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Mathematical modeling of diffusion problem

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Abstract

This work aims to introduce a numerical approximation procedure based on operational matrix of block pulse functions, which is employed in solving integral-algebraic equations arising from diffusion model. It is known that the integral-algebraic equations belong to the class of singular problems. The main advantage of this method is the reduction of these singular systems by using operational matrix to a linear lower triangular systems of algebraic equations, which is non-singular. An estimation of the error and illustrative instances are discussed to evaluate the validity and applicability of the presented method.

Keywords: Integral-algebraic equations, Diffusion model, Singular systems, Numerical treatment
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1. Introduction

A mixed system of Volterra integral equations consisting of the first and second kind often occurs as following

$$\begin{cases} x_1(t) = f_1(t) + \int_0^t k_{11}(t,s)x_1(s)ds + \int_0^t k_{12}(t,s)x_2(s)ds, \\ \quad 0 = f_2(t) + \int_0^t k_{21}(t,s)x_1(s)ds, \end{cases} \tag{1.1}$$

where f_1, f_2, k_{11}, k_{12} and k_{21} are known functions and $x_1(t)$ and $x_2(t)$ are solutions to be determined. These types of systems are called integral-algebraic equations (IAEs) and they are also well-known as singular systems of integral equations. IAEs arise in many applications and mathematical modeling processes such as problems of the theory of elasticity, dynamic processes in chemical reactors, evolution of a chemical reaction, neutron transport, the kernel identification in viscoelasticity and diffusion mechanism.(For further applications see [5, 6] and references therein.)

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In 1990, Gear introduced the theory of IAEs and he defined the "index reduction procedure" for these type of equations. Since then, Bulatov [1] presented the existence and uniqness results of solution for IAEs. The polynomial spline collocation method and its convergance results have been utilized for a semi-explicit IAEs in [2]. Hadizadeh et. al [3] has studied a Jacobi collocation method for integral-algebraic equations. Maleknejad et. al [4] in 2011, applied an operational matrix with block pulse functions for solving systems of Volterra integral equations of the first kind.

Diffusion mechanism models the movement of many individuals in media or environment. The individuals can be very small such as molecules, cells, basic particles in physics, bacteria, or also they can be very large objects such as plants, animals, or certain kind of events like epidemics, or rumors. The diffusion equation is a partial differential equation which describes density fluctations in a material undergoing diffusion. In order to give an application of IAEs, we consider the following (1D) diffusion equation with initial and mixed boundary conditions [7]:

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < 1, \quad 0 < t, \\ u(x,0) = h_1(x), \quad 0 < x < 1, \\ u_t(0,t) + \alpha(t)u_x(0,t) + \beta(t)u(0,t) = h_2(t), \quad 0 < t, \\ u_x(1,t) + \gamma(t)u(1,t) = h_3(t), \quad 0 < t. \end{cases} \tag{1.2}$$

(1.2) represents a boundary reaction in diffusion of chemicals. Also, the diffusive transport of material to the boundary is represented by $\alpha(t)u_x$. The solution of this equation, for continuous functions h_2, h_3, α, β, γ and for continuously differentiable function h_1, is

$$\begin{aligned} u(x,t) = & \int_0^1 \{\Theta(x-\xi,t) - \Theta(x+\xi,t)\} h_1(\xi) d\xi - 2\int_0^t \frac{\partial \Theta}{\partial x}(x,t-s)\{h_1(0) + \int_0^s \varphi_1(\eta)d\eta\}ds \\ & + 2\int_0^t \frac{\partial \Theta}{\partial x}(x-1,t-s)\{h_1(1) + \int_0^s \varphi_2(\eta)d\eta\}ds, \end{aligned} \tag{1.3}$$

if and only if φ_1 and φ_2 are continuous functions that satisfy in the following system:

$$\begin{cases} \varphi_1(t) = h_2(t) - 2\alpha(t)\int_0^t \Theta(\xi,t)h_1'(\xi)d\xi + 2\alpha(t)\int_0^t \Theta(0,t-s)\varphi_1(s)ds \\ \qquad - 2\alpha(t)\int_0^t \Theta(-1,t-s)\varphi_2(s)ds - \beta(t)h_1(0) + \beta(t)\int_0^t \varphi_1(s)ds, \\ h_3(t) = 2\int_0^t \Theta(1+\xi,t)h_1'(\xi)d\xi - 2\int_0^t \Theta(1,t-s)\varphi_1(s)ds \\ \qquad + 2\int_0^t \Theta(0,t-s)\varphi_2(s)ds + \gamma(t)h_1(1) + \gamma(t)\int_0^t \varphi_2(s)ds, \end{cases} \tag{1.4}$$

where $\Theta(x,t) = \sum_{n=-\infty}^{\infty} K(x+2n,t)$ is a well-known Theta function and $K(x,t) = \dfrac{e^{-x^2/4t}}{\sqrt{4\pi t}}$. Without loss of generality, by assuming $\gamma(t) = \frac{-2\int_0^t \Theta(0,t-s)\varphi_2(s)ds}{h_1(1)+\int_0^t \varphi_2(s)ds}$, this system reduces to a system of the form (1.1) as follows:

$$\begin{cases} \varphi_1(t) = h_2(t) - 2\alpha(t)\int_0^t \Theta(\xi,t)h_1'(\xi)d\xi + 2\alpha(t)\int_0^t \Theta(0,t-s)\varphi_1(s)ds \\ \qquad - 2\alpha(t)\int_0^t \Theta(-1,t-s)\varphi_2(s)ds - \beta(t)h_1(0) - \beta(t)\int_0^t \varphi_1(s)ds, \\ 0 = -h_3(t) + 2\int_0^t \Theta(1+\xi,t)h_1'(\xi)d\xi - 2\int_0^t \Theta(1,t-s)\varphi_1(s)ds, \end{cases} \tag{1.5}$$

The remainder of the paper consists of four sections. Section 2 describes some notations, properties and basic definitions of block pulse functions and itegral-algebraic equations. The general form of the integral-algebraic equation (1.1) is numerically investigated, then error bound of the proposed method is given in Section 3. In the final section, some numerical results are discussed to illustrate the efficiency and accuracy of our algorithm.

2. Preliminaries

In this section, we introduce some useful notations, definitions and also results concerning the integral-algebraic equations and the block pulse functions, which are used further in this paper [8, 4].

The general form of the linear integral- algebraic equations with variable coefficients is defined as follows:

$$A(t)X(t) = F(t) + \int_0^t K(t,s)X(s)ds, t \in I = [0,T], \tag{2.1}$$

with

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix}, K(t,s) = \begin{pmatrix} k_{11}(t,s) & k_{12}(t,s) \\ k_{21}(t,s) & k_{22}(t,s) \end{pmatrix}, F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \tag{2.2}$$

where A is a singular matrix. Also A, F, K are given functions and $X(t) = (x_1(t), x_2(t))$ is the solution to be determined. The conditions of existence and uniqueness of solutions related to the IAEs of (2.1) are considered by the following theorem [1]:

Theorem 2.1. *Assume that the system* (2.1) *with* det $A(t) = 0$ *and* $I = [0,1]$ *satisfies the following conditions:*
1) Rank $A(t)$=*deg(det*$[\lambda A(t) + K(t,t)]) = c \quad \forall t \in I$, *where* c *is a constant and* λ *is a scalar,*
2) Rank $A(0)$= *rank*$[A(0)|X(0)]$,
3)$A(t) \in C^1(I)$, $X(t) \in C^1(I)$ *and* $K(t,s) \in C^1(\triangle)$, *where* $\triangle = \{0 \le s \le t \le 1\}$,

then the system has a unique continuous solution.

Definition 2.2. *Let* $B(t) = [b_1(t), b_2(t), ..., b_n(t)]^T$ *and* b_i *is the ith block pulse function, then an* n*-set of block pulse functions is defined as:*

$$b_i(t) = \begin{cases} 1, & \frac{(i-1)T}{n} \le t < \frac{iT}{n}, \\ 0, & o.w, \end{cases}$$

where $t \in [0,T)$, $h = \frac{T}{n}, i = 1,2,...,n$. *In this work, it is assumed that the block pulse functions are defined over* $t \in [0,1)$. *i.e.* $T = 1$.

Definition 2.3. *Any function* $u(t) \in L^2[0,1)$ *can be approximated by block pulse functions as*

$$u(t) \simeq \sum_{i=1}^{n} u_i b_i(t) = U^T B(t) = B(t)^T U, \tag{2.3}$$

where $U = [u_1, u_2, ..., u_n]^T$ *and* $u_i = n\int_0^1 u(t)b_i(t)dt$.
Also for every $u(x,t) \in L^2([0,1) \times [0,1))$ *approximation of* $u(x,t)$ *can be written as:*

$$u(x,t) \simeq B^T(t)UB(s), \tag{2.4}$$

where U *is the block pulse coefficients matrix with* u_{ij}, $i = 1,2,...,n_1$, $j = 1,2,\ldots,n_2$ *and* $u_{ij} = n_1 n_2 \int_0^1 \int_0^1 u(x,t)b_i(t)b_j(s)dtds$.

Proposition 2.4. *Disjointness, orthogonality and completeness are the most important properties of block pulse functions. Also it is easy to verify the following properties:*

$$B(t)B^T(t) = \mathrm{diag}(B(t)), \qquad B(t)^T B(t) = 1, \tag{2.5}$$

$$B(t)B^T(t)W = \tilde{W}B(t), \tag{2.6}$$

where W be an n-vector, $\tilde{W} = \mathrm{diag}(W)$. And for every matrix $V_{n\times n}$, we can write:

$$B^T(t)VB(t) = \hat{V}^T B(t), \tag{2.7}$$

where $\hat{V}$ is an n-vector with elements equal to the diagonal entries of V. Another property of block pulse family is:

$$\int_0^t B(x)dx \simeq PB(t), \tag{2.8}$$

where $P_{n\times n}$ is well-known as an operational matrix and is given by

$$P = \frac{1}{2n}\begin{bmatrix} 1 & 2 & 2 & \dots & 2 \\ 0 & 1 & 2 & \dots & 2 \\ 0 & 0 & 1 & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 0 & 0 & 0 & \vdots & 1 \end{bmatrix}$$

3. Outline of the method for integral-algebraic equations

First part of our presentation in this section describes the algebraic structure which arises from the block pulse approximations of IAEs, and the second part is concerned with an error analysis of the proposed method.

3.1. Solution of the integral-algebraic equations

Consider the general form of integral-algebraic equations given by (2.1). For simplicity, we rewrite equation (2.1) as:

$$\sum_{j=1}^{2} a_{ij}(t)x_j(t) = f_i(t) + \sum_{j=1}^{2}\int_0^t k_{ij}(t,s)x_j(s)ds, \qquad i = 1, 2. \tag{3.1}$$

Due to (2.3) and (2.4), we have

$$a_{ij}(t) \simeq A_{ij}^T B(t) = B^T(t)A_{ij}, \qquad x_j(t) \simeq X_j^T B(t) = B^T(t)X_j,$$

$$f_i(t) \simeq F_i^T B(t) = B^T(t)F_i, \qquad k_{ij}(t,s) \simeq B^T(t)K_{ij}B(s).$$

Substituting the above relations in (3.1) yields

$$\sum_{j=1}^{2} B^T(t)A_{ij}X_j^T B(t) \simeq F_i^T B(t) + \sum_{j=1}^{2}\int_0^t B^T(t)K_{ij}B(s)B^T(s)X_j ds, \qquad i = 1, 2. \tag{3.2}$$

On the other hand, according to (2.6) and (2.8), the right-hand side of the above equation can be rewritten as

$$\sum_{j=1}^{2}\int_0^t B^T(t)K_{ij}B(s)B^T(s)X_j ds \simeq \sum_{j=1}^{2} B^T(t)K_{ij}\int_0^t B(s)B^T(s)X_j ds = \sum_{j=1}^{2} B^T(t)K_{ij}\int_0^t \tilde{X}_j B(s)ds$$

$$= \sum_{j=1}^{2} B^T(t)K_{ij}\tilde{X}_j \int_0^t B(s)ds = \sum_{j=1}^{2} B^T(t)K_{ij}\tilde{X}_j P B(t), \tag{3.3}$$

and by rearranging (3.2) due to (3.3) we get

$$B^T(t)\left[\sum_{j=1}^{2} A_{ij}X_j^T\right]B(t) \simeq F_i^T B(t) + B^T(t)\left[\sum_{j=1}^{2} K_{ij}\tilde{X}_j P\right]B(t). \tag{3.4}$$

let us set $D_i = \sum_{j=1}^{2} A_{ij}X_j^T$ and $E_i = \sum_{j=1}^{2} K_{ij}\tilde{X}_j P$. Using (2.7) in Proposition 1, the last equation can be transformed to the following matrix form:

$$\widehat{D}_i^T B(t) \simeq F_i^T B(t) + \widehat{E}_i^T B(t), \quad i = 1, 2. \tag{3.5}$$

or equivalently

$$\widehat{D}_i \simeq F_i + \widehat{E}_i, \quad i = 1, 2, \tag{3.6}$$

where

$$\widehat{D}_i = \begin{pmatrix} \sum_{j=1}^{2} a_{ij_{(1)}}x_{i(1)} \\ \sum_{j=1}^{2} a_{ij_{(2)}}x_{i(2)} \\ \vdots \\ \sum_{j=1}^{2} a_{ij_{(n)}}x_{i(n)} \end{pmatrix}, \widehat{E}_i = \begin{pmatrix} \frac{1}{2n}\sum_{j=1}^{2} k_{ij_{(1,1)}} & 0 & \cdots & 0 \\ \frac{1}{n}\sum_{j=1}^{2} k_{ij_{(2,1)}} & \frac{1}{2n}\sum_{j=1}^{2} k_{ij_{(2,2)}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{m}\sum_{j=1}^{2} k_{ij_{(n,1)}} & \frac{1}{n}\sum_{j=1}^{2} k_{ij_{(n,2)}} & \cdots & \frac{1}{n}\sum_{j=1}^{2} k_{ij_{(n,n)}} \end{pmatrix} \begin{pmatrix} x_{i(1)} \\ x_{i(2)} \\ \vdots \\ x_{i(n)} \end{pmatrix}.$$

Clearly, system (3.6) can be represented as a simple form:

$$A_{(r)}X_{(r)} = F_{(r)} + \frac{1}{n}\sum_{s=1}^{r-1} K_{(r,s)}X_{(s)} + \frac{1}{2n}K_{(r,r)}X_{(r)}, \quad r = 1, 2, \ldots, n, \tag{3.7}$$

and so

$$X_{(r)} = \left[A_{(r)} - \frac{1}{2n}K_{(r,r)}\right]^{-1}\left[F_{(r)} + \frac{1}{n}\sum_{s=1}^{r-1} K_{(r,s)}X_{(s)}\right], \quad r = 1, 2, \ldots, n, \tag{3.8}$$

where $X_{(r)} = [x_{1(r)}, x_{2(r)}]^T$, $F_{(r)} = [f_{1(r)}, f_{2(r)}]^T$, $A_{(r)} = [a_{ij(r)}]$ and $K_{(r,s)} = [k_{ij(r,s)}]$. Since $\left[A_{(r)} - \frac{1}{2n}K_{(r,r)}\right]^{-1}$ is non-singular, so (3.8) enables us to compute block pulse coefficients. Finally, the desired approximation to the solution $X(t)$ of (2.1) can be obtained from $X(t) = [X_{(1)}, X_{(2)}, \ldots, X_{(n)}]B(t)$.

The following algorithm summarizes our proposed method:

Algorithm 1. The construction of proposed method for integral-algebraic equation (2.1)

Step 1. Input:
n, f_i, a_{ij}, k_{ij}, for $i, j = 1, 2$, and block pulse bases $B(t)$.
Step 2. Compute $A_{(r)}$, $k_{(r,s)}$ and $F_{(r)}$ from matrices representation of (3.6).
Step 3. Compute non-singular matrix of $\left[A_{(r)} - \frac{1}{2n}K_{(r,r)}\right]^{-1}$.
Step 4. Compute $X_{(r)}$ from the following equation:

$$\left[A_{(r)} - \frac{1}{2n}K_{(r,r)}\right]^{-1}\left[F_{(r)} + \frac{1}{n}\sum_{s=1}^{r-1} K_{(r,s)}X_{(s)}\right].$$

Step 5. Set $x_1(t) = [x_{1(1)}, x_{1(2)}, \ldots, x_{1(n)}]B(t)$, $x_2(t) = [x_{2(1)}, x_{2(2)}, \ldots, x_{2(n)}]B(t)$, and then an approximated solution will be obtained.

3.2. Error analysis

Theorem 3.1. *Let $X_n(t)$ be an approximate solution of the exact solution $X(t)$ of integral-algebraic equation (2.1) by block pulse functions and suppose there exists $(A(t)+I)^{-1}$. If $L = \underbrace{\sup}_{0\le t\le 1} |(A(t)+I)^{-1}|$ and $M = \underbrace{\sup}_{0\le t\le 1} \int_0^t |k_{i,j}(t,s)|ds$, then for $0 < L < 1$ we can conclude $||X(t) - X_n(t)|| \to 0$ as $n \to \infty$.*

Proof . According to the (2.1) and given assumptions, we have

$$A(t)(X(t) - X_n(t)) = \int_0^t K(t,s)(X(s) - X_n(s))ds, \tag{3.9}$$

by adding $(X(t) - X_n(t))$ to the above equation, we obtain

$$(A(t) + I)(X(t) - X_n(t)) = \int_0^t K(t,s)(X(s) - X_n(s))ds + (X(t) - X_n(t)), \tag{3.10}$$

or equivalently

$$(X(t) - X_n(t)) = (A(t) + I)^{-1}\left[\int_0^t K(t,s)(X(s) - X_n(s))ds + (X(t) - X_n(t))\right], \tag{3.11}$$

then we can write

$$\|X(t) - X_n(t)\| \le \|(A(t) + I)^{-1}\|\left[\int_0^t \|K(t,s)\|\|(X(s) - X_n(s))\|ds + \|X(t) - X_n(t))\|\right], \tag{3.12}$$

suppose $M = \underbrace{\sup}_{0\le t\le 1} \int_0^t |k_{i,j}(t,s)|ds$ and $L = \underbrace{\sup}_{0\le t\le 1} |(A(t) + I)^{-1}|$, then we get

$$||X(t) - X_n(t)|| \le L||(X(t) - X_n(t))||\,(M + 1)\,, \tag{3.13}$$

let $N = L(M + 1)$ and we rewrite the above equation as

$$(1 - N)||X(t) - X_n(t)|| \le 0, \tag{3.14}$$

finally, for $0 < N < 1$ we have $||X(t) - X_n(t)|| \to 0$ as $n \to \infty$. □

4. Numerical experiments

In this section, two special cases of the integral-algebraic equations will be investigated to demonstrate the reliability and efficiency of the proposed numerical method. All calculations will be done by the Mathematica.

Example 4.1. *Consider the following diffusion equation* (1.2) *where* $\alpha(t) = 1$, $\beta(t) = 0$, $h_1(x) = 1$, $h_2(t) = \frac{1}{\sqrt{\pi t}}$, $h_3(t) = \frac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi}t^{\frac{3}{2}}}$, $\gamma(t) = \dfrac{e^{\frac{-49}{4t}}(7-12e^{\frac{13}{4t}}+5e^{\frac{6}{t}})(9+\pi^2 t)}{9\pi t(-erf(\frac{5}{2\sqrt{t}})+2erf(\frac{3}{\sqrt{t}})-erf(\frac{7}{2\sqrt{t}}))}$,

$$\begin{cases} u_t = u_{xx}, & 0<x<1,\ 0<t, \\ u(x,0) = 1, & 0<x<1, \\ u_t(0,t)+u_x(0,t) = \frac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi}t^{\frac{3}{2}}}, & 0<t, \\ u_x(1,t)+\gamma(t)u(1,t) = \frac{1}{\sqrt{\pi t}}, & 0<t, \end{cases} \tag{4.1}$$

and the exact solution of this equation is $u(x,t) = erf(\dfrac{1-x}{2\sqrt{t}})$.

Using the given assumption and due to (1.4), we have

$$\begin{cases} \varphi_1(t) = \frac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi}s^{\frac{3}{2}}} + 2\int\limits_0^t \Theta(0,t-s)\varphi_1(s)ds - 2\int\limits_0^t \Theta(-1,t-s)\varphi_2(s)ds \\ \quad 0 = \frac{-1}{\sqrt{\pi t}} - 2\int\limits_0^t \Theta(1,t-s)\varphi_1(s)ds \end{cases}$$

Or equivalently

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix} = \begin{pmatrix} \frac{e^{\frac{-1}{4t}}(1+2t)}{2\sqrt{\pi}t^{\frac{3}{2}}} \\ \frac{-1}{\sqrt{\pi t}} \end{pmatrix} + \int_0^t \begin{pmatrix} 2\Theta(0,t-s) & -2\Theta(-1,t-s) \\ -2\Theta(1,t-s) & 0 \end{pmatrix}\begin{pmatrix} \varphi_1(s) \\ \varphi_2(s) \end{pmatrix} ds$$

Table 1: The absolute errors of Example 1 for different values of n.

n	$\lvert u(0.95,0.05)-\widehat{u}(0.95,0.05)\rvert$ 11 points of interpolation for x and t	$\lvert u(0.95,0.05)-\widehat{u}(0.95,0.05)\rvert$ 101 points of interpolation for x and t
2	5.031×10^{-2}	2.0865×10^{-3}
4	5.022×10^{-2}	2.0861×10^{-3}
8	4.899×10^{-2}	2.0802×10^{-3}
16	1.595×10^{-2}	1.9908×10^{-3}

For computational and numerical implementation of the proposed algorithm, due to complexity of $\Theta(x,t)$, we take $\Theta(x,t) \simeq \sum_{n=-3}^{3} K(x+2n,t)$ and also we should interpolate it by appropriate function. Using Algorithm 1, unknowns $\varphi_1(t)$ and $\varphi_2(t)$ will be obtained. For finding the desired approximation, we substitute $\varphi_1(t)$ and $\varphi_2(t)$ in the following equation

$$\widehat{u}(x,t) = \int_0^1 \{\Theta(x-\xi,t)-\Theta(x+\xi,t)\}d\xi - 2\int_0^t \frac{\partial\Theta}{\partial x}(x,t-s)\left\{1+\int_0^s \varphi_1(\eta)d\eta\right\}ds + 2\int_0^t \frac{\partial\Theta}{\partial x}(x-1,t-s)\left\{1+\int_0^s \varphi_2(\eta)d\eta\right\}ds. \tag{4.2}$$

In Table 1, we report numerical results for different n-set of block pulse functions which $\Theta(x,t)$ is interpolated in 11-points and 101-points of x and t.

Example 4.2. *Let us consider the IAEs system,*

$$\begin{pmatrix} 0 & 0 \\ t & -2t \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{t^3}{6} \\ 3t^2 - \frac{t^3}{6} \end{pmatrix} + \int_0^t \begin{pmatrix} 3s & t+s \\ t+1 & s+1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(s) \end{pmatrix} ds,$$

with the exact solution $x_1(t) = t$ *and* $x_2(t) = -t$.

As we expected, proposed scheme has produced good numerical results. The error estimates of the method for different n-set of block pulse are represented in Table 2. Numerical results for $x_1(t)$ and $x_2(t)$ with $n = 32$ have been shown in Figs. 1, 2, respectively. An approximate solutions of $x_1(t)$ and $x_2(t)$ are shown by $\widehat{x}_1(t)$ and $\widehat{x}_2(t)$, respectively.

Figure 1: Numerical results for $x_1(t)$ of Example 2 using proposed method with $n = 32$

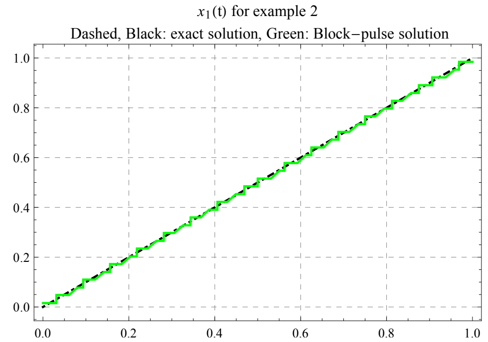

Figure 2: Numerical results for $x_2(t)$ of Example 2 using proposed method with $n = 32$

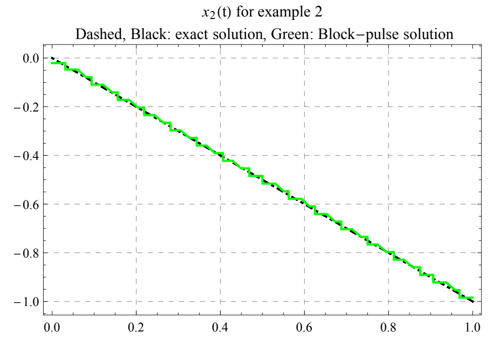

Table 2: The absolute errors of Example 2 for different values of n.

n	$\lvert x_1(0.55) - \widehat{x}_1(0.55)\rvert$	$\lvert x_2(0.55) - \widehat{x}_2(0.55)\rvert$
2	2.098×10^{-1}	1.960×10^{-1}
4	7.192×10^{-2}	7.850×10^{-2}
8	6.192×10^{-2}	8.240×10^{-2}
16	3.251×10^{-2}	4.145×10^{-2}

5. Conclusion

In this research, an efficient numerical algorithm based on the block pulse functions was proposed for the new class of the system of integral equation as the integral-algebraic equations. The difficulty which we will be faced with is the singularity of $A(t)$, which is done by proposed method and also using triangular matrix, we may conclude the low computational complexity. Two numerical examples was introduced to show that the numerical method is applicable with a good accuracy.

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