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A meshfree radial basis function method for nonlinear phi-four equation

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Abstract

Radial basis function pseudospectral method is applied to obtain the solution for nonlinear Phi-four time dependant equation with nonhomogeneous initial and boundary conditions. In this method, the efficient pseudospectral technique is combined with radial basis function to get the best of it. In the proposed method, the radial basis kernels are used to discretize the space derivatives in the Phi-four equation where as a time stepping technique is used to accord with the temporal part of the solution. The given Phi-four equation is transformed into a set of ordinary equations. An ode solver is used to solve the ordinary equations. An effective approach is used to choose the value of the shape parameter for radial basis function. Numerical results are presented to check the validity and accuracy of the method to solve the Phi-four equation.

Keywords: Meshfree, Radial Basis function, Nonlinear Phi-four equations
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1. Introduction

In recent years, Radial basis functions (RBFs) are used extensively in various numerical methods for solving the differential equations due to their mesh free nature and easy to apply nature. The valuable RBF methods is a substitute for classical methods where they are hard to apply or even fail to perform. It all started when Kansa [1] firstly used RBF interpolation and collocation technique to solve differential equation. Due to its simplicity in implementation, researchers use Kansa method to solve a vast variety of problems [2][3][4][5]. Many well-known methods combine with RBF approximation to take the advantage of its mesh free nature. Some of the methods are of weak formulation

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and other are of strong formulation. One such method in strong formulation is Radial Basis function pseudospectral method (RBFPS). Fasshauer [6] proposed this method in which RBF collocation is used in pseudospectral mode. Various researchers applied RBFPS method to solve problems in the areas of science and engineering [7][8][9]. Recently Abbasbandy et al. [10] applied RBFPS to solve two dimensional telegraph equation.

Many Phenomena in engineering and other fields of sciences are governed by nonlinear partial differential equations. These equations play a great role in modelling a variety of physical problems arises in the field of fluid mechanics, hydrodynamics, plasma physics and quantum mechanics. The Phi-four equation is one such nonlinear partial differential equation. We study this equation in the context of quartic interaction theory arises in the field of quantum mechanics.

In the literature, few numerical techniques have been presented to solve the Phi-four equation. Chowdhury and Biswas [11] obtained the singular soliton solution of the Phi-four equation with the help of ansatz method. Numerical solution is also obtained by using rational chebyshev basis function in spectral mode. Bhrawy et al. [12] established Jacobi-Gauss-Lobatto Collocation method for solving Phi-Four equation. Recently Zahra et al.[13] developed a numerical scheme based on Cubic B-spline collocation method to solve the Phi-four equation. On the other hand, several analytic methods are also used to obtain the solution for the equation. Wazwaz and Triki [14] applied anstaz method to obtain the soliton solution of the equation. Najafi [15] applied the He's Variational method to obtain the analytic solution of the Phi-four equation. Demiray and Bulut [16] used modified exp $(-\Omega(\xi))$-expansion function method for the same.

In this present work, the RBFPS method is applied to solve Phi-four equation

$$u_{tt} = \lambda_1 u_{xx} + \lambda_2 u + \lambda_3 u^{\theta}, (x,t) \in [A,B] \times [0,T] \tag{1.1}$$

with initial and boundary condition as

$$u(A,t) = f_1(t), u(B,t) = f_2(t) \tag{1.2}$$

$$u(x,0) = g_1(t), u_t(x,0) = g_2(t), x \in [A,B] \tag{1.3}$$

where $u(x,t)$ represent the wave profile at spatial and temporal independent variables x and t respectively. The coefficients λ_1, λ_2 and λ_3 are real valued parameters. In this method, the partial differential equation is transformed into a system of ordinary differential equations with the help of RBF kernels. An ODE solver is used to solve the resultant system of ordinary differential equations. Many RBFs contains a free parameter known as shape parameter. It is already proved that shape parameter effects the accuracy and stability of the method so a special consideration is given to choose the good value of shape parameter. Fasshauer and Zhang [17] proposed an algorithm which is the modified version of Rippa approach for choosing the good value of the shape parameter. It depends on the leave-one out cross validation (LOOCV) technique used in statistics. In this present study, we have used the Fasshauer criteria for selecting the shape parameter. The paper is organized as follows: in section 2, implementation of the proposed method is given. The given Phi-four equation is transformed into a system of ordinary differential equations. An ODE solver is used to solve the resultant ODEs. The Fasshauer criteria for selecting the shape parameter is also discussed. In section 3, to check the validity of the method, we present the results of three numerical problems. In the end, a conclusion is drawn in section 4.

2. Implementation of the proposed method

In this section, we will use the RBFPS method to obtain a numerical solution of Phi-four equation 1.1 with initial and boundary conditions 1.21.3. We will transform the given Phi-four equation 1.1

as coupled equation

$$u_t = v, v_t = \lambda_1 u_{xx} + \lambda_2 u + \lambda_3 u^\theta \tag{2.1}$$

The given domain $[A, B]$ is divided into nodes $x_k = 1, 2, 3, ...N$. The RBF approximation for $v(x, t)$ can be written in the form as

$$u_N = \sum_{k=1}^{N} {\zeta^1}_k \phi_k(\| x - x_k \|), v_N = \sum_{k=1}^{N} {\zeta^2}_k \phi_k(\| x - x_k \|) \tag{2.2}$$

where $\phi_k = \phi(r)$ and r=$\| x - x_k \|$ denotes the Euclidean distance between the points x and x_k and ϕ_k is the radial basis function.

Equation 2.2 evaluated at various nodes $x_k = 1, 2, 3, ...N$ we get

$$u_N(x_i) = \sum_{k=1}^{N} {\zeta^1}_k \phi_k(\| x_i - x_k \|), v_N(x_i) = \sum_{k=1}^{N} {\zeta^2}_k \phi_k(\| x_i - x_k \|) \tag{2.3}$$

In the matrix form, we can write equation 2.3 as

$$U = LC_1, V = LC_2 \tag{2.4}$$

where $L_{ik} = \phi_k(\| x_i - x_k \|)$ are the radial basis functions at nodes and $C_1 = [{\zeta^1}_1, {\zeta^1}_2, ..., {\zeta^1}_N]^T$ and $C_2 = [{\zeta^2}_1, {\zeta^2}_2, ..., {\zeta^2}_N]^T$ are the unknown interpolation coefficients. Now, the derivative of u_N of 2.3 by differentiating the basis functions, as

$$\frac{d}{dx_i} u_N(x_i) = \sum_{k=1}^{N} {\zeta^1}_k \frac{d}{dx_i} \phi_k(\| x_i - x_k \|) \tag{2.5}$$

Again, evaluate 2.5 at the grid points, $x_k = 1, 2, 3, ...N$, we get

$$U_x = L_x C_1 \tag{2.6}$$

where the entries of the derivative matrix L_x are $\frac{d}{dx_i}\phi_k(\| x_i - x_k \|)$ The condition that the evaluation matrix L in 2.4 is invertible, depends on various factors like RBF selected and the chosen grid points. The matrix generated by using a positive definite RBFs is always lead to a non-singular matrix and hence invertible. Since L is invertible so from 2.4 $C_1 = L^{-1}U$ and equation 2.6 becomes

$$U_x = L_x L^{-1} U = D_x U \tag{2.7}$$

Similarly, one can find the differentiation matrix concerning the second and higher order derivatives, i.e

$$U_{xx} = L_{xx} L^{-1} U = D_{xx} U \tag{2.8}$$

Similarly, $V_x = D_x V$ and $V_{xx} = D_{xx} V$ Using the approximations, the coupled equation 2.1 can be written as

$$\frac{dU_N}{dt} = V_N, \frac{dV_N}{dt} = \lambda_1 D_{xx} U_N + \lambda_2 U_N + \lambda_3 {U_N}^\theta \tag{2.9}$$

By RBF-PS scheme, 1.1 reduces to a system of ODEs. The obtained ODEs can be discretized in time using any ODE solver like ode113, ode45 from MATLAB. We have used ode45 ODE solver to solve the resultant ODEs.

2.1. Rippa's Leave-One-Out Cross-Validation Approach

The first algorithm based on a well-known method in statistics as LOOCV was proposed by Rippa [18] to determine an optimal value for the shape parameter. LOOCV is a cross-validation technique used in statistics for a variety of parameter identification problems. Rippa determine the value of the shape parameter by minimizing the error function as

$$E = [E_1, E_2, E_3, ..., E_n]^t$$

where $E_k = |u(x_k) - P^k(x)|$ and $P^k(x)$ is the radial basis function interpolation to all the data points except x_k. Rippa use a simplified formula for finding the error to avoid the high cost

$$E_k = \frac{\alpha_k}{{A_{kk}}^{-1}}$$

where α_k is the k^{th} coefficient of the interpolation and ${A_{kk}}^{-1}$ is the k^{th} diagonal element of the inverse of the interpolation matrix. As the problem becomes an optimization problem, Rippa used Brent's method to find the optimal value of the shape parameter for which the error function is minimum. Fasshauer and Zang [17] modified the Rippa's approach to finding the good value of the shape parameter for the RBF-PS method. They suggest the use of MATLAB function fminbnd to find the minimum of the error function.

3. Numerical simulation and Discussion

In this section, we consider three numerical problems to check the applicability of the proposed method on Phi-four equation. We compare and analyse the obtained numerical solution of the equation with respect to the exact solution. For discussion, we used the following error norms for the solution.

$$L_\infty = \max_{1 \le i \le N} |u(x_i, t) - U_N(x_i, t)|$$

$$L_2 = \frac{\sum_{i=1}^{N} |u(x_i, t) - U_N(x_i, t)|}{N}$$

$$RMS = \sqrt{\frac{\sum_{i=1}^{N} (u(x_i, t) - U_N(x_i, t))^2}{N}}$$

where $u(x_i, t)$ is the exact solution and $U_N(x_i, t)$ is the numerical solution for the given equation. In the present paper for discretize the space derivatives, the positive definite Cubic Matern RBF given by $\phi(r) = (15 + 15\epsilon r + 6(\epsilon r)^2 + (\epsilon r)^3)e^{-\epsilon r}$

Example 3.1. *Consider the nonlinear Phi-four equation from [12] as*

$$u_{tt} = \lambda_1 u_{xx} + \lambda_2 u + \lambda_3 u^\theta, (x, t) \in [A, B] \times [0, T]$$

The exact solution of the equation is

$$u(x, t) = \frac{3\lambda_2}{2\lambda_3}\left(1 - tanh^2\left[\sqrt{\frac{\lambda_2}{(4v^2 - \lambda_1)}}(x - vt)\right]\right)$$

The initial and boundary conditions can be obtained from the exact solution. The above equation is solved with $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 1$. First we solve the above equation in the computational domain $[0, 10]$ with $N = 51$ and $\delta t = 0.001$.The absolute errors are calculated for different values of t in the interval $[0, 10]$ with $v = 2$ and are represented in Table 1. We calculate different error norms for $t = 0.01, t = 0.03$ and $t = 0.05$ in the interval $[0, 1]$ with $v = 2$ and results are reported in table 2. In Figure 1, a graph comparing the numerical and exact solutions at $t = 0.1, t = 0.5$ and $t = 0.9$ is represented in the interval $[-5, 5]$. The comparison between the exact and numerical solution for $t \leq 0.1$ is also depicted by figure 2. It is found that the obtained numerical results are comparable to exact solution.

Table 1: Absolute error for Example 3.1 in the interval $[0, 10]$ for different values of t

	Time = 0.1	Time = 0.2	Time = 0.3
x	Absoulte Error	Absoulte Error	Absoulte Error
1	1.15E-08	1.34E-08	7.52E-08
2	1.30E-08	1.46E-08	2.34E-08
3	3.68E-09	9.59E-09	4.08E-09
4	1.17E-08	1.66E-08	1.15E-08
5	8.86E-09	1.24E-08	1.16E-08
6	1.48E-08	1.87E-08	2.20E-08
7	1.51E-09	4.24E-10	7.92E-09
8	1.05E-08	2.07E-08	5.67E-09
9	6.40E-09	4.51E-09	1.99E-08
10	9.02E-08	4.26E-07	1.21E-06

Table 2: Error norms for different values of t for Example 3.1 with $v = 2$ and $N = 51$

t	0.1	0.2	0.3
L_∞	2.3160E-10	3.2839E-09	1.4780E-08
L_2	4.0461E-12	5.7882E-11	2.7112E-10
RMS	7.1604E-11	9.6323E-10	4.2558E-09

Example 3.2. *As second example, we consider equation 1.1 with the value of the parameters as $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$ and $\theta = 3$ which is of the form*

$$u_{tt} = u_{xx} + u - u^3, (x, t) \in [A, B] \times [0, T]$$

with initial and boundary conditions as

$$u(x, 0) = tanh\left[\sqrt{\frac{1}{2(1 - v^2)}}(x)\right], u_t(x, 0) = sech^2\left[\sqrt{\frac{1}{2(1 - v^2)}}(x)\right]\left(\frac{-v}{\sqrt{2(1 - v^2)}}\right)$$

and

$$u(A, t) = tanh\left[\sqrt{\frac{1}{2(1 - v^2)}}(A - vt)\right], u(B, t) = tanh\left[\sqrt{\frac{1}{2(1 - v^2)}}(B - vt)\right]$$

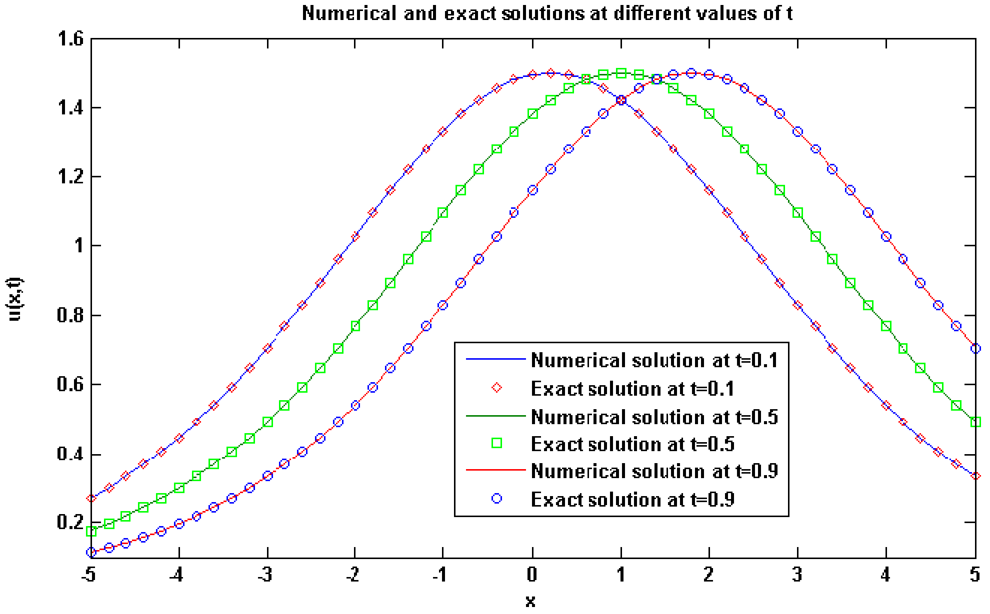

Figure 1: Approximate solution of Example 3.1 for different values of t

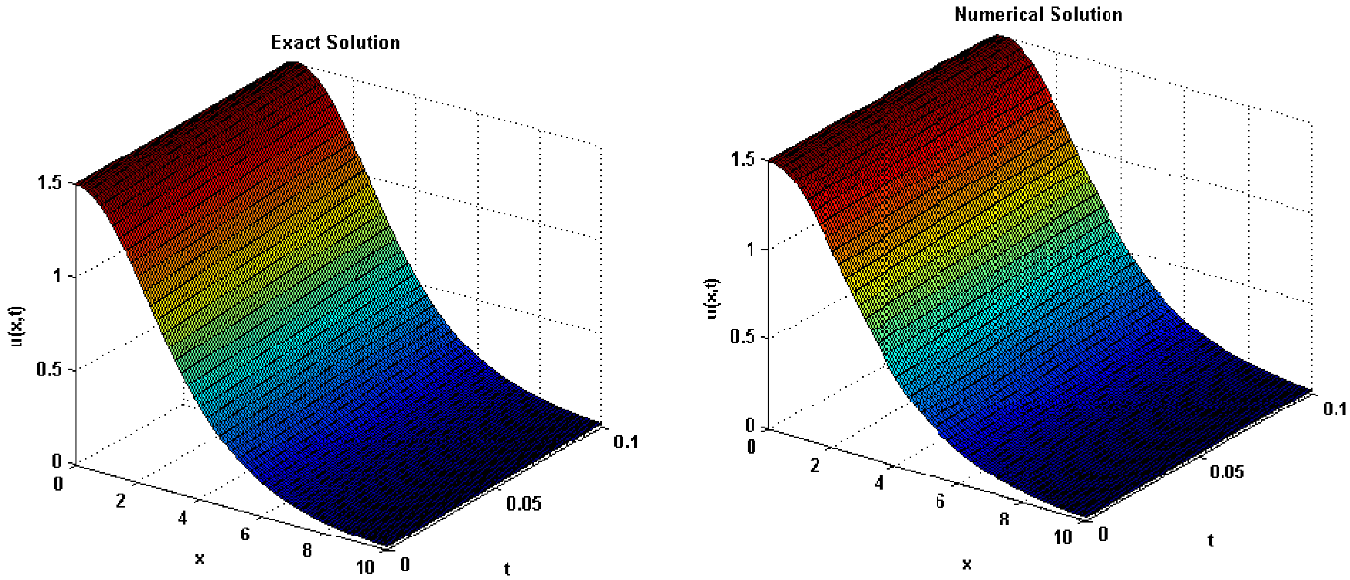

Figure 2: Exact and Numerical solution of Example 3.1 for $t \leq 0.1$

The exact solution is

$$u(x,t) = tanh\left[\sqrt{\frac{1}{2(1-v^2)}}(x-vt)\right]$$

We solve the above equation in the computational domain $[0,1]$ *with* $N = 51$*.The absolute errors are calculated for different values of* t *in the interval* $[0,1]$ *with* $v = 0.01$ *and are represented in Table 3 In Table 4 , we represent* L_∞*,* L_2 *and RMS errors for different values of* t *In figure 3, a graph comparing the numerical and exact solutions at* $t = 0.5$ *is represented in the interval* $[-5,5]$*. In figure 4, we display the comparison between the exact and numerical solution for* $t \leq 1$ *in the interval* $[-5,5]$ *with*

$N = 51$. *It is found that the obtained numerical results are comparable to the exact solution.*

Table 3: Absolute error for Example 3.2 in the interval $[0, 1]$ for different values of t

	Time = 0.01	Time = 0.02	Time = 0.03
x	Absoulte Error	Absoulte Error	Absoulte Error
0.1	2.39E-09	1.17E-08	3.41E-08
0.2	5.74E-12	2.07E-12	2.21E-12
0.3	1.79E-12	3.68E-11	4.81E-11
0.4	3.23E-11	3.01E-11	2.17E-11
0.5	2.13E-11	2.93E-11	2.24E-11
0.6	2.28E-11	2.87E-11	1.55E-12
0.7	5.14E-11	1.03E-10	5.17E-11
0.8	3.23E-13	2.67E-12	3.81E-11
0.9	4.91E-11	5.19E-11	2.02E-11
1	2.22E-11	1.12E-11	5.04E-11

Table 4: Error norms for different values of t for Example 3.2 with $v = 0.01$ and $N = 51$

t	0.1	0.2	0.3
L_∞	7.1857E-10	8.0328E-10	8.0170E-10
L_2	1.9504E-10	3.1702E-09	1.6544E-08
RMS	1.2163E-09	1.7204E-08	7.8882E-08

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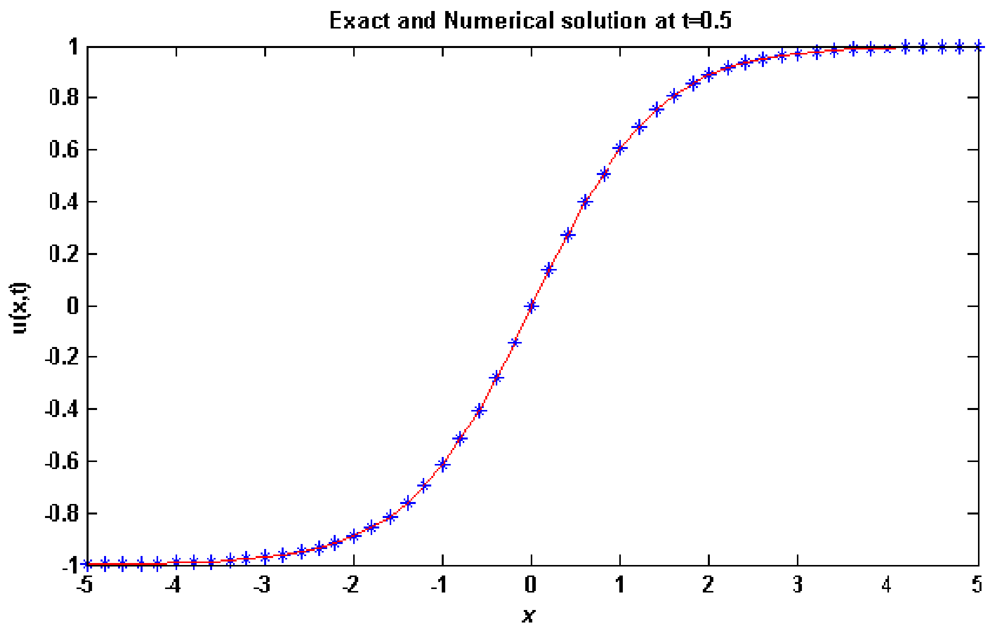

Figure 3: Exact and numerical solution of Example 3.2 at $t = 0.5$ in the interval $[-5, 5]$

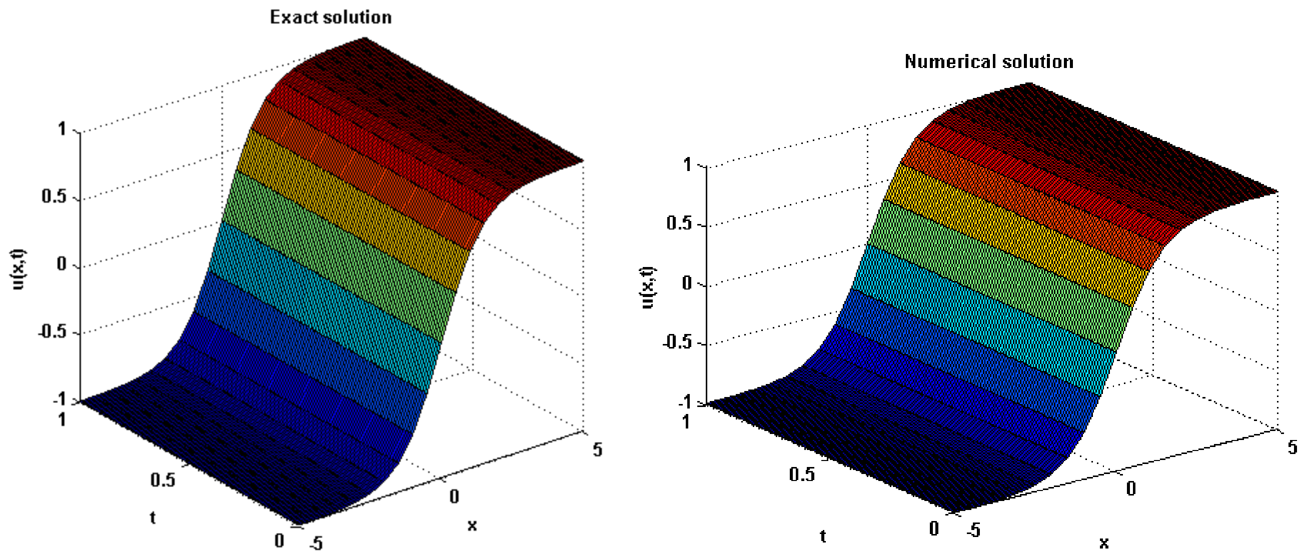

Figure 4: Exact and Numerical solution of Example 3.2 for $t \leq 1$ in the interval $[-5, 5]$

Example 3.3. *Consider the equation of the form*

$$u_{tt} = u_{xx} + u - u^4, (x,t) \in [A, B] \times [0, T]$$

The initial and boundary conditions are

$$u(x,0) = 2\left(1 - tanh^2\left[\frac{3}{2}(x)\right]\right)^{\frac{1}{3}}, u_t(x,0) = 4v\left(1 - tanh^2\left[\frac{3}{2}(x)\right]\right)^{\frac{-2}{3}} tanh\left[\frac{3}{2}(x)\right] sech^2\left[\frac{3}{2}(x)\right]$$

$$u(A,t) = 2\left(1 - tanh^2\left[\frac{3}{2}(A - vt)\right]\right)^{\frac{1}{3}}, u(B,t) = 2\left(1 - tanh^2\left[\frac{3}{2}(B - vt)\right]\right)^{\frac{1}{3}}$$

The exact solution of the equation is given as

$$u(x,t) = 2\left(1 - tanh^2\left[\frac{3}{2}(x - vt)\right]\right)^{\frac{1}{3}},$$

In Table 5 , we report L_∞, L_2 and RMS errors for different values of t with $N = 51$ and $\triangle t = 0.0001$. Figure 5 represent the comparison between the exact and numerical solution of the equation in the interval$[-5, 5]$ for $t \leq 0.01$ with $N = 31$. This assertion that the obtained numerical solution is in good agreement with the exact solution.

Table 5: Error norms for different values of t of Example 3.3 with $v = 0.01$ and $N = 51$

t	0.001	0.002	0.003
L_∞	1.0414E-05	3.1225E-05	5.5174E-05
L_2	6.0865E-07	5.4778E-06	1.5215E-05
RMS	5.8134E-06	2.5507E-05	6.1089E-05

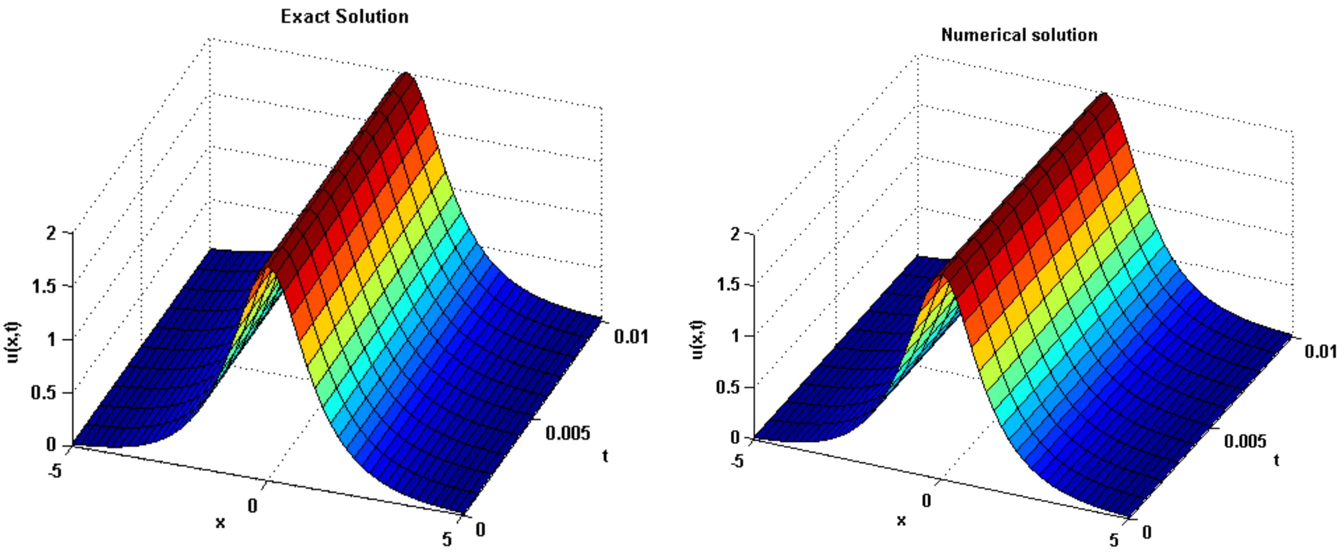

Figure 5: Exact and Numerical solution of Example 3.3 for $t \leq 0.01$ in the interval $[-5, 5]$

4. Conclusion

In this paper, we investigate the application of Radial basis pseudospectral method to produce the solution of Phi-four equation. The Radial kernel reduce the given into a system of ordinary differential equations. The obtained equations then solved further with the help of an ODE solver in MATLAB. To check the applicability and accuracy of the proposed method, three numerical problems are solved. The obtained results are compared with the exact solutions. Different error norms are obtained to check the accuracy. It is found that the proposed method is easy to implement, can be used without any linearization process and produces good results in comparison with the exact solution.

References

[1] E. J. Kansa, *Multiquadrics—A scattered data approximation scheme with applications to computational fluid-dynamics—II solutions to parabolic, hyperbolic and elliptic partial differential equations*, Comput. Math. with Appl. 19(8) (1990) 147–161.

[2] J. Li and C. S. Chen, *Some observations on unsymmetric radial basis function collocation methods for convection-diffusion problems*, Int. J. Numer. Meth. Engng. 57 (2003) 1085–1094.

[3] S. Chantasiriwan, *Multiquadric collocation method for time-dependent heat conduction problems with temperature-dependent thermal properties*, J Heat Transf Trans ASME 129(2) (2007) 109–113.

[4] Y. Duan, P.F. Tang, T.Z. Huang and S.J. Lai, *Coupling projection domain decomposition method and Kansa's method in electrostatic problems*, Comput. Phys. Commun. 180(2) (2009) 209–214.

[5] W. Chen, L. Ye and H. Sun, *Fractional diffusion equations by the Kansa method*, Comput. Math. Appl. 59(5) (2010) 1614–1620.

[6] G. E. Fasshauer, *RBF collocation methods as pseudospectral methods*, WIT transactions on modelling and simulation. 2005.

[7] M. Uddin, S. Haq, and M. Ishaq, *RBF-Pseudospectral Method for the Numerical Solution of Good Boussinesq Equation*, Appl. Math. Sci. 6(49) (2012) 2403-2410.

[8] M. Uddin, *RBF-PS scheme for solving the equal width equation*, Appl. Math. Comput. 222 (2013) 619-631.

[9] A. Krowiak, *Radial basis function-based pseudospectral method for static analysis of thin plates*, Eng. Anal. Bound Elem. 71 (2016) 50-58 .

[10] D. Rostamy, M. Emamjome and S. Abbasbandy, *A meshless technique based on pseudospectral radial basis functions method for solving the two- dimensional hyperbolic telegraph equation*, Eur. Phys. J. Plus 132 (2017) 263.

[11] A. Chowdhury and A. Biswas, *Singular solitons and numerical analysis of ϕ–$four$ equation*, Math. Sci. 6 (2012) 42.

[12] A.H. Bhrawy, L.M. Assas, and M.A. Alghamdi, *An efficient spectral collocation algorithm for nonlinear Phi-four equations*, Bound. Value Probl. 2013 (2013) 87.

[13] W. K. Zahra, W. A. Ouf, and M. S. El-Azab, *A robust uniform B-spline collocation method for solving the generalized PHI-four equation*, Appl. Appl. Math. 11(1) (2016) 384-396.

[14] H. Triki and AM. Wazwaz, *Envelope solitons for generalized forms of the phi-four equation*, J. King Saud Univ. Sci. 25 (2013) 129–133.

[15] M. Najafi, *Using He's Variational Method to Seek the Traveling Wave Solution of PHI-Four Equation*, Int. J. Appl. Math. Res. 1(4) (2012) 659.

[16] S.T. Demiray and H. Bulut, *Analytical solutions of Phi-four equation*, Int. J. Optim. Control, Theor. Appl. 7(3) (2017) 275-280.

[17] G. E. Fasshauer and J. G. Zhang, *On choosing "optimal" shape parameters for RBF approximation*, Numer. Algorithms 45(1) (2007) 345-368.

[18] S. Rippa, *An algorithm for selecting a good value for the parameter c in radial basis function interpolation*, Adv. Comput. Math. 11(2) (1999) 193-210.