \Omega\times\Omega \to H$ and $g(t,.,.) : \Omega\times\Omega \to \mathcal{H}$ are $\mathcal{F}_t$-measurable, a.e. $t \in [0,T]$ and $\mathcal{F}_0$-measurable, a.e. $t \in [-r,0]$. Let $M\mathcal{C}_\alpha(0,p), p > 2$, denote the space of all $\mathcal{F}_0$-measurable functions that belong to $L_p(\Omega,\mathcal{C}_\alpha)$, that is, $M\mathcal{C}_\alpha(0,p), p > 2$, is the space of all $\mathcal{F}_0$-measurable $\mathcal{C}_\alpha$-valued functions $\psi : \Omega \to \mathcal{C}_\alpha$with the norm $E\|\psi\|^p_{\mathcal{C}_\alpha} = E\{sup_{-r\leq s\leq 0}\|A^\alpha(t_0)\psi(s)\|^p\} < \infty$.

Denote $J_0 = [0,t_1], J_k = (t_k,t_{k+1}], k = 1,2,...,m$. Let $J = [0,T]$ be an interval. We define the following classes of functions: Let $\mathcal{PC}([-r,T],L_p(\Omega,F,P,\mathcal{H}))$ be the Banach space of piecewise continuous function from $[-r,T]$ into $L_p(\Omega,F,P,\mathcal{H})$ satisfying the condition $sup_{t\in[-r,T]}E\|\mathcal{X}(t)\|^p < \infty$. Let $\mathcal{H}_p$ be closed subspace of all continuous processes X that belong to the space $\mathcal{PC}([-r,T],L_p(\Omega,F,P,\mathcal{H})) = \{\mathcal{X}(t) is\ continuous\ everywhere\ except\ for\ some\ t_k\ at\ which\ x(t_k^-)\ and\ x(t_k^+)\ exist\ and\ x(t_k^-) = x(t_k), k = 1,2,...,m\}$ consisting of measurable and $\mathcal{F}_t$-adapted $\|\mathcal{X}\|_{\mathcal{H}p} = (sup_{t\in[0,T]}E\|\mathcal{X}(t,\omega)\|^P_{\mathcal{C}})^{\frac{1}{p}} = (sup_{t\in[0,T]}sup_{-r\leq s\leq 0}\|\mathcal{X}(t+s)\omega\|^P_{\mathcal{C}})^{\frac{1}{p}},\quad p > 2$.

p and r are conjugate indices:$\frac{1}{p}+\frac{1}{r} = 1$. For brevity, we suppress the dependence of all mappings on ω throughout the manuscript. Here $D = \{t_1,t_2,...,t_m\} \subset J,\ 0 = t_0 < t_1 < ... < t_m < t_{m+1} = T, I_k(K = 1,2,...,m)$ is a nonlinear map and $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = x(t_k^+) - x(t_k)$.

Assumption A. $-\mathcal{A}(t)$ generates a linear evolution operators $\mathcal{U}(t,s),\quad t,s > 0$ on a separable Hilbert space $\mathcal{H}$ and $0 \in \rho(A(t))$.

The following results relating to Assumption A are obtained. $A^\alpha(t)$ and the $U(t,s)$ linear evolution operators created by $\mathcal{A}(t)$ hold (see [21]).

$(B1)$ The domain $D(A)$ of $\{\mathcal{A}(t) : 0 \leq t \leq T\}$ is dense in $\mathcal{H}$ and independent of t, $\mathcal{A}(t)$ is closed linear operator;

$(B2)$ For each $t \in [0,T]$, the resolvent $R(\lambda,\mathcal{A}(t))$ exists for all λ with $Re\lambda \leq 0$ and there exists $C > 0$ so that $\|R(\lambda,\mathcal{A}(t)\| \leq C/(|\lambda|+1)$;

$(B3)$ There exists $0 < \delta < 1$ and $C > 0$ such that $\|\mathcal{A}(t) - \mathcal{A}(s)A^{-1}(\tau)\| \leq C|t-s|^\delta$ for all $t,s,\tau \in [0,T]$.

Lemma 2.1. *Let $-\mathcal{A}(t)$ generates linear evolution operators $\mathcal{U}(t,s)$. If $0 \in \rho(A(t))$, then:*

1. *There exist constants $M > 0$ and $a > 0$ such that $\|\mathcal{U}(t,s)x\| \leq Me^{-a(t-s)}\|x\|,\quad t \geq 0$, for any $x \in \mathcal{H}$.*
2. *The fractional power A^α satisfies that $\|A^\alpha(t)\mathcal{U}(t,s)x\| \leq M_\alpha e^{-a(t-s)}(t-s)^\alpha\|x\|, t > 0$, for any $x \in H$ and $M_\alpha > 0$.*
3. *Let $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$, then $\|\mathcal{U}(t,s)x - x\| \leq N_\alpha(t-s)^\alpha\|A^\alpha(t)x\|, N_\alpha > 0$.*

Lemma 2.2. *Assume that* $(B1)-(B3)$ *hold. If* $0 \le \gamma \le 1,\ 0 \le \beta \le \alpha < 1+\delta,\ 0 < \alpha-\gamma \le 1$, *then for any*
$0 \le \tau < t+\Delta t \le t_0,\ 0 \le \zeta \le T,\ \|A^{\gamma}(\zeta)(\mathcal{U}(t+\Delta t,\tau)-\mathcal{U}(t,\tau))A^{-\beta}(\tau)\| \le C(\beta,\gamma,\alpha)(\Delta t)^{\alpha-\gamma}|t-\tau|^{\beta-\alpha}$.

We can refer to for more information on the theory of linear evolution systems, operator semigroups, and fraction powers of operators [10, 21].

Assumption B. For arbitrary $\gamma_i, \xi_i \in \mathcal{C}_\alpha, i = 1,2,$ *and* $0 \le t_0 \le t_1 \le t \le t_2 \le T$, suppose that there exist positive real constants $N_1, N_2, K, d_k > 0$ and $\alpha \in (0,1]$ such that $g(t_i,\gamma_i,\xi_i) \in D(A^\alpha)$ and
$\|f(t,\gamma_1,\xi_1) - f(t,\gamma_2,\xi_2)\|^p + \|\sigma(t,\gamma_1,\xi_1) - \sigma(t,\gamma_2,\xi_2)\|_Q^p \le N_1[\|\gamma_1-\gamma_2\|^p + \|\xi_1-\xi_2\|^p]$,
$\|A^\alpha(t_0)g(t_1,\gamma_1,\xi_1) - A^\alpha(t_0)g(t_2,\gamma_2,\xi_2)\|^p \le N_2[|t_1-t_2|^p + \|\gamma_1-\gamma_2\|^p + \|\xi_1-\xi_2\|^p]$,
$\|h(t_1,s,\gamma_1) - h(t_2,s,\gamma_2)\|^p \le K[|t_1-t_2|^p + \|\gamma_1-\gamma_2\|^p]$, and $\|I_k(\gamma_1) - I_k(\gamma_2)\|^p \le d_k\|\gamma_1-\gamma_2\|^p$.

Assumption B_1**.** For arbitrary $\gamma, \xi \in \mathcal{C}_\alpha,\ 0 \le t_0 \le t \le T$, suppose that there exist positive real constants $\tilde{N}_1, \tilde{N}_2, \tilde{K}, \tilde{d}_k > 0$ such that
$\|f(t,\gamma,\xi)\|^p + \|\sigma(t,\gamma,\xi)\|_Q^p \le \tilde{N}_1$, and $\|A^\alpha(t_0)g(t,\gamma,\xi)\|^p \le \tilde{N}_2$,
$\|h(t,s,\xi)\|^p \le \tilde{K}$, and $\|I_k(\gamma)\|^p \le \tilde{d}_k$, $(k = 1,2,...,m)$.

Assumption B_2**.** For arbitrary $\gamma, \xi \in \mathcal{C}_\alpha$, and $0 \le t_0 \le t \le T$ suppose that there exist positive real constants $\hat{N}_1, \hat{N}_2, \hat{K} > 0,\ \hat{d}_k > 0$ such that
$\|f(t,\gamma,\xi)\|^p + \|\sigma(t,\gamma,\xi)\|_Q^p \le \hat{N}_1(1 + \|\gamma\|^p + \|\xi\|^p)$ and
$\|A^\alpha(t_0)g(t,\gamma,\xi)\|^p \le \hat{N}_2(1 + \|\gamma\|^p + \|\xi\|^p)$,
$\|h(t,s,\xi)\|^p \le \hat{K}(1 + \|\xi\|^p)$, and $\|I_k(\xi)\|^p \le \hat{d}_k(1 + \|\xi\|^p)$.

Assumption C. For each $0 \le s < T$ the operator $\lambda(\lambda I + \Gamma_s^T)^{-1} \to 0$ in the strong operator topology as $\lambda \to 0^+$, where $\Gamma_s^T = \int_s^T \mathcal{U}(T,r)\mathcal{B}\mathcal{B}^*\mathcal{U}^*(T,r)dr$ is the controllability Grammian. Notice that the deterministic linear system corresponding to (1) is approximately controllable on $[s,T]$ if and only if the operator $\lambda(\lambda I + \Gamma_s^T)^{-1} \to 0$ strongly as $\lambda \to 0^+$ (see [7]). For simplicity, we denote $H_0^s * \mathcal{X} := \int_0^s h(s,\tau,\mathcal{X}_\tau)d\tau$.

Definition 2.3. *A stochastic process* $\mathcal{X}$ *is said to be a mild solution of (1.1) if the following conditions are satisfied:*

1. $\mathcal{X}(t,\omega)$ *is measurable as a function from* $[0,T] \times \Omega$ *to* H *and* $\mathcal{X}(t)$ *is* $\mathcal{F}_t$*-adapted;*
2. $E\|\mathcal{X}(t)\|^p < \infty$, *for each* $t \in [-r,T]$*;*
3. *For each* $u \in L_p^{\mathcal{F}}(0,T;\mathcal{U})$ *the process* $\mathcal{X}$ *satisfies the following integral equation:*

$$\begin{aligned}
\mathcal{X}(t) =& \,\mathcal{U}(t,0)(\phi(0) + g(0,\phi,0)) - g(t,\mathcal{X}_t, \int_0^t h(t,s,\mathcal{X}_s)ds) \\
&- \int_0^t \mathcal{A}(t_0)\mathcal{U}(t,s)g(s,\mathcal{X}_s,H_0^s * X)ds + \sum_{0<t_k<t} \mathcal{U}(t,t_k)I_k(\mathcal{X}_{t_k}) \\
&+ \int_0^t \mathcal{U}(t,s)(\mathcal{B}u(s) + f(s,\mathcal{X}_s,H_0^s * \mathcal{X}))ds + \int_0^t \mathcal{U}(t,s)\sigma(s,\mathcal{X}_s,H_0^s * \mathcal{X})dW(s) \\
\mathcal{X}_0 =& \,\phi \in M\mathcal{C}_\alpha(0,p), \quad t \ge 0.
\end{aligned} \tag{2.1}$$

Definition 2.4. *System (1.1) is approximately(exactly)controllable on* $[0, T]$ *if* $\overline{\mathcal{R}(T)} = L_p(\Omega, \mathcal{F}, P, H), (\mathcal{R}(T) = L_p(\Omega, \mathcal{F}, P, H))$*, where* $\mathcal{R}(T) = \{\mathcal{X}(T) = \mathcal{X}(T; u) : u(\cdot) \in L_p^{\mathcal{F}}(J, U)\}$*. We also need the following lemmas (Proposition 4.15 and Lemma 7.2 in [9] and Lemmas 7-9 in [7]) to prove our main results.*

Lemma 2.5. *If* $\Phi \in L_2^{\mathcal{F}}(0, T; L_2^0(K, H)), A^\alpha(t_0)\Phi \in L_2^{\mathcal{F}}(0, T; L_2^0(K, H))$ *and* $\Phi(t)k \in H_\alpha, 0 < t_0 \leq t$, *for arbitrary* $k \in K$*, then* $A^\alpha(t_0)\int_0^t \Phi(s)dW(s) = \int_0^t A^\alpha(t_0)\Phi(s)dW(s)$*.*

Lemma 2.6. *For any* $p > 2, \Phi \in L_p^{\mathcal{F}}(\Omega, L_2(0, T; L_2^0(K, H)))$ *we have*

$$E(sup_{0\leq s\leq t}\|\int_0^s \Phi(r)dW(r)\|^p) \leq c_p sup_{0\leq s\leq t}E\|\int_0^s \Phi(r)dW(r)\|^p \leq C_p E(\int_0^t \|\Phi(r)\|_Q^2 dr)^{\frac{p}{2}}$$

for all $t \in [0, T]$*, where* $c_p = \left(\frac{p}{p-1}\right)^p, C_p = \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\left(\frac{p}{p-1}\right)^{\frac{p^2}{2}}$*.*

Lemma 2.7. *For any* $h \in L_p(\Omega, \mathcal{F}, P; H)$ *there exists* $\varphi \in L_p^{\mathcal{F}}(\Omega, L_2(0, T; L_2^0(K, H)))$ *such that* $h = Eh + \int_0^T \varphi(s)dW(s)$*.*

Lemma 2.8. *Let* $p > 2$ *and let* $\sigma \in L_p^{\mathcal{F}}(0, T; L_2^0(K, H))$*. Then there exists a constant* $N_3 > 0$ *such that* $E\ sup_{-r\leq\theta\leq 0}\|\int_0^{t+\theta} A^\alpha(t_0)U(t+\theta, \tau)\sigma(\tau)dW(\tau)\|^p \leq N_3 E\int_0^t \|\sigma(\tau)\|_Q^p d\tau$*, where* $N_3 = M_\alpha^p(\Gamma(1 + q(\beta - 1 - \alpha))(aq)^{q(1+\alpha-\beta)})^{\frac{p}{q}} C_p \frac{t^{p(1-2\beta)/2}}{(1-2\beta)^{p/2}}, 1/p + \alpha < \beta < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*.*

For any $\lambda > 0$ and $h \in L_p(\Omega, \mathcal{F}, P; H)$ define the control

$$\begin{aligned} u^\lambda(t, \mathcal{X}) = {} & B^*U^*(T, t)(\lambda I + \Gamma_0^T)^{-1}(Eh - U(T, 0)(\phi(0) + g(0, \phi, 0)) \\ & - g(T, \mathcal{X}_T, \int_0^T h(T, s, \mathcal{X}_s))ds - \sum_{0<t_k<t} U(T, t_k)I_k(\mathcal{X}_{t_k})) \\ & - B^*U^*(T, t)\int_0^t (\lambda I + \Gamma_s^T)^{-1}A(t_0)\mathcal{U}(t, s)g(s, \mathcal{X}_s, H_0^s * \mathcal{X})ds \\ & - B^*U^*(T, t)\int_0^t (\lambda I + \Gamma_s^T)^{-1}\mathcal{U}(t, s)f(s, \mathcal{X}_s, H_0^s * \mathcal{X})ds \\ & - B^*U^*(T, t)\int_0^t (\lambda I + \Gamma_s^T)^{-1}\mathcal{U}(t, s)\sigma(s, \mathcal{X}_s, H_0^s * \mathcal{X})dW(s) \\ & + B^*U^*(T, t)\int_0^t (\lambda I + \Gamma_s^T)^{-1}\varphi(s)dW(s). \end{aligned} \tag{2.2}$$

Lemma 2.9. *There exist a positive real constants* $N_4, \hat{N}_4 > 0$ *such that for all* $X, Y \in \mathcal{H}_p$ $E\|u^\lambda(t, \mathcal{X}) - u^\lambda(t, \mathcal{Y})\|^p \leq \frac{1}{\lambda^p}N_4\int_0^t E\|\mathcal{X}_s - \mathcal{Y}_s\|_{\mathcal{C}_\alpha}^p ds$ *and* $E\|u^\lambda(t, \mathcal{X})\|^p \leq \frac{1}{\lambda^p}\hat{N}_4(1 + \int_0^t E\|\mathcal{X}_s\|_{\mathcal{C}_\alpha}^p ds)$*.*

3. Approximate Controllability

This section presents our main result on approximate controllability of system (1.1). Let us fix $\lambda > 0$ and introduce the following mapping Φ on $\mathcal{H}_p$:

$$\begin{aligned}(\Phi Z)(t) =& \mathcal{U}(t,0)A^\alpha(t_0)(\phi(0)+g(0,\phi,0)) + \int_0^t A^\alpha(t_0)\mathcal{U}(t,s)\mathcal{B}u^\lambda(s,A^{-\alpha}(t_0)Z_s)ds \\ &- A^\alpha(t_0)g\left(t, A^{-\alpha}(t_0)Z_t, \int_0^t h(t,s,A^{-\alpha}(t_0)Z_s)ds\right) + A^{-\alpha}(t_0)\sum_{0<t_k<t}\mathcal{U}(t,t_k)I_k(A^{-\alpha}(t_0)Z_{t_k}) \\ &+ \int_0^t A^\alpha(t_0)\mathcal{A}(t_0)\mathcal{U}(t,s)g(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))ds \\ &+ \int_0^t A^\alpha(t_0)\mathcal{U}(t,s)f(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))ds \\ &+ \int_0^t A^\alpha(t_0)\mathcal{U}(t,s)\sigma(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))dW(s) \qquad (3.1)\end{aligned}$$

$$\begin{aligned}(\Phi Z)(t) =& A^\alpha(t_0)\phi(t), -r \le t_0 \le t \le 0 \\ u^\lambda(t, A^{-\alpha}(t_0)Z) =& \mathcal{B}^*\mathcal{U}^*(T,t)(\lambda I + \Gamma_0^T)^{-1}(Eh - \mathcal{U}(T,0)(\phi(0)+g(0,\phi,0)) \\ &- g\left(T, A^{-\alpha}(t_0)Z_T, \int_0^T h(T,s,A^{-\alpha}(t_0)Z_s)ds\right) - \sum_{0<t_k<t}\mathcal{U}(T,t_k)I_k(A^{-\alpha}(t_0)Z_{t_k})) \\ &- \mathcal{B}^*\mathcal{U}^*(T,t)\int_0^T(\lambda I + \Gamma_s^T)^{-1}A(t_0)\mathcal{U}(T,s)g(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s)) \\ &- \mathcal{B}^*\mathcal{U}^*(T,t)\int_0^T(\lambda I + \Gamma_s^T)^{-1}\mathcal{U}(T,s)f(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))ds \\ &- \mathcal{B}^*\mathcal{U}^*(T,t)\int_0^T(\lambda I + \Gamma_s^T)^{-1}\mathcal{U}(T,s)\sigma(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))dW(s) \\ &+ \mathcal{B}^*\mathcal{U}^*(T,t)\int_0^T(\lambda I + \Gamma_s^T)^{-1}\varphi(s)dW(s). \qquad (3.2)\end{aligned}$$

Lemma 3.1. *Assume* $0 < \alpha < (p-2)/2p$. *For any* $Z \in \mathcal{H}_p, (\Phi Z)(t)$ *is continuous on the interval* $[0,T]$ *in the* L_p*-sense.*

Proof . Let $0 \le t_0 \le t_1 < t_2 < T$. Then for any fixed $Z \in \mathcal{H}_p$.
$E\|(\Phi Z)(t_1) - (\Phi Z)(t_2)\|^p \le 8^{p-1}(E\|(\mathcal{U}(t_2,0) - \mathcal{U}(t_1,0))A^\alpha(t_0)(\phi(0)+g(0,\phi,0))\|^p$
$+E\|\int_0^{t_2} A^\alpha(t_0)\mathcal{U}(t_2,s)Bu^\lambda(s,A^{-\alpha}(t_0)Z_s)ds - \int_0^{t_1} A^\alpha(t_0)\mathcal{U}(t_1,s)Bu^\lambda(s,A^{-\alpha}(t_0)Z_s)ds\|^p$
$+E\|A^\alpha(t_0)g(t_2,A^{-\alpha}(t_0)Z_{t_2},\int_0^{t_2} h(t_2,s,A^{-\alpha}(t_0)Z_s)ds) - A^\alpha(t_0)g(t_1,A^{-\alpha}(t_0)Z_{t_1},\int_0^{t_1} h(t_1,s,A^{-\alpha}(t_0)Z_s)ds)\|$
$+E\|[(\int_0^{t_2}\mathcal{A}(t_0)\mathcal{U}(t_2,s) - \int_0^{t_1}\mathcal{A}(t_0)\mathcal{U}(t_1,s))A^\alpha(t_0)g(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))ds]\|^p$
$+E\|[(\int_0^{t_2} A^\alpha(t_0)\mathcal{U}(t_2,s) - \int_0^{t_1} A^\alpha(t_0)\mathcal{U}(t_1,s))f(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z_s))ds]\|^p$
$+E\|[(\int_0^{t_2} A^\alpha(t_0)\mathcal{U}(t_2,s) - \int_0^{t_1} A^\alpha(t_0)\mathcal{U}(t_1,s))\sigma(s,A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}Z_s))dW(s)]\|^p$
$+\sum_{0<t_k<t} E\|A^\alpha(t_0)[\mathcal{U}(t_2,t_k) - \mathcal{U}(t_1,t_k)]I_k(A^{-\alpha}(t_0)Z(t_k))\|^p) = \sum_{i=1}^8 I_i$.
Thus, we obtain by Lemma 2.1, that

$I_1 = 8^{p-1}E\|(\mathcal{U}(t_2,t_1)\mathcal{U}(t_1,0) - \mathcal{U}(t_1,0))A^\alpha(t_0)\phi(0)\|^p \le 8^{p-1}N_p^\alpha(t_2-t_1)^{p\alpha}E\|A^\alpha(t_0)\mathcal{U}(t_1,0)A^\alpha(t_0)\phi(0)\|^p$,
$I_2 = 8^{p-1}E\|(\mathcal{U}(t_2,t_1)\mathcal{U}(t_1,0) - \mathcal{U}(t_1,0))A^\alpha(t_0)g(0,\phi,0)\|^p$
$\le 8^{p-1}N_p^\alpha(t_2-t_1)^{p\alpha}E\|A^\alpha(t_0)\mathcal{U}(t_1,0)A^\alpha(t_0)g(0,\phi,0)\|^p$,

$I_3 \leq 16^{p-1}E(\int_{t_1}^{t_2} \|A^\alpha(t_0)\mathcal{U}(t_2,s)\mathcal{B}u^\lambda(s, A^{-\alpha}(t_0)Z)\|ds)^p$
$+16^{p-1}E(\int_0^{t_1} \|A^\alpha(t_0)\mathcal{U}(t_1,s)(\mathcal{U}(t_2,t_1)-I)\mathcal{B}u^\lambda(s, A^{-\alpha}(t_0)Z)\|ds)^p = I_{31} + I_{32}$.
Therefore, there exist positive constants $l_{31}, l_{32} > 0$ and $\epsilon_1 = p(1-\alpha) > 0$ such that
$I_{31} \leq l_{31}(t_2-t_1)^{\epsilon_1}(1+\|Z_s\|^p_{\mathcal{H}_p})$ and $I_{32} \leq l_{32}(t_2-t_1)^{p\alpha}(1+\|Z_s\|^p_{\mathcal{H}_p})$.
Therefore, there exists a positive constant $l_{41} > 0$ such that $I_4 \leq l_{41}(1+KT^p)(\|t_2-t_1\|^p+\|Z_{t_2}-Z_{t_1}\|^p)$.
In a similar way, for I_5 and I_6, there exist positive constants l_{51}, l_{52}, l_{61} and $l_{62} > 0$ such that
$I_5 \leq (l_{51}(t_2-t_1)^{\epsilon_1} + l_{52}(t_2-t_1)^{p\alpha})(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$,
$I_6 \leq (l_{61}(t_2-t_1)^{\epsilon_1} + l_{62}(t_2-t_1)^{p\alpha})(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$.
Now by using Lemma 2.5 for some C'_p we have
$I_7 \leq 16^{p-1}C'_pE(\int_{t_1}^{t_2} \|A^\alpha(t_0)\mathcal{U}(t_2,s)\sigma(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))\|^2_Q ds)^{p/2}$
$+16^{p-1}C'_pE(\int_0^{t_1} \|A^\alpha(t_0)\mathcal{U}(t_1,s)[\mathcal{U}(t_2,t_1)-I]\sigma(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))\|^2_Q ds)^{p/2} = I_{71} + I_{72}$.
Then, it follows that there exist positive constants $l_{71} > 0$ and $\epsilon_2 = (p-2-2p\alpha)/2 > 0$ such that
$I_{71} \leq l_{71}(t_2-t_1)^{\epsilon_2}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$.
Let $\{e_n\}, n \geq 1$, be a complete orthonormal basis of the separable Hilbert space K such that $Q^{1/2}e_n = \sqrt{\lambda_n}e_n$, where Q is the covariance operator of Wiener process W. Then we obtain that there exists a positive constant $l_{62} > 0$ such that
$I_{72} \leq l_{72}(t_2-t_1)^{p\alpha}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$.
In a similar way, for I_8 there exist positive constants $l_{81}, l_{82} > 0$ such that
$I_8 \leq (l_{81}(t_2-t_1)^{\epsilon_1} + l_{82}(t_2-t_1)^{p\alpha})\sum_{k=1}^m \hat{d}_k(1+\|Z_s\|^p_{\mathcal{H}_p})$.
Since $Z \in \mathcal{H}_p$, it follows that I_i, for $i = 1, ..., 8$ tend to zero as $t_2 \to t_1$. Hence $(\Phi Z)(t)$ is continuous from the right in $[0,T)$. A similar argument demonstrates that it is similarly continuous from left $(0,T]$ As a result, the lemma's proof is complete. □

Lemma 3.2. *The operator Φ sends $\mathcal{H}_p$ into itself.*

Proof . Let $Z \in \mathcal{H}_p$. Then we have
$E\|(\Phi Z)_t\|^p_C \leq 8^{p-1}E\ sup_{-r\leq\theta\leq 0}\|\mathcal{U}(t+\theta,0)A^\alpha(t_0)(\Phi(0)+g(0,\Phi,0))\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\|A^\alpha(t_0)g(t, A^{-\alpha}(t_0)Z_t, \int_0^t h(t,s,A^{-\alpha}(t_0)Z_s)ds\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\| \int_0^{t+\theta} A^\alpha(t_0)\mathcal{A}(t_0)\mathcal{U}(t+\theta,s)g(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))ds\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\| \int_0^{t+\theta} A^\alpha(t_0)\mathcal{U}(t+\theta,s)\mathcal{B}u^\lambda(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))ds\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\| \int_0^{t+\theta} A^\alpha(t_0)\mathcal{U}(t+\theta,s)f(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))ds\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\| \int_0^{t+\theta} A^\alpha(t_0)\mathcal{U}(t+\theta,s)\sigma(s, A^{-\alpha}(t_0)Z_s, H_0^s * (A^{-\alpha}(t_0)Z))dW(s)\|^p$
$+8^{p-1}E\ sup_{-r\leq\theta\leq 0}\sum_{0<t_k<t}\|A^\alpha(t_0)(\mathcal{U}(t+\theta),t_k)I_k(A^{-\alpha}(t_0)Z(t_k))\| = I_1+I_2+I_3+I_4+I_5+I_6+I_7+I_8$.
where
$I_1 \leq 8^{p-1}M^pE\|\phi\|^p_{C_\alpha}$,
$I_2 \leq 8^{p-1}M^pE\|g(0,\Phi,0)\|^p_{C_\alpha}$,
$I_3 \leq 8^{p-1}\hat{N}_2(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$,
$I_4 \leq 8^{p-1}M^p\hat{N}_2T(\Gamma(1-q)(aq)^q)^{p/q}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$,
$I_5 \leq \frac{1}{\lambda}4^{p-1}M^p_\alpha\hat{N}_4T(\Gamma(1-\alpha q)(aq)^{q\alpha})^{p/q}(1+\|Z_s\|^p_{\mathcal{H}_p})$,
$I_6 \leq 8^{p-1}M^p_\alpha\hat{N}_1T(\Gamma(1-\alpha q)(aq)^{q\alpha})^{p/q}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$,
By Lemma 2.7 for any β satisfying the inequality $1/p+\alpha < \beta < 1/2$ we have
$I_7 \leq 8^{p-1}M^p_\alpha\hat{N}_1T(\Gamma(1+(\rho-1-\alpha)q)(aq)^{q(1+\alpha-p)})^{p/q}c_pM^p\frac{T^{-p\beta+\frac{p}{2}}}{(1-2\beta)^{\frac{p}{2}}}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$,
$I_8 \leq 8^{p-1}M^p_\alpha\sum_{k=1}^m \hat{d}_kT(\Gamma(1-\alpha q)(aq)^{q\alpha})^{p/q}(1+\hat{K}T^p)(1+\|Z_s\|^p_{\mathcal{H}_p})$.
Therefore, we obtain that $\|\Phi Z\|_{\mathcal{H}_p} < \infty$. By Lemma 3.1, $(\Phi Z)(t)$ is continuous on $[0,T]$ and so Φ maps $\mathcal{H}_p$ into $\mathcal{H}_p$. This completes the proof. □

Theorem 3.3. *Assume* $0 < \alpha < (p-2)/2p$ *and let* $f : [0, \infty) \times \mathcal{C}_\alpha \times H \to H, g : [0, \infty) \times \mathcal{C}_\alpha \times H \to H$, *and* $\sigma : [0, \infty) \times \mathcal{C}_\alpha \times H \to L_2^0$, *, satisfy Assumptions A, B and* B_2. *Then the operator* Φ *has a unique fixed point in* $\mathcal{H}_p$.

Proof . We prove the theorem through the classical Banach fixed point theorem that for each fixed $\lambda > 0$ the operator Φ has a unique fixed point in $\mathcal{H}_p$. By Lemma 3.2, Φ maps $\mathcal{H}_p$ into $\mathcal{H}_p$. To show that there exists a natural n such that Φ^n is contraction, let $\mathcal{X}, \mathcal{Y} \in \mathcal{H}_p$; then for any fixed $t \in [0, T]$,
$E\|(\Phi\mathcal{X})_t - (\Phi\mathcal{Y})_t\|_C^p \leq E\ sup_{-r\leq\theta\leq 0}\|(\Phi\mathcal{X})(t+\theta) - (\Phi\mathcal{Y})(t+\theta)\|^p = J_1 + J_2 + J_3 + J_4 + J_5 + J_6$.
$J_1 \leq 6^{p-1}N_2(1 + KT^{p/q})sup_{t\in[0,T]}E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p$,
$J_2 \leq \frac{1}{\lambda^p}6^{p-1}M_\alpha^p N_4 T\|\mathcal{B}\|^p(\Gamma(1-\alpha q)(aq)^{q\alpha})^{p/q}E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p ds$,
$J_3 \leq 6^{p-1}M^p N_2 T(\Gamma(1-q)(aq)^q)^{p/q}(1 + KT^{p/q})E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p ds$,
$J_4 \leq 6^{p-1}M_\alpha^p N_1 T(\Gamma(1-\alpha q)(aq)^{q\alpha})^{p/q}(1 + KT^{p/q})E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p ds$,
Let $1/p + \alpha < \beta < 1/2$, by Lemma 2.7, we have J_5
$J_5 \leq 6^{p-1}M_\alpha^p N_1 T(\Gamma(1+(\rho-1-\alpha)q)(aq)^{q(1+\alpha-p)})^{p/q} \times C_p M^p \frac{T^{-p\beta+p/2}}{(1-2\beta)^{p/2}}(1+KT^{p/q})E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p ds$,
$J_6 \leq 6^{p-1}M_\alpha^p \sum_{k=1}^m d_k sup_{t\in[0,T]}E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p$.
Hence, we obtain a positive real number $\mathcal{B}(\lambda) > 0$ such that

$$E\|(\Phi\mathcal{X})_t(\Phi\mathcal{Y})_t\|_C^p \leq \mathcal{B}(\lambda)E\int_0^t \|\mathcal{X}_s - mathcalY_s\|_C^p ds, for\ any\ \mathcal{X}, \mathcal{Y} \in \mathcal{H}_p. \tag{3.3}$$

For any integer $n \geq 1$, by iteration, it follows from (3.3) that
$\|\Phi^n\mathcal{X} - \Phi^n\mathcal{Y}\|_{\mathcal{H}_p}^p \leq \frac{(T\mathcal{B}(\lambda))^n}{n!}\|\mathcal{X} - \mathcal{Y}\|_{\mathcal{H}_p}^p$.
Since for sufficiently large n, $\frac{(T\mathcal{B}(\lambda))^n}{n!} < 1$, Φ^n is a contraction map in $\mathcal{H}_p$ and therefore Φ itself has a unique fixed point Z in $\mathcal{H}_p$. The theorem is proved. □

Thus, by Theorem 3.3 for any $\lambda > 0$ the operator Φ_λ has a unique fixed point $Z^\lambda \in \mathcal{H}_p$ which setting $\mathcal{X}^\lambda(t) = A^\alpha(t_0)Z^\lambda(t)$ immediately yields

$$\begin{aligned}
\mathcal{X}^\lambda(t) =& \mathcal{U}(t,0)(\phi(0) + g(0,\phi,0)) - g(t, \mathcal{X}_t, \int_0^t h(t,s,\mathcal{X}_s)ds) \\
&+ \sum_{0<t_k<t}\mathcal{U}(t,t_k)I_k(\mathcal{X}_{t_k}) + \Gamma_0^t\mathcal{U}^*(T,t)(\lambda I + \Gamma_0^T)^{-1}(Eh - \mathcal{U}(T,0)(\phi(0) \\
&+ g(0,\phi,0) - g(T, \mathcal{X}_T, \int_0^T h(T,s,\mathcal{X}_s)ds) + \sum_{0<t_k<T}\mathcal{U}(T,t_k)I_k(\mathcal{X}_{t_k})) \\
&+ \int_0^t [I - \Gamma_s^t\mathcal{U}^*(T,t)(\lambda I + \Gamma_0^T)^{-1}\mathcal{U}(T-t)]\mathcal{A}(t_0)\mathcal{U}(t,s)g(s, \mathcal{X}_s^\lambda, H_0^s * \mathcal{X}^\lambda)ds \\
&+ \int_0^t [I - \Gamma_s^t\mathcal{U}^*(T,t)(\lambda I + \Gamma_0^T)^{-1}\mathcal{U}(T-t)]\mathcal{U}(t,s)f(s, \mathcal{X}_s^\lambda, H_0^s * \mathcal{X}^\lambda)ds \\
&+ \int_0^t [I - \Gamma_s^t\mathcal{U}^*(T,t)(\lambda I + \Gamma_0^T)^{-1}\mathcal{U}(T-t)]\mathcal{U}(t,s)\sigma(s, \mathcal{X}_s^\lambda, H_0^s * \mathcal{X}^\lambda)dW(s) \\
&+ \int_0^t \Gamma_s^t\mathcal{U}^*(T-t)(\lambda I + \Gamma_0^T)^{-1}\varphi(s)dW(s).
\end{aligned} \tag{3.4}$$

$$\mathcal{X}(t) = \phi(t), \quad -r \leq t \leq 0$$

Now our main result in this paper can be stated as follows.

Theorem 3.4. *Under Assumptions A, B, B_1 and C the system(1.1) is approximately controllable on* $[0, T]$.

Proof . Let $\mathcal{X}^\lambda$ be a solution of Eq.(2.2). Then writing Eq.(3.4) at $t = T$ yields

$$\begin{aligned} \mathcal{X}^\lambda(T) &= h - \lambda(\lambda I + \Gamma_0^T)^{-1}(Eh - \mathcal{U}(T,0)(\phi(0) + g(0,\phi,0)) \\ &- g(T, \mathcal{X}_T, \int_0^T h(T,s,\mathcal{X}_s)ds) + \sum_{0<t_k<t} \mathcal{U}(T,t_k)I_k(\mathcal{X}_{t_k})) \\ &- \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{A}(t_0)\mathcal{U}(T,\tau)g(s,\mathcal{X}_\tau^\lambda, H_0^\tau * X^\lambda)ds \\ &- \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}\mathcal{U}(T,\tau)f(s,\mathcal{X}_\tau^\lambda, H_0^\tau * X^\lambda)ds \\ &- \int_0^T \lambda(\lambda I + \Gamma_0^T)^{-1}[\mathcal{U}(T,\tau)\sigma(s,\mathcal{X}_\tau^\lambda, H_0^\tau * X^\lambda) + \varphi(\tau)]dW(\tau). \end{aligned} \tag{3.5}$$

By Assumption B_1,
$\|f(t,\gamma,\xi)\|^p + \|\sigma(t,\gamma,\xi)\|_Q^p \le \tilde{N}_1$, $\|A^\alpha(t_0)g(t,\gamma,\xi)\|^p \le \tilde{N}_2$, and $\|h(t,s,\xi)\|^p \le \tilde{K}$
in $I \times \Omega$. Then there is a sub-sequence, still denoted by
$\{f(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu), g(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu), \sigma(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu)\}$, weakly converging to, say,
$(f(\tau,\omega,\mu), g(\tau,\omega,\mu), \sigma(\tau,\omega,\mu))$ in $H \times H \times L_0^2$. The condition on $\mathcal{U}(T,0), t > 0$ implies that

$$\begin{cases} \mathcal{U}(T,\tau)f(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu) \to U(T,\tau)f(\tau,\omega,\mu) \\ \mathcal{U}(T,\tau)g(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu) \to U(T,\tau)g(\tau,\omega,\mu) \\ \mathcal{U}(T,\tau)\sigma(\tau,\mathcal{X}_\tau^\lambda, -\int_0^\tau h(\tau,\mu,\mathcal{X}_\mu^\lambda)d\mu) \to U(T,\tau)\sigma(\tau,\omega,\mu) \end{cases} \tag{3.6}$$

a.e. in $I \times \Omega$. On the other hand, by Assumption C, for all $0 \le \tau < T$

$$\lambda(\lambda I + \Gamma_\tau^T)^{-1} \to 0 \quad \textit{strongly as } \lambda \to 0^+ \tag{3.7}$$

and moreover

$$\|\lambda(\lambda I + \Gamma_\tau^T)^{-1}\| \le 1 \tag{3.8}$$

Thus from (3.5),(3.6),(3.7),(3.8) by the Lebesgue dominated convergence theorem it follows that $E\|\mathcal{X}^\lambda(T) - h\|^p \to 0 \quad as \quad \lambda \to 0^+$.
This gives the approximate controllability and hence the theorem is proved. □

4. Exact Controllability

Here the exact controllability for the impulsive neutral stochastic evolution functional integrodifferential equation (1.1) without a compactness assumption are taken into account.

Assumption D. $\{\mathcal{U}(t,s); t,s > 0\}$ and there exists a family of bounded linear operators $\{R(t,\tau)|0 \le \tau \le t \le T\}$ with $\|R(t,\tau)\| \le K|t-\tau|^{\delta-1}$ such that $\mathcal{U}(t,s)$ has the representation
$\mathcal{U}(t,s) = e^{-(t-s)\mathcal{A}(t)} + \int_s^t e^{-(t-\tau)a(\tau)}R(\tau,s)d\tau$,
where $exp(-\tau\mathcal{A}(t))$ denotes the analytic semigroup having the infinitesimal generator $-\mathcal{A}(t)$ and $\max_{0\le t\le T}\|\mathcal{U}(t,0)\| \le M$.

Assumption L. The linear operator L_0^T from $L_p^{\mathcal{F}}(0,T;U)$ into $L_p(\Omega,\mathcal{F},P;U)$, defined by
$L_0^T = \int_0^T U(T,s)Bu(s)ds$,
induces a bounded invertible operator $\tilde{L}$ defined on $L_p^{\mathcal{F}}(0,T;U)/\ker L_0^T$.

Assumption E.

$$6^{p-1}N_2(1+KT^{p/q}) + 6^{p-1}M^p\|B\|^pT^{p/q} + 6^{p-1}M^pN_2T^{p/q}(1+KT^{p/q}) + 6^{p-1}M^p\sum_{k=1}^{m} d_k$$

$$+6^{p-1}M^p\frac{T^{p(\beta-1)+\frac{p}{q}}}{(q\beta-q+1)^{\frac{p}{q}}}C_p\frac{T^{-p\beta+\frac{p}{2}}}{(1-2\beta)^{\frac{p}{2}}}N_1(1+KT^{p/q}) + 6^{p-1}M^pN_1T^{p/q}(1+KT^{p/q}) < 1$$

.

Using Assumptions D, L, B and B_2, for an arbitrary process Z_s, define the control process
$u(t,Z) = E\{(\tilde{L})^{-1}(h - \mathcal{U}(T,0)(\Phi(0)+g(0,\Phi,0)) - g(T,Z_T,\int_0^T h(T,s,Z_s))ds$
$+\sum_{0<t_k<T}\mathcal{U}(T,t_k)I_k(X_{t_k}) + \int_0^t \mathcal{A}(t)\mathcal{U}(T,s)g(s,Z_s,H_0^s*Z)$
$-\int_0^T \mathcal{U}(T,s)f(s,Z_s,H_0^s*Z)ds - -\int_0^T \mathcal{U}(T,s)\sigma(s,Z_s,H_0^s*Z)dW(s))|F_t\}$.

Lemma 4.1. *There exist positive real constants $N_5, \hat{N}_5 > 0$ such that $\forall X, Y \in \mathcal{H}_p$*
$E\|u(t,\mathcal{X}) - u(t,\mathcal{Y})\|^p \leq N_5\int_0^T E\|\mathcal{X}_s - \mathcal{Y}_s\|_{C_\alpha}^p ds$
$E\|u(t,\mathcal{X}\|^p \leq \hat{N}_5(1+\int_0^T E\|\mathcal{X}_s\|_{C_\alpha}^p ds)$.
We will show that, when using this control, the operator Ψ, defined by
$(\Psi Z)(t) = \mathcal{U}(t,0)(\phi(0)+g(0,\phi,0)) - g(t,Z_t,\int_0^t h(t,s,Z_s)ds)$
$-\int_0^t \mathcal{A}(t)\mathcal{U}(t,s)g(t,Z_t,H_0^s*Z) + \int_0^t\mathcal{U}(t,s)(Bu(s,Z)+f(s,Z_t,H_0^s*Z)ds$
$+\int_0^t\mathcal{U}(t,s)\sigma(s,Z_t,H_0^s*Z)dW(s) + \sum_{0\leq t_k\leq t}\mathcal{U}(t,t_k)I_k(\mathcal{X}_{t_k})$
has a fixed point Z, which is a solution of (1.1).

Theorem 4.2. *Assume that Assumptions D, L, B, B_2 and E are satisfied. Then the system (1.1) is exactly controllable on $[0,T]$.*

Proof . The proof is based on the application of the Banach fixed point theorem. First, we have to show that Ψ maps $\mathcal{H}_p$ into itself and has a unique fixed point in $\mathcal{H}_p$. It is similar to that of Lemma 3.2 and Theorem 3.3 and is omitted. □

5. Example

Let $H = L_2[0,\pi], U = L_2[0,T]$ and $\mathcal{A}(t)$ be defined by $\mathcal{A}(t)\xi = (-\partial^2/\partial x^2)\xi$, where $D(\mathcal{A}(t)) = \{\xi \in H : \xi, \frac{d\xi}{ds}$ *are absolutely continuous, and* $\frac{d^2\xi}{dx^2} \in H, \xi(0) = \xi(\pi) = 0\}$. Let $\mathcal{B} \in L(\mathcal{R}, X)$ be defined as $(\mathcal{B}u)(x) = b(x)u, 0 \leq x \leq \pi, u \in \mathcal{R}, b(x) \in L_2[0,\pi]$. Let $p > 2, 0 < \alpha < (p-2)/2p$ and suppose $r > 0$ is a real number. Set $H_\alpha = D(A^\alpha(t))$ and $C_\alpha = PC([-r,0],H_\alpha)$. It is well known that $\mathcal{A}(t)$ is a closed, densely defined linear operator. Let $\beta(t)$ denote a one-dimensional standard Brownian motion and the constant α_k, $(k = 1,2,...,m)$ are small.
Consider the neutral stochastic delay diffusion equation with impulses of the form

$$
\begin{aligned}
d[Z(t,x) &+ G(t, Z(-r_1(t), \int_{-r}^{t}\int_0^{\pi} h(s-t,y,x)Z(s,y)dyds)] \\
&= [(\frac{\partial^2}{\partial\theta^2} + a(t))Z(t,x) + b(x)u(t) + F(t, \mathcal{X}(t-r_2(t), \int_{-r}^{t}\int_0^{\pi} h(s-t,y,x)Z(s,y)dyds)]dt \\
&+ \mathbb{K}(t, X(t-r_3(t)\int_{-r}^{t}\int_0^{\pi} h(s-t,y,x)Z(s,y)dyds)d\beta(t) \\
Z(t,0) &= Z(t,\pi) = 0, t \in J = [0,T], \\
Z(s,x) &= \phi(s,x), \quad -r \le s \le 0,\ 0 \le x \le \pi, \\
\Delta Z(t_k,x) &= I_k(Z(t_k,x)) = (\alpha_k|Z(x)| + t_k)^{-1}, x \in U,\ 1 \le k \le m. \qquad (5.1)
\end{aligned}
$$

where r_1, r_2, r_3 are continuous with $0 < r_i(t) < r, i = 1, ..., 3$ for all $t \ge 0$ and $\phi \in C_\alpha$. Suppose $F : [0,\infty)\times C_\alpha \times H \to H, G : [0,\infty)\times C_\alpha\times H \to H, \mathbb{K} : [0,\infty)\times C_\alpha\times H \to H, h : [0,\infty)\times[0,\infty)\times C_\alpha \to H$ are both ongoing and global Lipschitz is uniformly bounded in the second variable and continuous in the first. Furthermore, in the first variable, G is continuously differentiable.

Then it is not difficult to verify that $\mathcal{A}(t)$ generates an evolution operator $\mathcal{U}(t,s)$ satisfying assumptions (B1)-(B3) [10] and $\mathcal{U}(t,s) = T(t-s)e^{-\int_s^t a(\tau)d\tau}$, where $T(t)$ is the analytic semigroup generated by the operator $-A$ with $-A\xi = -\xi''$ for $\xi \in D(A)$. It is easy to compute that, A has a discrete spectrum, and note that there exists a complete orthonormal set $\{e_n\}, n \ge 1$, of eigenvectors of A with $e_n(x) = \sqrt{2/\pi}\sin\ nx$ and the analytic semigroup $T(t)$, $t \ge 0$, generated by A such that

$$
-\mathcal{A}(t)\xi = \sum_{n=1}^{\infty}(-n^2 - a(t))(\xi, e_n)e_n, \xi \in D(A)
$$

and clearly, the common domain and the operator A's domain are the same. In addition, we can define $A^\alpha(t_0)(t_0 \in [0,a])$ for self-adjoint operator $A(t_0)$ by the classical spectral theorem and it is easy to deduce that

$$
A^\alpha(t_0)\xi = \sum_{n=1}^{\infty}(n^2 + a(t))^\alpha(\xi, e_n)e_n
$$

on the domain

$$
D(A^\alpha(t_0)) = \{\xi \in H, \sum_{n=1}^{\infty}(n^2 + a(t))^\alpha(\xi, e_n)e_n \in H\}.
$$

Particularly,

$$
A^{\frac{1}{2}}(t_0)\sum_{n=1}^{\infty}\sqrt{(n^2 + a(t))^\alpha}(\xi, e_n)e_n.
$$

Therefore, we have that, for each $\xi \in H$,

$$
\mathcal{U}(t,s)\xi = \sum_{n=1}^{\infty} e^{-n^2(t-s)-\int_s^t a(\tau)d\tau}(\xi, e_n)e_n, A^\alpha(t_0)A^{-\beta}(t_0)\xi = \sum_{n=1}^{\infty}(n^2 + a(t))^{\alpha-\beta}(\xi, e_n)e_n
$$

and

$$
A^\alpha(t_0)\mathcal{U}(t,s)\xi = \sum_{n=1}^{\infty}(n^2 + a(t))^\alpha e^{-n^2(t-s)-\int_s^t a(\tau)d\tau}(\xi, e_n)e_n.
$$

Then,

$$\|A^{\alpha}(t)A^{-\beta}(s) \leq (1+\|a(\cdot)\|)^{\alpha}, \|A^{\beta}(t)\mathcal{U}(t,s)A^{-\beta}(s)\| \leq (1+\|a(\cdot)\|)^{\beta} \quad \forall t,s\in[0,T], 0<\alpha<\beta,$$

$$\|A^{\alpha}(t)\mathcal{U}(t,s)\xi\|^2 \leq (t-s)^{-2\alpha}e^{-2a(t-s)}\sum_{n=1}^{\infty}\alpha^{2\alpha}e^{a(t)(t-s)-2\int_s^t a(\tau)d\tau}|(\xi,e_n)|^2$$

note that $c\log x - x \leq c\log c - c$, which shows that

$$\|A^{\alpha}(t)\mathcal{U}(t,s)\| \leq M_{\alpha}e^{-a(t-s)}(t-s)^{-\alpha},$$

for $M_{\alpha} = \alpha^{\alpha}\max\{e^{a(t)(t-s)-\int_s^t a(\tau)d\tau} : t,s\in[0,T]\} > 0$. Now let

$$\int_0^t h(t,s,\psi)(x)ds = \int_{-r}^0\int_0^{\pi} h(s,y,x)\psi(s,y)dyds, g(t,\phi,\psi)(x) = G(t,\phi(-r_1(t)),h(t,s,\psi)(x))$$

and

$$f(t,\phi,\psi)(x) = F(t,\phi(-r_2(t)),h(t,s,\psi)(x)), \sigma(t,\phi,\psi)(x) = \mathbb{K}(t,\phi(-r_3(t)),h(t,s,\psi)(x)),$$

for all $\phi,\psi\in C_{\alpha}$ and any $x\in[0,\pi]$.

Then we have for any fixed $s\in[-r,0], \phi_i,\psi_i\in C_{\alpha}, i=1,2,$

$$f(t,\phi_1,\psi_1) - f(t,\phi_2,\psi_2)\|^2 \leq N_1(\|\phi_1-\phi_2\|^2_{C_{\alpha}} + \|\psi_1-\psi_2\|^2_{C_{\alpha}}),$$

$$\|A^{\alpha}(t)g(t,\phi_1,\psi_1) - A^{\alpha}(t)g(t,\phi_2,\psi_2)\|^2 N_2(\|\phi_1-\phi_2\|^2_{C_{\alpha}} + \|\psi_1-\psi_2\|^2_{C_{\alpha}}),$$

$$\|\sigma(t,\phi_1,\psi_1) - \sigma(t,\phi_2,\psi_2)\|^2 N_3(\|\phi_1-\phi_2\|^2_{C_{\alpha}} + \|\psi_1-\psi_2\|^2_{C_{\alpha}}),$$

where $N_i > 0, i = 1,2,3$ are constants. Further, $f(t,\phi,\psi), g(t,\phi,\psi)$ and $\sigma(t,\phi,\psi)$ are uniformly bounded. On the other hand, it is known that the deterministic linear system corresponding to (5.1) is approximately controllable on every $[0,T], t>0$, provided that $\int_0^{\pi} b(x)e_n(x)dx \neq 0$, for $n = 1,2,3,....$ As a result of this choice of A,B,f,g and I k, it is clear that the assumptions in Theorem 3.4 are met, and that (1.1) is the abstract formulation of (5.1), implying that the system (5.1) is roughly controllable on $[0,T]$.

References

[1] N.U. Ahmed and X. Ding, *A semilinear Mckean-Vlasov stochastic evolution equation in Hilbert spaces*, Stochastic Process Appl. 60 (1995) 65–85.

[2] D.D. Bainov and P.S. Simenov, *Systems with Impulse Effect*, Ellis Horwood, Chichester,1989.

[3] K. Balachandran, S. Karthikeyan and J.Y. Park, *Controllability of stochastic systems with distributed delays in control*, Int. J. Control 82 (2009) 1288–1296.

[4] G. Ballinger, X. Liu, *Boundness for impulsive delay differential equations and applications to population growth models*, Nonlinear Anal. Theory Methods Appl. 53 (2003) 1041–1062.

[5] Y.K. Chang, *Controllability of impulsive functional differential systems with infinite delay in Banach spaces*, Chaos Solitons & Fractals 33 (2007) 1601–1609.

[6] R. Curtain, H.J. Zwart, *An Introduction to Infinite Dimensional Linear Systems Theory*, New York: Springer, 1995.

[7] J.P. Dauer and N.I. Mahmudov, *Controllability of stochastic semilinear functional differential equations in Hilbert spaces*, J. Math. Anal. Appl. 290 (2004) 373–394.

[8] J.P. Dauer, N.I. Mahmudov and M.M. Matar, *Approximate controllability of backward stochastic evolution equations in Hilbert spaces*, J. Math. Anal. Appl. 323 (2006) 42–56.

[9] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Uni. Press, Cambridge, 1992.

[10] A. Friedman, *Partial Differential Equations*, Holt, Rinehart and Winston, New York, 1969.

[11] T.E. Govindan, *Stability of mild solutions of stochastic evolution equations with variable delay*, Stochastic Anal. Appl. 21 (2003) 1059–1077.

[12] E. Hernandez, M. Rabello and H.R. Henriquez, *Existence of solutions for impulsive partialneutral functional differential equations*, J. Math. Anal. Appl. 331 (2007) 1135–1158.

[13] Z. He, X. He, *Periodic boundary value problems for first order impulsive integrodifferential equations of mixed type*, J. Math. Anal. Appl. 296 (2004) 8–20.

[14] J. Klamka, *Constrained controllability of semilinear systems with multiple delays in control*, Bull. Polish Academy Sci. Technical Sci. 52 (2004) 25–30.

[15] J. Klamka, *Stochastic controllability of linear systems with delay in control*, Bull. Polish Academy Sci. Technical Sci. 55 (2007) 55 23–29.

[16] V. Lakshmikantham, D.D. Bainov and P.S. Simenov, *Theory of Impulsive Differential Equations*, World scientific, Singapore, 1989.

[17] M.L. Li, M.S. Wang and F.Q. Zhang, *Controllability of impulsive functional differential systems in Banach spaces*, Chaos, Solitons & Fractals 29 (2006) 175–181.

[18] N.I. Mahmudov, *On controllability of linear stochastic systems in Hilbert spaces*, J. Math. Anal. Appl. 259 (2001) 64–82.

[19] N.I. Mahmudov, *Approximate controllability of semilinear deterministic and stochastic evolution equations in abstract spaces*, SIAM J. Control Optim. 42 (2003) 1604–1622.

[20] N.I. Mahmudov, *Controllability of stochastic semilinear functional differential equations in Hilbert spaces*, J. Math. Anal. Appl. 290 (2004) 373–394.

[21] A. Pazy, *Semigroups of Linear Operators and Applications to Partial DifferentialEquations*, Springer-Verlag, Berlin, 1983.

[22] B. Radhakrishnan and P. Chandru, *Boundary controllability of impulsive integrodifferential evolution systems with time-varying delays*, J. Taibah Uni. Sci. 12 (2018) 520–531.

[23] B. Radhakrishnan and T. Sathya, *Controllability and periodicity results for neutral impulsive evolution system in Banach spaces*, Dynamics of Continuous, Discrete Impulsive Syst. 26(4) (2019) 261–277.

[24] B. Radhakrishnan and M. Tamilarasi, *Existence of solutions for quasilinear random impulsive neutral differential evolution equation*, Arab J. Math. Sci. 24(2) (2018) 235–246.

[25] B. Radhakrishnan and M. Tamilarasi, *Existence, uniqueness and stability results for fractional hybrid pantograph equation with random impulse*, Dynamics Continuous, Discrete Impulsive Systems Series B: Appl. & Algorithms 28(3) (2021) 165–181.

[26] B. Radhakrishnan, M. Tamilarasi and P. Anukokila, *Existence, uniqueness and stability results for semilinear integrodifferential non-local evolution equations with random impulse*, Filomat 32(19) (2018) 6615–6626.

[27] R. Subalakshmi and K. Balachandran, *Approximate controllability of neutral stochastic integrodifferential systems in Hilbert spaces*, Elect. J. Diff. Equ. 162 (2008) 1–15.

[28] R. Subalakshmi and K. Balachandran, *Approximate controllability of nonlinear stochastic impulsive integrodifferential systems in Hilbert spaces*, Chaos, Solitons & Fractals 42 (2009) 2035–2046.

[29] H. Zhang, Z. Hang and G. Feng, *Reliable dissipative control for stochastic impulsive systems*, Automatica 44 (2008) 1004–1010.