Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 2025-2042
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.22017.2315

Non-linear contractions via auxiliary functions and fixed point results with some consequences

Ankush Chanda[a,*], Lakshmi Kanta Dey[b], Arslan Hojat Ansari[c]

[a] Department of Mathematics, Vellore Institute of Technology, Vellore, India
[b] Department of Mathematics, National Institute of Technology Durgapur, India
[c] Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this manuscript, we bring into play the essence of a new class of auxiliary functions, C-class functions, and exhibit some fixed point results. Notably, in this article, we come up with the idea of modified $\mathcal{Z}_F$-contractions and enquire the existence and uniqueness of fixed points of such operators in the framework of θ-metric spaces. Concerning the interpretation of the achieved results, some non-trivial examples are also studied. From obtained theorems, we derive several related fixed point results in usual metric spaces and θ-metric spaces.

Keywords: Fixed point, C-class function, $\mathcal{Z}_F$-contraction, modified $\mathcal{Z}_F$-contraction.
2010 MSC: 47H10,54H25.

1. Introduction

Mathematical analysis with its innumerable disciplines furnishes effective tools in the study of various real world problems emerging in applied sciences. Particularly, this assertion is implemented to the progress of metric fixed point theory. As it happens, the Banach contraction principle [5] is a theoretical result on the existence and uniqueness of a fixed point in metric spaces, but having said that, is an iterative algorithm to approximate this fixed point. This wonderful result has been generalized and extended in numerous abstract spaces using various contractive conditions. Besides, the fixed point outcomes and techniques have engaged many scientists and hence there are extensive findings at hand in various metric settings [4, 3, 14, 9, 11, 15, 6].

*Corresponding author
Email addresses: ankushchanda8@gmail.com (Ankush Chanda), lakshmikdey@yahoo.co.in (Lakshmi Kanta Dey), analsisamirmath2@gmail.com (Arslan Hojat Ansari)

Received: December 2020 *Accepted:* April 2021

In recent past, Khojasteh, Shukla and Radenović [12] put forward the concept of $\mathcal{Z}$-contractions employing simulation functions. This family of functions includes a large number of non-linear contractions present in fixed point theory. Lately, Ansari [1] introduced the notion of C-class functions and the motive behind defining this new idea is to generalize many fixed point theorems in the literature. Afterwards, Liu et al. [13] made use of these functions to generalize the idea of simulation functions, namely C_F-simulation functions and explored the existence and uniqueness of coincidence points for two non-linear operators.

Driven by the notion of fuzzy metric, in a recent research article Khojasteh et al. [10] coined θ-metric by introducing a more generalized triangle inequality. Later on, Chanda et al. [7] attained some fixed point results via simulation functions on the θ-metric spaces.

The purpose of this paper is to employ $\mathcal{Z}_F$-contractions to derive the results on existence and uniqueness of fixed points of some self-maps in θ-metric spaces. Besides, we originate the idea of a modified $\mathcal{Z}_F$-contraction and achieve a fixed point theorem employing this notion on the said spaces. Examples are furnished which illustrate our results and their applicability.

In this article, first of all we look back on some requisite definitions, examples and noteworthy results in the preliminaries section. In the main results section, we define $\mathcal{Z}_F$-contractions and modified $\mathcal{Z}_F$-contractions in the setting of θ-metric spaces and derive some fixed point results. Indeed, these results complement, extend and enrich many a number of results in the existing literature. Moreover, several non-trivial examples are equipped to evoke the relevancy of the obtained theorems. As consequences of this study, we infer a few related fixed point results in usual metric spaces and θ-metric spaces.

2. Preliminaries

Before all else, we look back on some definitions and auxiliary notions that can be found in [7, 8, 12, 10]. Precisely, all through this paper, $\mathbb{N}$ will represent the set of all naturals and $\mathbb{R}$ will mean the set of all reals.

Firstly, we put down the ideas of B-actions and θ-metrics here. As a proper generalization of a metric, Khojasteh et al. [10] offered the concept of a θ-metric.

Definition 2.1. *[10] Suppose $\theta : [0,\infty)^2 \to [0,\infty)$ be a continuous map in both variables. Let $Im(\theta) = \{\theta(m,n) : m \geq 0, n \geq 0\}$. The map θ is said to be an B-action iff the following conditions hold:*

(B1) $\theta(0,0) = 0$ *and* $\theta(m,n) = \theta(n,m)$ *for all* $m,n \geq 0$,

(B2)

$$\theta(m,n) < \theta(x,y) \;\;\Rightarrow\;\; \begin{cases} \text{either } \;m < x, n \leq y \\ \text{or } \;m \leq x, n < y, \end{cases}$$

(B3) for each $r_1 \in Im(\theta)$ *and for each* $m \in [0,r_1]$, *there exists* $n \in [0,r_1]$ *such that* $\theta(n,m) = r_1$,

(B4) $\theta(m,0) \leq m$, *for all* $m > 0$.

Example 2.2. *[10] We give some examples from the existing literature.*

1. $\theta_1(s,t) = \frac{ts}{1+ts}$.
2. $\theta_2(s,t) = \sqrt{ts} + t + s$.

Y denotes the set of all B-actions.

The notion of B-action has been handy to conceive the idea of θ-metric spaces [10]. We here recollect the definition of the said spaces.

Definition 2.3. *[10] Let Y be a non-empty set. A mapping $d_\theta : Y \times Y \to [0, \infty)$ is called a θ-metric on Y with respect to B-action $\theta \in Y$ if d_θ satisfies the following:*

(θ1) $d_\theta(u, v) = 0$ if and only if $u = v$ for all $u, v \in Y$,
(θ2) $d_\theta(u, v) = d_\theta(v, u)$ for all $u, v \in Y$,
(θ3) $d_\theta(u, v) \leq \theta(d_\theta(u, w), d_\theta(w, v))$ for all $u, v, w \in Y$.

Then the pair (Y, d_θ) is called a θ-metric space.

Example 2.4. *[10] Here we provide a non-trivial example of θ-metric space.*

Let $Y = \{a, b, c\}$ and $d_\theta : Y \times Y \to [0, \infty)$ is defined as:

$$d_\theta(a, b) = 5, d_\theta(b, c) = 12, d_\theta(c, a) = 13, d_\theta(a, b) = d_\theta(b, a),$$

$$d_\theta(b, c) = d_\theta(c, b), d_\theta(c, a) = d_\theta(a, c), d_\theta(a, a) = d_\theta(b, b) = d_\theta(c, c) = 0.$$

Considering $\theta(s, t) = \sqrt{s^2 + t^2}$, the mapping d_θ forms a θ-metric. And so the pair (Y, d_θ) is a θ-metric space.

Remark 2.5 (cf. [10]). *If (Y, d_θ) is a θ-metric space and $\theta(s, t) = t + s$, for all $s, t \in [0, \infty)$, then (Y, d_θ) is a metric space.*

For more diction and attained results, see [10].

Ansari [1] considered the concept of C-class functions as the following:

Definition 2.6. *[1] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is said to be a C-class function if it is continuous and satisfies following conditions:*

1. *$F(s, t) \leq s$,*
2. *$F(s, t) = s$ implies that either $s = 0$ or $t = 0$, for all $s, t \in [0, \infty)$.*

Note that for some F, we have $F(0, 0) = 0$. We denote the family of C-class functions as $\mathcal{C}$. For additional examples of C-class functions, see [1, 2].

Definition 2.7. *[13] A mapping $F : [0, \infty)^2 \to \mathbb{R}$ has the property C_F, if there exists a $C_F \geq 0$ such that*

1. *$F(s, t) > C_F \implies s > t$,*
2. *$F(t, t) \leq C_F$, for all $t \in [0, \infty)$.*

Example 2.8. *[13] The following functions $F_i : [0, \infty)^2 \to \mathbb{R}$ are some elements of $\mathcal{C}$ having the property C_F, for all $s, t \in [0, \infty)$.*

1. *$F_1(s, t) = s - t, \quad C_F = r, \quad r \in [0, \infty)$.*
2. *$F_2(s, t) = \frac{s}{(1+t)^r}, \quad r \in (0, \infty), \quad C_F = 1$.*

Now we define a C_F-simulation function using C-class functions with property C_F.

Definition 2.9. *[13] A C_F-simulation function is a mapping $\xi : [0,\infty)^2 \to \mathbb{R}$ satisfying the following axioms:*

(ξ1) $\xi(0,0) = 0$,
(ξ2) $\xi(t,s) < F(s,t)$ *for all* $t,s > 0$, *where* $F \in \mathcal{C}$ *satisfying property* C_F,
(ξ3) if $\{t_n\}$, $\{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

then

$$\limsup_{n\to\infty} \xi(t_n, s_n) < C_F.$$

The third condition is symmetric in both the arguments of ξ. But, in the proofs, this property is not at all necessary. Practically, the arguments of ξ stand for different meanings and represent different roles. So the authors slightly modified the previous definition with a view to underline this feature and to expand the family of C_F-simulation functions.

Definition 2.10. *[13] A C_F-simulation function is a mapping $\xi : [0,\infty)^2 \to \mathbb{R}$ satisfying the following conditions:*

(ξ_a) $\xi(0,0) = 0$,
(ξ_b) $\xi(t,s) < F(s,t)$ *for all* $t,s > 0$, *where* $F \in \mathcal{C}$ *satisfying property* C_F,
(ξ_c) if $\{t_n\}, \{s_n\}$ *are sequences in* $(0,\infty)$ *such that*

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0,$$

and $t_n < s_n$, *then*

$$\limsup_{n\to\infty} \xi(t_n, s_n) < C_F.$$

Let $\mathcal{Z}_F$ be the family of C_F-simulation functions. Every simulation function is also a C_F-simulation function. But the converse is not true, in general. The following example illustrates the claim.

Example 2.11. *[13] Let $\xi : [0,\infty)^2 \to \mathbb{R}$ be a function defined by $\xi(t,s) = kF(s,t)$, where $t,s \in [0,\infty)$ and $k \in \mathbb{R}$ be such that $k < 1$ and for all $t,s \in [0,\infty)$. We consider $C_F = 1$.*

Here, using Definitions 2.6 and 2.7, we have

$$\begin{aligned} \xi(t,s) &= kF(s,t) \\ &\leq ks \\ &< s \end{aligned}$$

and

$$\begin{aligned} \xi(t,t) &= kF(t,t) \\ &< 1. \end{aligned}$$

One can easily check that, ξ satisfies (ξ_a) and (ξ_b). Now if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \delta > 0$$

and $t_n < s_n$ for all $n \in \mathbb{N}$, then by Definition 2.7,

$$\begin{aligned}\limsup_{n\to\infty} \xi(t_n, s_n) &= \limsup_{n\to\infty} kF(s_n, t_n) \\ &\leq kF(\delta, \delta) \\ &< 1.\end{aligned}$$

Therefore ξ is a C_F-simulation function. Also, if we take $F(s,t) = \frac{s}{1+t}$, then also $\xi(t,s) = \frac{ks}{1+t}$ is a C_F-simulation function. In fact, we can check that the aforementioned ξ is not a simulation function.

Example 2.12. *[13] Let $F : [0,\infty)^2 \to \mathbb{R}$ be a C-class function such that*

$$F(\psi(s), \varphi(t)) - t < F(s,t), \quad \psi(t) < t,$$

and let $\xi : [0,\infty)^2 \to \mathbb{R}$ be the function defined as

$$\xi(t,s) = F(\psi(s), \varphi(t)) - t.$$

Then $\xi(t,s)$ is a C_F-simulation function with $C_F = 0$.

The subsequent example is also an example of a C_F-simulation function.

Example 2.13. *Let $\xi : [0,\infty)^2 \to \mathbb{R}$ be a function defined by $\xi(t,s) = s - 2t$, for all $t, s \in [0,\infty)$. We consider $F(s,t) = s - t$ and $C_F = 1$. Then ξ is a C_F-simulation function.*

3. Main Results

In the following section, we deduce a couple of fixed point theorems concerning self-maps via C_F-simulation functions on account of the notion of θ-metric spaces and also we present suitable examples. We set up with the definition of a $\mathcal{Z}_F$-contraction in the setting of a θ-metric space.

Definition 3.1. *Let (X, d_θ) be a θ-metric space and $T : X \to X$ be a self-mapping. A mapping T is called a $\mathcal{Z}_F$-contraction if there exists $\xi \in \mathcal{Z}_F$ such that*

$$\xi(d_\theta(Tx, Ty), d_\theta(x,y)) \geq C_F \tag{3.1}$$

for all $x, y \in X$ such that $x \neq y$.

Remark 3.2. *1. From (ξ_b), it is clear that a C_F-simulation function must verify $\xi(r,r) < C_F$ for all $r > 0$.*

2. If T is a $\mathcal{Z}_F$-contraction with respect to $\xi \in \mathcal{Z}_F$, then

$$d_\theta(Tx, Ty) < d_\theta(x,y),$$

for all $x, y \in X$ such that $x \neq y$.

At the very beginning, we establish some lemmas which are essential to pull off our main results.

Lemma 3.3. *Let $T : X \to X$ be any $\mathcal{Z}_F$-contraction with respect to any C_F-simulation function $\xi \in \mathcal{Z}_F$ defined on a complete θ-metric space (X, d_θ). This implies that T is asymptotically regular at each arbitrary element of X.*

Proof. Let $x \in X$. Without loss of generality, consider $T^n x \neq T^{n+1}x$ for all $n \in \mathbb{N}$. This implies

$$d_\theta(x_n, x_{n+1}) > 0$$

for all $n \in \mathbb{N}$. Since T is a $\mathcal{Z}_F$-contraction and ξ is any C_F-simulation function, we obtain,

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tx_{n+1}), d_\theta(x_n, x_{n+1})) \\ &< F(d_\theta(x_n, x_{n+1}), d_\theta(Tx_n, Tx_{n+1})) \end{aligned} \tag{3.2}$$

for some $C_F \geq 0$, $F \in \mathcal{C}$ and for all $n \in \mathbb{N}$. From (3.2) using Definition 2.7, we get

$$0 < d_\theta(Tx_n, Tx_{n+1}) = d_\theta(x_{n+1}, x_{n+2}) < d_\theta(x_n, x_{n+1}).$$

So $\{d_\theta(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative real numbers. Thus there exists some $r \geq 0$ such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = r.$$

We claim that $r = 0$. To prove this, we choose $t_n = d_\theta(Tx_n, Tx_{n+1})$ and $s_n = d_\theta(x_n, x_{n+1})$ and we know that $t_n < s_n$ for all $n \in \mathbb{N}$. Since T is a $\mathcal{Z}_F$-contraction with respect to ξ, applying (ξ_c), we obtain

$$\limsup_{n\to\infty} \xi(d_\theta(Tx_n, Tx_{n+1}), d_\theta(x_n, x_{n+1})) < C_F.$$

But from (3.1), we have

$$\xi(d_\theta(Tx_n, Tx_{n+1}), d_\theta(x_n, x_{n+1})) \geq C_F,$$

which is a contradiction. So we can conclude that $r = 0$ and hence

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

Therefore T is asymptotically regular at each $x \in X$. □

Lemma 3.4. *Assume that T is any $\mathcal{Z}_F$-contraction with respect to $\xi \in \mathcal{Z}_F$ and is defined on any complete θ-metric space (X, d_θ). Then whenever T possesses a fixed point in X, it is unique.*

Proof. Take $u, v \in X$ be two distinct fixed points of T. Therefore $Tu = u$, $Tv = v$ and

$$d_\theta(u, v) = d_\theta(Tu, Tv) > 0.$$

Since T is a $\mathcal{Z}_F$-contraction with respect to $\xi \in \mathcal{Z}_F$, we have

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tu, Tv), d_\theta(u, v)) \\ &= \xi(d_\theta(Tu, Tv), d_\theta(Tu, Tv)) \\ &< F(d_\theta(Tu, Tv), d_\theta(Tu, Tv)). \end{aligned}$$

From Definition 2.7, we find

$$d_\theta(Tu, Tv) < d_\theta(Tu, Tv),$$

which is impossible. Hence the assertion is proved. □

Here, we put down a new version of Lemma 2.1 of [16] and moreover, we generalize it in our context.

Lemma 3.5. *Let (X, d_θ) be a θ-metric space and $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence in (X, d_θ), then there exist $\epsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that $n_k > m_k > k$ and such that the following sequences $\{d_\theta(x_{m_k}, x_{n_k})\}$ and $\{d_\theta(x_{m_k+1}, x_{n_k+1})\}$ tend to ϵ as $k \to \infty$.

Proof. If $\{x_n\}$ is not a Cauchy sequence in (X, d_θ), then there exist $\epsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that $n_k > m_k > k$ and

$$d_\theta(x_{m_k}, x_{n_k-1}) < \epsilon, \quad d_\theta(x_{m_k}, x_{n_k}) \geq \epsilon$$

for all $k \in \mathbb{N}$. Hence we have,

$$\begin{aligned} \epsilon &\leq d_\theta(x_{m_k}, x_{n_k}) \\ &\leq \theta(d_\theta(x_{n_k}, x_{n_k-1}), d_\theta(x_{n_k-1}, x_{m_k})). \end{aligned}$$

So passing to the limit when $k \to \infty$ in the above inequality and employing $(B4)$, we obtain

$$\begin{aligned} \epsilon &\leq \lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k}) \\ &\leq \lim_{k\to\infty} \theta(d_\theta(x_{n_k}, x_{n_k-1}), d_\theta(x_{n_k-1}, x_{m_k})) \\ &\leq \theta(\lim_{k\to\infty} d_\theta(x_{n_k}, x_{n_k-1}), \lim_{k\to\infty} d_\theta(x_{n_k-1}, x_{m_k})) \\ &\leq \theta(0, \lim_{k\to\infty} d_\theta(x_{n_k-1}, x_{m_k})) \\ &\leq \lim_{k\to\infty} d_\theta(x_{n_k-1}, x_{m_k}) \\ &\leq \epsilon. \end{aligned}$$

As a result,

$$\lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.3}$$

We observe that,

$$d_\theta(x_{m_k}, x_{n_k}) \leq \theta(d_\theta(x_{m_k}, x_{n_k+1}), d_\theta(x_{n_k+1}, x_{n_k})) \tag{3.4}$$

and also

$$d_\theta(x_{m_k}, x_{n_k+1}) \leq \theta(d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{n_k}, x_{n_k+1})). \tag{3.5}$$

So passing to the limit when $k \to \infty$, and considering (3.4) and (3.5), we obtain

$$\lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k+1}) = \epsilon.$$

Again we notice that

$$d_\theta(x_{m_k+1}, x_{n_k+1}) \leq \theta(d_\theta(x_{m_k+1}, x_{m_k}), d_\theta(x_{m_k}, x_{n_k+1}))$$

and also

$$d_\theta(x_{m_k}, x_{n_k+1}) \leq \theta(d_\theta(x_{m_k+1}, x_{n_k+1}), d_\theta(x_{m_k+1}, x_{m_k})).$$

So by the previous inequalities, when $k \to \infty$, we get

$$\lim_{k\to\infty} d_\theta(x_{m_k+1}, x_{n_k+1}) = \epsilon. \tag{3.6}$$

□

By means of these lemmas, now we are in a position to state our first main result here.

Theorem 3.6. *Suppose that $T : X \to X$ is a $\mathcal{Z}_F$-contraction concerning a C_F-simulation function $\xi \in \mathcal{Z}_F$ in a complete θ-metric space (X, d_θ). Then T has a unique fixed point u in X.*

Proof. By Lemma 3.3, the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, is such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

Now, by Lemma 3.5, if $\{x_n\}$ is not a Cauchy sequence in (X, d_θ), then there exist $\epsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that $n_k > m_k > k$ and such that

$$\lim_{k\to\infty} d_\theta(x_{m_k}, x_{n_k}) = \epsilon = \lim_{k\to\infty} d_\theta(x_{m_k+1}, x_{n_k+1}).$$

However, there exists $n_1 \in N$ such that

$$d_\theta(x_{m_k}, x_{n_k}) > \frac{\epsilon}{2} > 0$$

and

$$d_\theta(x_{m_k+1}, x_{n_k+1}) > \frac{\epsilon}{2} > 0$$

for all $k \geq n_1$. Now, that T is a $\mathcal{Z}_F$-contraction with respect to ξ and using the axiom (ξ_b), we obtain that

$$\begin{aligned} C_F &\leq \xi(d_\theta(x_{m_k+1}, x_{n_k+1}), d_\theta(x_{m_k}, x_{n_k})) \\ &= \xi(d_\theta(Tx_{m_k}, Tx_{n_k}), d_\theta(x_{m_k}, x_{n_k})) \\ &< F(d_\theta(x_{m_k}, x_{n_k}), d_\theta(Tx_{m_k}, Tx_{n_k})). \end{aligned} \tag{3.7}$$

Using Definition 2.7, this implies

$$0 < d_\theta(x_{m_k+1}, x_{n_k+1}) < d_\theta(x_{m_k}, x_{n_k}) \tag{3.8}$$

for all $k \geq n_1$. Employing the sequences $\{t_k\} = \{d_\theta(x_{m_k+1}, x_{n_k+1})\}$ and $\{s_k\} = \{d_\theta(x_{m_k}, x_{n_k})\}$, which have the same positive limit by (3.3) and (3.6) and using (3.8) in axiom (ξ_c), we conclude that

$$\limsup_{k\to\infty} \xi(t_k, s_k) < C_F$$

which is a contradiction to (3.7). Hence $\{x_n\}$ is a Cauchy sequence. Since (X, d_θ) is complete, there exists $z \in X$, such that

$$\lim_{n\to\infty} x_n = z \tag{3.9}$$

or equivalently

$$\lim_{n\to\infty} d_\theta(Tx_n, z) = 0.$$

Next we confirm that z is a fixed point of T. To the contrary, let $Tz \neq z$. Then $0 < d_\theta(z, Tz) = \delta$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} d_\theta(Tx_n, z) &< \delta \\ &= d_\theta(Tz, z) \end{aligned}$$

for all $n \geq n_0$. This leads us to

$$Tx_n \neq Tz$$

which implies

$$d_\theta(Tx_n, Tz) \quad > \quad 0 \tag{3.10}$$

for all $n \geq n_0$. Now, there does not exist some $n_3 \in \mathbb{N}$ such that for all $n \geq n_3$

$$x_n = z.$$

Therefore, there exists a subsequence $\{x_{p_k}\}$ of $\{x_n\}$ such that

$$x_{p_k} \quad \neq \quad z \tag{3.11}$$

for all $k \in \mathbb{N}$. Now, let $n_2 \in \mathbb{N}$ be such that $p_{n_2} \geq n_0$. Hence by (3.10) and (3.11), we have $d_\theta(Tx_{p_n}, Tz) > 0$ and $d_\theta(x_{p_n}, z) > 0$ for all $n \geq n_2$. Using (ξ_b),

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_{p_n}, Tz), d_\theta(x_{p_n}, z)) \\ &< F(d_\theta(x_{p_n}, z), d_\theta(Tx_{p_n}, Tz)) \end{aligned}$$

for all $n \geq n_2$. In view of Definition 2.7, this means

$$0 < d_\theta(Tx_{p_n}, Tz) < d_\theta(x_{p_n}, z)$$

for all $n \geq n_2$. In particular, using sandwich theorem and (3.9), we have

$$\lim_{n\to\infty} Tx_{p_n} = Tz.$$

Again, $\{Tx_{p_n}\} = \{x_{p_n+1}\}$ is a subsequence of $\{x_n\}$, which converges to z. By the unicity of the limit, we conclude that $Tz = z$. Hence z is a fixed point of T. Lemma 3.4 guarantees the uniqueness of the fixed point. □

Here we affirm the previous result by subsequent examples.

Example 3.7. *Let $X = \{1, 3, 5, 7, 9\}$ be equipped with the Euclidean metric*

$$d_\theta(x, y) = |x - y|.$$

We consider $\theta(s, t) = st+s+t$ and define a mapping T on X such that $T1 = T3 = T5 = T7 = T9 = 3$. It is easy to check that

$$d_\theta(Tx, Ty) = 0$$

for all $x, y \in X$. Here, we take $\xi(t, s) = \frac{ks}{1+t}$ for all $t, s \in [0, \infty)$, as the C_F-simulation function, where $k \in [\frac{1}{2}, 1)$ and $C_F = 1$. Then it is easy to check that T satisfies

$$\xi(d_\theta(Tx, Ty), d_\theta(x, y)) \geq 1$$

$x, y \in X$. Hence T is a $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ. Making use of Theorem 3.6 we get, T has a unique fixed point and it is $u = 3 \in X$.

Example 3.8. *Consider the metric space l^∞ equipped with the sup metric. Take $C = \{e_0, e_i : i \in \mathbb{N}\}$ where e_0 is the zero sequence and e_i is the sequence whose i-th term is 3^i and all the other terms are 0. Then one can easily check that C is a closed subset of l^∞ and hence complete.*

We define $T : C \to C$ such that

$$Tx = \begin{cases} e_0, & \text{if } x = e_i,\ i = 1, \dots, 5; \\ e_{i-5}, & \text{if } x = e_i,\ i \geq 6. \end{cases}$$

We also consider $\theta(s,t) = s+t+st$ and $\xi(t,s) = \frac{ks}{1+t}$, $t, s \in [0, \infty)$, as the extended C_F-simulation function, where $k = \frac{1}{2}$ and $C_F = 1$. Let $x, y \in C$ be arbitrary with $x \neq y$ and the three cases may arise.

Case I: *$x, y \in \{e_i, i = 1, \dots, 5\}$. It is easy to check that*

$$\begin{aligned} Tx =& e_0 = Ty \\ \Rightarrow d_\theta(Tx, Ty) = 0. \end{aligned}$$

But, $d_\theta(x,y) \geq 3$ and taking $k = \frac{1}{2}$, we obtain

$$\begin{aligned} kd_\theta(x,y) \geq& \frac{3}{2} \\ >& 1 \\ =& 1 + d_\theta(Tx, Ty) \\ \frac{kd_\theta(x,y)}{1 + d_\theta(Tx, Ty)} \geq& 1 \\ \xi(d_\theta(Tx, Ty), d_\theta(x,y)) = \frac{kd_\theta(x,y)}{1 + d_\theta(Tx, Ty)} \geq& 1. \end{aligned}$$

Case II: *$x, y \in \{e_i, i \geq 6\}$. We take $x = e_i$ and $y = e_j$. Therefore*

$$Tx = e_{i-5},\ Ty = e_{j-5}.$$

Without loss of generality, $i > j$. Then we get

$$\begin{aligned} d_\theta(Tx, Ty) &= 3^{i-5} \\ d_\theta(x,y) &= 3^i \\ kd_\theta(x,y) &= \frac{3^i}{2} \\ &> 1 + d_\theta(Tx, Ty) \\ \frac{kd_\theta(x,y)}{1 + d_\theta(Tx, Ty)} &\geq 1 \\ \xi(d_\theta(Tx, Ty), d_\theta(x,y)) = \frac{kd_\theta(x,y)}{1 + d_\theta(Tx, Ty)} &\geq 1. \end{aligned}$$

Case III: *$x \in \{e_i, i = 1, \dots, 5\}, y \in \{e_i, i \geq 6\}$. Considering $x = e_i$ and $y = e_i$, we have*

$$Tx = e_0,\ Ty = e_{i-5}.$$

Hence we get

$$\begin{aligned} d_\theta(Tx,Ty) &= 3^{i-5} \\ d_\theta(x,y) &= 3^i \\ kd_\theta(x,y) &= \frac{3^i}{2} \\ &> 1 + d_\theta(Tx,Ty) \\ \frac{kd_\theta(x,y)}{1+d_\theta(Tx,Ty)} &\geq 1 \\ \xi(d_\theta(Tx,Ty), d_\theta(x,y)) = \frac{kd_\theta(x,y)}{1+d_\theta(Tx,Ty)} &\geq 1. \end{aligned}$$

So, in all cases T satisfies all the hypotheses of Theorem 3.6 and employing the theorem, T possesses a unique fixed point and it is $w = e_0 \in C$.

Here we propose the notion of modified $\mathcal{Z}_F$-contractions in the framework of θ-metric spaces.

Definition 3.9. *Let a mapping T defined on a θ-metric space (X, d_θ) such that*

$$\xi(d_\theta(Tu,Tv), M(u,v)) \geq C_F$$

holds for all $u, v \in X$, with $u \neq v$ where,

$$M(u,v) = \max\{d_\theta(u,v), d_\theta(u,Tu), d_\theta(v,Tv)\}.$$

Then T is regarded as a modified $\mathcal{Z}_F$-contraction with respect to ξ.

In this context, we are going to present another fixed point result related to these modified $\mathcal{Z}_F$-contractions. The following result ensures us the existence and uniqueness of a fixed point of a modified $\mathcal{Z}_F$-contraction. The following lemma is crucial to claim the assertion.

Lemma 3.10. *Let (X, d_θ) is a complete θ-metric space. Also suppose that $T : X \to X$ is a modified $\mathcal{Z}_F$-contraction with respect to any C_F-simulation function $\xi \in \mathcal{Z}_F$. Then if T has a fixed point in X, it is unique.*

Proof. Let $a, b \in X$ be two distinct fixed points of T. Therefore $Ta = a, \quad Tb = b$ and

$$d_\theta(a,b) = d_\theta(Ta,Tb) > 0.$$

From Definition 3.9 and using the previous facts, we observe that

$$\begin{aligned} M(a,b) &= \max\{d_\theta(a,b), d_\theta(a,Ta), d_\theta(b,Tb)\} \\ &= \max\{d_\theta(a,b), d_\theta(a,a), d_\theta(b,b)\} \\ &= d_\theta(a,b). \end{aligned}$$

Using the definition of modified $\mathcal{Z}_F$-contractions, we attain that

$$\begin{aligned} C_F &\leq \xi(d_\theta(Ta,Tb), M(a,b)) \\ &= \xi(d_\theta(Ta,Tb), d_\theta(a,b)) \\ &= \xi(d_\theta(a,b), d_\theta(a,b)) \\ &< F(d_\theta(a,b), d_\theta(a,b)). \end{aligned}$$

From this, using Definition 2.7 we get,

$$0 < d_\theta(a,b) < d_\theta(a,b),$$

which is impossible. Hence the lemma is done. □

Here, we assert one more fixed point result.

Theorem 3.11. *Let a mapping T defined on a complete θ-metric space (X, d_θ) be a modified $\mathcal{Z}_F$-contraction with respect to a C_F-simulation function $\xi \in \mathcal{Z}_F$. Then for each $x_0 \in X$, the Picard iteration $\{x_n\}$ converges to u, which is the unique fixed point of T in X.*

Proof. Suppose (X, d_θ) be any θ-metric space and $T : X \to X$ be some modified $\mathcal{Z}_F$-contraction with respect to $\xi \in \mathcal{Z}_F$. Take x_0 be any arbitrary element and $\{x_n\}$ be the corresponding Picard iterate, i.e., $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. Now we suppose that $d_\theta(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. On the other hand, if there is some $n_p \in \mathbb{N}$ such that $x_{n_p} = x_{n_p+1}$, then x_{n_p} is a fixed point of T and the theorem is proved. We consider $d_\theta^n = d_\theta(x_n, x_{n+1})$. Then,

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{d_\theta(x_n, x_{n-1}), d_\theta(x_n, x_{n+1}), d_\theta(x_{n-1}, x_n)\} \\ &= \max\{d_\theta^n, d_\theta^{n-1}\}. \end{aligned}$$

So if $M(x_n, x_{n-1}) = d_\theta^n$, we get,

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})) \\ &= \xi(d_\theta^n, \max\{d_\theta^n, d_\theta^{n-1}\}) \\ &= \xi(d_\theta^n, d_\theta^n) \\ &< F(d_\theta^n, d_\theta^n) \\ \Rightarrow d_\theta^n &< d_\theta^n \end{aligned}$$

and this is impossible. So, we obtain,

$$M(x_n, x_{n-1}) = d_\theta^{n-1}.$$

Therefore we have,

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tx_{n-1}), M(x_n, x_{n-1})) \\ &= \xi(d_\theta^n, \max\{d_\theta^n, d_\theta^{n-1}\}) \\ &= \xi(d_\theta^n, d_\theta^{n-1}) \\ &< F(d_\theta^{n-1}, d_\theta^n) \\ \Rightarrow d_\theta^n &< d_\theta^{n-1} \end{aligned}$$

for all $n \in \mathbb{N}$. Here $\{d_\theta^n\}$ is a non-increasing sequence of positive reals and so, is convergent. Let

$$\lim_{n\to\infty} d_\theta^n = r.$$

If $r > 0$, we choose $\{t_n\} = \{d_\theta^n\}$ and $\{s_n\} = \{d_\theta^{n-1}\}$ and we know that $t_n < s_n$ for all $n \in \mathbb{N}$. Since T is a $\mathcal{Z}_F$-contraction involving ξ, we apply (ξ_c) and obtain

$$\begin{aligned} C_F &\leq \limsup_{n\to\infty} \xi(d_\theta^n, M(x_n, x_{n-1})) \\ C_F &\leq \limsup_{n\to\infty} \xi(d_\theta^n, d_\theta^{n-1}) \\ &< C_F, \end{aligned}$$

which is impossible and so $r = 0$, i.e.,

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0.$$

Now we will show that $\{x_n\}$ is Cauchy. Now, by Lemma 3.5, since $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} d_\theta(x_n, x_{n+1}) = 0$$

holds, then whenever $\{x_n\}$ is not a Cauchy sequence, there exist $\epsilon > 0$ and two sequences $\{n_k\}$ and $\{m_k\}$ of natural numbers such that $n_k > m_k > k$ and the sequences $\{d_\theta(x_{m_k}, x_{n_k})\}$ and $\{d_\theta(x_{m_k+1}, x_{n_k+1})\}$ tend to ϵ as $k \to \infty$. In particular, there exists $n_1 \in N$ such that

$$d_\theta(x_{m_k}, x_{n_k}) > \frac{\epsilon}{2} > 0$$

and

$$d_\theta(x_{m_k+1}, x_{n_k+1}) > \frac{\epsilon}{2} > 0$$

for all $k \geq n_1$. Now, that T is a $\mathcal{Z}_F$-contraction with respect to ξ, and also from (ξ_b), we obtain that

$$\begin{aligned} C_F &\leq \xi(d_\theta(x_{m_k+1}, x_{n_k+1}), M(x_{m_k}, x_{n_k})) \\ &< F(M(x_{m_k}, x_{n_k}), d_\theta(x_{m_k+1}, x_{n_k+1})) \\ \Rightarrow M(x_{m_k}, x_{n_k}) &> d_\theta(x_{m_k+1}, x_{n_k+1}) \end{aligned} \tag{3.12}$$

where,

$$M(x_{m_k}, x_{n_k}) = \max\{d_\theta(x_{m_k}, x_{n_k}), d_\theta(x_{m_k}, x_{m_k+1}), d_\theta(x_{n_k}, x_{n_k+1})\}.$$

Now if

$$M(x_{m_k}, x_{n_k}) = d_\theta(x_{m_k}, x_{m_k+1}),$$

we get,

$$d_\theta(x_{m_k}, x_{m_k+1}) > d_\theta(x_{m_k+1}, x_{n_k+1}).$$

Letting $k \to \infty$ and using Lemma 3.5, we obtain

$$0 \geq \epsilon,$$

which contradicts our assumption. Using a similar argument, we can prove that

$$M(x_{m_k}, x_{n_k}) \neq d_\theta(x_{n_k}, x_{n_k+1}).$$

Hence

$$M(x_{m_k}, x_{n_k}) = d_\theta(x_{m_k}, x_{n_k})$$

and

$$C_F \leq \xi(d_\theta(x_{m_k+1}, x_{n_k+1}), d_\theta(x_{m_k}, x_{n_k})). \tag{3.13}$$

So we obtain from (3.12),

$$\begin{aligned} 0 &< d_\theta(x_{m_k}, x_{n_k}) \\ &= M(x_{m_k}, x_{n_k}) \\ &> d_\theta(x_{m_k+1}, x_{n_k+1}). \end{aligned} \tag{3.14}$$

Employing the sequences $\{t_k\} = \{d_\theta(x_{m_k+1}, x_{n_k+1})\}$ and $\{s_k\} = \{d_\theta(x_{m_k}, x_{n_k})\}$, which have the same positive limit ϵ and using (3.14) in axiom (ξ_c), we conclude that

$$\limsup_{k\to\infty} \xi(t_k, s_k) < C_F$$

which is a contradiction to (3.13). As a result, $\{x_n\}$ is Cauchy. As (X, d_θ) is a complete metric space, we can find some $z \in X$ with

$$\lim_{n\to\infty} x_n = z$$

or equivalently

$$\lim_{n\to\infty} d_\theta(Tx_n, z) = 0. \tag{3.15}$$

Next we check that z is a fixed point of T. To the contrary, we assume $Tz \neq z$. So

$$d_\theta(z, Tz) = \delta > 0. \tag{3.16}$$

Therefore from (3.15), there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} d_\theta(Tx_n, z) &< \delta \\ &= d_\theta(Tz, z) \end{aligned}$$

for all $n \geq n_0$. This implies

$$Tx_n \neq Tz$$

and hence,

$$d_\theta(Tx_n, Tz) > 0 \tag{3.17}$$

for all $n \geq n_0$. Now, there does not exist any $n_3 \in \mathbb{N}$ such that for all $n \geq n_3$

$$x_n = z.$$

Therefore, we can find a subsequence $\{x_{p_k}\}$ of $\{x_n\}$ such that

$$x_{p_k} \neq z \tag{3.18}$$

for all $k \in \mathbb{N}$. Now, let $n_2 \in \mathbb{N}$ be such that $p_{n_2} \geq n_0$. Hence by (3.17) and (3.18), we have

$$d_\theta(Tx_{p_n}, Tz) > 0$$

and

$$d_\theta(x_{p_n}, z) > 0$$

for all $n \geq n_2$. Now we employ Definition 3.9 to get,

$$C_F \leq \xi(d_\theta(Tx_n, Tz), M(x_n, z)),$$

where

$$M(x_n, z) = \max\{d_\theta(x_n, z), d_\theta(x_n, x_{n+1}), d_\theta(z, Tz)\}.$$

Now if

$$M(x_n, z) = d_\theta(x_n, z),$$

then,

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tz), d_\theta(x_n, z)) \\ &< F(d_\theta(x_n, z), d_\theta(Tx_n, Tz)) \\ \Rightarrow d_\theta(x_n, z) &> d_\theta(Tx_n, Tz). \end{aligned} \tag{3.19}$$

As $n \to \infty$ we get from (3.16) and (3.19),

$$0 \geq d_\theta(z, Tz) = \delta,$$

which is a contradiction. Now if

$$M(x_n, z) = d_\theta(z, Tz),$$

then,

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tz), d_\theta(z, Tz)) \\ &< F(d_\theta(z, Tz), d_\theta(Tx_n, Tz)). \end{aligned} \tag{3.20}$$

From (3.20), we derive

$$d_\theta(z, Tz) > d_\theta(Tx_n, Tz).$$

Also,

$$\lim_{n\to\infty} d_\theta(Tx_n, Tz) = d_\theta(z, Tz) = \delta.$$

Hence considering the sequences $\{t_n\} = \{d_\theta(Tx_n, Tz)\}$ and $\{s_n\} = \{d_\theta(z, Tz)\}$, which have the same positive limit δ and using (3.14) in (ξ_c), we conclude that

$$\limsup_{n\to\infty} \xi(t_n, s_n) < C_F$$

which is a contradiction to (3.20). Now if

$$M(x_n, z) = d_\theta(x_n, x_{n+1}),$$

then from Definition 2.7 and (ξ_b),

$$\begin{aligned} C_F &\leq \xi(d_\theta(Tx_n, Tz), d_\theta(x_n, x_{n+1})) \\ &< F(d_\theta(x_n, x_{n+1}), d_\theta(Tx_n, Tz)) \\ \Rightarrow d_\theta(x_n, x_{n+1}) &> d_\theta(Tx_n, Tz). \end{aligned}$$

As $n \to \infty$ we get,

$$0 \geq d_\theta(z, Tz) = \delta,$$

which is also impossible. These contradictions confirm that $d_\theta(z, Tz) = 0$, and therefore, $Tz = z$. Therefore z is a fixed point of T. The uniqueness of the fixed point is confirmed from the Lemma 3.10. □

Corollary 3.12. *If $M(x, y) = d(x, y)$, then Theorem 3.11 coincides with Theorem 3.6.*

To authenticate our previous result, we construct the succeeding examples which elucidate Theorem 3.11.

Example 3.13. *Let $X = \{1, 3, 5, 6\}$ be furnished with the Euclidean metric and $\theta(s,t) = st + s + t$, for all $t, s \in [0, \infty)$. We consider T on X by*

$$T1 = 3, \quad T3 = 3, \quad T5 = 1, \quad T6 = 1.$$

Now, we consider $\xi(t,s) = \frac{ks}{1+t}$, $t, s \in [0, \infty)$, as the C_F-simulation function, where $k \in [0, 1)$ and $C_F = 1$. Then T is not a $\mathcal{Z}_F$-contraction with respect to ξ, since $d_\theta(T3, T5) = 2$, $d_\theta(3, 5) = 2$ and

$$\xi(d_\theta(T3, T5), d_\theta(3, 5)) = \frac{2k}{3} \ngeq 1,$$

for any $k \in [0, 1)$. Also it is very simple to verify that T satisfies

$$\xi(d_\theta(Tx, Ty), M(x, y)) \geq 1$$

$x, y \in X$. Therefore T is a modified $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ. By Theorem 3.11, $u = 3$ is that required fixed point.

Example 3.14. *Let $X = \mathbb{N}$ be furnished with the metric defined by*

$$d_\theta(x, y) = \begin{cases} 0, & \text{if } x = y; \\ 3 + \left|\frac{1}{x} - \frac{1}{y}\right|, & \text{otherwise} \end{cases}$$

and we take $\theta(s,t) = s + t + st$, for all $t, s \in [0, \infty)$. Then clearly (X, d_θ) is a complete metric space. We consider T on X by

$$Tx = 1,$$

for all $x \in \mathbb{N}$. Now, we consider $\xi(t,s) = \frac{ks}{1+t}$, for all $t, s \in [0, \infty)$. as the C_F-simulation function, where $k = \frac{1}{3}$ and $C_F = 1$. For any $x, y \in X$ with $x \neq y$, we have

$$\begin{aligned} M(x, y) \geq& d_\theta(x, y) \\ =& 3 + \left|\frac{1}{x} - \frac{1}{y}\right| \end{aligned}$$

and $d_\theta(Tx, Ty) = 0$. We take $k = \frac{1}{3}$, and obtain

$$\begin{aligned} kM(x, y) \geq& kd_\theta(x, y) \\ =& 1 + \frac{1}{\left|\frac{1}{x} - \frac{1}{y}\right|} \\ \geq& 1 \\ =& 1 + d_\theta(Tx, Ty) \\ \frac{kM(x, y)}{1 + d_\theta(Tx, Ty)} \geq& 1 \\ \xi(d_\theta(Tx, Ty), M(x, y)) = \frac{kM(x, y)}{1 + d_\theta(Tx, Ty)} \geq& 1. \end{aligned}$$

Therefore T is a modified $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ. By Theorem 3.11, $u = 1$ is that required fixed point.

4. Consequences

In this section, as consequences of our obtained results, we provide some fixed point results in the literature.

Theorem 4.1. *[7] Let T be a $\mathcal{Z}$-contraction with respect to a simulation function $\xi \in \mathcal{Z}$ and is defined on a complete θ-metric space (X, d_θ). Then u is a unique fixed point of T in X.*

Proof. We consider $F(s,t) = s - t$ and $C_F = 0$. Therefore T is a $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ and by Theorem 3.6, it possesses a unique fixed point. □

Theorem 4.2. *[12] Let T be a $\mathcal{Z}$-contraction with respect to a simulation function $\xi \in \mathcal{Z}$ and is defined on a complete metric space (X, d). Then T has a unique fixed point u in X.*

Proof. In Theorem 3.6, we take $\theta(s,t) = s + t$. Then (X, d_θ) is actually the metric space endowed by the usual metric. Now, taking $F(s,t) = s - t$ and $C_F = 0$, we have T is a $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ and so, by Theorem 3.6, it has a unique fixed point. □

Theorem 4.3. *[7] Let T be any modified $\mathcal{Z}$-contraction with respect to a simulation function $\xi \in \mathcal{Z}$ and is defined on a complete θ-metric space (X, d_θ). Then T possesses a unique fixed point u in X.*

Proof. We take $F(s,t) = s - t$ and $C_F = 0$. Therefore T is a modified $\mathcal{Z}_F$-contraction with respect to the C_F-simulation function ξ and by Theorem 3.11, it has a unique fixed point. □

Acknowledgement:
We thank the learned referee for careful reading and constructive suggestions which undoubtedly improve the first draft of the article. The authors are indebted to Tanusri Senapati and Hiranmoy Garai for the useful inputs on the construction of the examples.

References

[1] A.H. Ansari, *Note on ϕ-ψ-contractive type mappings and related fixed point*, The 2nd Regional Conf. Math. Appl. Payame Noor Univ. (2014) 377–380.

[2] A.H. Ansari, M. Berzig and S. Chandok, S*ome fixed point theorems for (CAB)-contractive mappings and related results*, Math. Morav. 19(2) (2015) 97–112.

[3] A.H. Ansari, S. Chandok and C. Ionescu, *Fixed point theorems on b-metric spaces for weak contractions with auxiliary functions*, J. Inequal. Appl. 2014 (2014) 429.

[4] H. Argoubi, B. Samet and C. Vetro, *Nonlinear contractions involving simulation functions in a metric space with a partial order*, J. Nonlinear Sci. Appl. 8(6) (2015) 1082–1094.

[5] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. 2006 (2006) Article ID 74503.

[6] A. Bera, H. Garai, B. Damjanović and A. Chanda, *Some interesting results on $\mathcal{F}$-metric spaces*, Filomat 33(10) (2019) 3257-3268.

[7] A. Chanda, B. Damjanović and L.K. Dey, *Fixed point results on θ-metric spaces via simulation functions*, Filomat 31(11) (2017) 3365–3375.

[8] S. Chandok, A. Chanda, L.K. Dey, M. Pavlović and S. Radenović, *Simulations functions and Geraghty type results*, Bol. Soc. Paran. Mat. 39(1) (2021) 35-50.

[9] S. Karmakar, L.K. Dey, P. Kumam and A. Chanda, *Best proximity results for cyclic α-implicit contractions in quasi-metric spaces and its consequences*, Adv. Fixed Point Theory 7(3) (2017) 342–358.

[10] F. Khojasteh, E. Karapinar and S. Radenović, *θ-metric space: A generalization*, Math. Probl. Eng. 2013 (2013) Article ID 504609.

[11] Z. Kadelburg, S. Radenović and S. Shukla, *Boyd-Wong and Meir-Keeler type theorems in generalized metric spaces*, J. Adv. Math. Stud. 9(1) (2016) 83–93.

[12] F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat 29(6) (2015) 1189–1194.

[13] X. Liu, A. H. Ansari, S. Chandok and S. Radenović, *On some results in metric spaces using auxiliary simulation functions via new functions*, J. Comput. Anal. Appl. 24(6) (2018) 1103–1114.

[14] S. Mondal, A. Chanda and S. Karmakar, *Common fixed point and best proximity point theorems in C^*-algebra-valued metric spaces*, Int. J. Pure Appl. Math. 115(3) (2017) 477–496.

[15] Z. Ma, L. Jiang and H. Sun. C^*-algebra-valued metric spaces and related fixed point theorems. *Fixed Point Theory Appl.*, (2014) 2014:206.

[16] S. Radenović, Z. Kadelburg, D. Jandrlić and A. Jandrlić, *Some results on weakly contractive maps*, Bull. Iranian Math. Soc. 38(3) (2012) 625–645.