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A novel scheme for solving multi-delay fractional optimal control problems

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Abstract

In this paper, we consider the problems of suboptimal control for a class of fractional-order optimal control problems with multi-delay argument. The fractional derivative in these problems is in the Caputo sense. To solve the problem, first by a suitable approximation, we replace the Caputo derivative to integer order derivative. The optimal control law consists of an accurate linear feedback term and a nonlinear compensation term which is the limit of an adjoint vector sequence, is obtained by a sensitivity approach. The feed back term is determined by solving Riccati matrix differential equation. By using a finite sum of the series, we can obtain a suboptimal control law. Finally, numerical results are included to demonstrate the validity and applicability of the present technique.

Keywords: Delay optimal control problems, Fractional order, Riccati differential equation, Caputo deriavitive.
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1. Introduction

The fractional calculus is more than 300 years old. The idea of differential order calculus a fraction of the correct order differential calculus was taken at the request of Mr. Leibniz from the hospital in 1965 to calculate the half-order derivative. In the last three decades, the applications of fractional and integral calculus have grown significantly. This mathematical theory makes it possible to describe the model more accurately than the classical method for real systems. The main reason for using the correct order models was the lack of response method for fractional differential equations (FDEs). Currently, there are many methods for approximating fractional derivatives and integrals,

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and fractional differential calculus can be used in many physical applications. Fractional order differential calculus plays a very important role in the laws of physics, power electrical engineering, control systems, robotics, biology, economics and many more [16, 39].

Fractional optimal control problems (FOCPs) are a subclass of optimal control problems (OCPs) whose dynamics are described by FDEs. Recently, the applications of this equations have included in various classes of FOCPs that refers to the minimization of a performance index subject to the FDEs are used as the dynamic constraints [1]. With the emerging number of the applications of FOCPs, the solution of these kind of problems has become an important topic for researchers. Using necessary optimality conditions, the FOCP reduced to a system of FDEs so that by finding its solution, one approximates the solutions of the original problem. A general formulation of FOCPs was extended by Agrawal [2], where the necessary conditions of optimization are achieved with the Caputo and Reimann-Liouvile derivatives. Since, it is difficult to obtain the exact solutions of FOCPs, approximate and numerical methods are used extensively that can be seen in [1, 45].

Among dynamic systems, time-delay systems are very important. In many physical and biological phenomena, the coefficient of variation in system variables depends on the past values of system variables. This feature is called time delay. Delay is first discovered in biological systems and later in many engineering systems, such as mechanical transmission lines, fluid transmission lines, grid control systems, and metal smelting processes; see [8, 20, 26]. Time-delay is often a source of instability and leads to poor control performance, which is why the persistent issue of time-delayed systems has attracted much attention in recent research. Stabilizing time-delayed systems is also a complex issue [10]. In the meantime, several techniques for time-delayed systems have been proposed. Of course, all the existing articles in this field have designed control rules assuming that the exact amount of delay is known, and the proposed control rules depend on the amount of delay. Optimal control of time-delay systems is one of the most challenging mathematical problems in control theory. We briefly review some resent papers that are relevant to the method developed in the current work for time-delay optimal control problem. The method based on biortogonal cubic Hermit spline multiwavelets [30], variational iteration method (VIM) [25], DARE solutions and suboptimal control of systems with multiple input-output delays [13], method in the work of Basin [3], hybrid of block-pulse functions and orthogonal Taylor series [6], composite Chebyshev finite difference method [22], semi-infinite programming approach to nonlinear time delay optimal control problem (TDOCP) [17], adaptive pseudospectral method [21], hybrid of block-pulse functions and Legendre polynomials [23, 24], the Homotopy perturbation method [27], an iterior-point algorithm [40, 41], sliding mode control [37], a novel neural network discrete-time optimal control [18] and a numerical approach based upon Fibonacci wavelets and Petrov-Galerkin method [35].

The delay fractional optimal control problems (DFOCPs) are an extension form of fractional optimal problems which at least one of their variables in the objective function or in the dynamic system has the delay term. These problems appear in engineering, economics, power systems, transportation, biological, electronics, manufacturing, chemical, and many other fields [42, 43]. In the recent years, however, fewer articles in the delay fractional optimal control problems have been proposed such as the Legendre operational technique is suggested by Bhrawy and Ezz-Eldien [5]. Effati et al. [7] designed a Grunwald-Letnikov approximation for the fractional derivatives and the Euler-Lagrange equation is used for finding its state and control variables. Binazadeh and Yousefi [4] designed a cascade-control structure using fractional-order controllers based on the fractional-order sliding-mode controller. Moradi et al. [31] presented a numerical approach based on the Chelyshkov wavelets and the operational matrices for DOCPs with fractional order. Kheyrinataj and Nazemi [15] proposed an artificial inteligence approach using Müntz–Legendre neural network construction for solving delay optimal control problems of fractional order with equality and inequality constraints. Hosseinpour et al. [11] solved the DOCPs of fractional order by using a Müntz–Legendre spectral collocation method. moreover, Nemati et al. [32] proposed Legendre wavelet collocation method combined with the Gauss–Jacobi quadrature for solving fractional delay-type integro-differential equations. Yousefi and Binazadeh [44] solved the DFOC by using a delay-independent sliding mode control.

Motivated by the above discussions, in this paper, we consider a particular numerical scheme based on sensitivity approach which can be used to solve fractional-order optimal control problems with multi-delay argument, and develop an approximation algorithm to obtain the optimal control law. The optimal control law obtained is composed of a state feedback term and a compensation term. Using the adjoint vector solution for the finite-step iteration, a suboptimal control law can be obtained. The method is especially suitable for the synthesis of small time-delay systems. The arithmetic of solving the suboptimal control law is given. The simulation results show the effectiveness of our method.

Based on the above review, the structure of this paper is arranged as follows: In Section 2, some important definitions and necessary preliminaries of fractional derivatives are described. In Section 3, we introduce problem statement of DFOCP and transform the DFOCP to a DOCP. In this section, we replace the Caputo derivative to integer order derivative with the help of a good approximation and state the necessary optimality conditions of the obtained problem. The solution of the optimal control problem is given in Section 4, including the optimal control law design and an implementation algorithm. In Section 5 the numerical examples are simulated to show the reasonableness of our theory and demonstrate the performance of our network. Finally, we end this paper with conclusions in Section 6.

2. Fractional derivatives and integrals

In this section, we recall some basic definitions and propertied of fractional calculus theory, which will be used in this paper. The more basic and detailed information can be obtained from [1, 36]. Let $f : [a, b] \to \mathbb{R}$ be a function, $\alpha > 0$ is a real number, and $m = [\alpha] + 1$.

Definition 2.1. *The fractional left and right Riemann-Lioville integrals of order α are defined as follows, respectively:*

$$\begin{aligned} {}_{t_0}I_t^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-z)^{\alpha-1} f(z)dz, \qquad t > t_0, \\ {}_{t}I_{t_f}^{\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_{t}^{t_f} (z-t)^{\alpha-1} f(z)dz, \qquad t < t_f, \end{aligned} \tag{2.1}$$

where Γ is the Euler-Gamma function, that is

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \qquad z > 0. \tag{2.2}$$

Definition 2.2. *The left Caputo fractional derivative (LCFD) and the right Caputo fractional derivative (RCFD) of $f(t)$ of order α, when it exists, are defined as:*

$${}_{t_0}^{C}D_t^{\alpha} f(t) = {}_{t_0}I_t^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-z)^{m-\alpha-1} f^m(z)dz, \tag{2.3}$$

$${}_{t}^{C}D_{t_f}^{\alpha} f(t) = (-1)^m {}_{t}I_{t_f}^{m-\alpha} f(t) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{t}^{t_f} (z-t)^{m-\alpha-1} f^m(z)dz, \tag{2.4}$$

where $f^{(m)}(t)$ and $D^m f(t)$ are the usual m-th derivative of $f(t)$.

Definition 2.3. *The fractional integral and Caputo fractional derivative of t^k are given by*

$$ {}_{t_0}I_t^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(\alpha+k+1)} t^{\alpha+k}, \quad k \in \mathbb{N} \cup \{0\}, \ t > 0, \tag{2.5} $$

and if $0 \leq m-1 < \alpha \leq m < k+1, k > 0$ *then we have*

$$ {}_{t_0}^{C}D_t^{\alpha} t^k = \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, \quad t > 0. \tag{2.6} $$

The following theorem, helps us to apply a fractional integral over a fractional derivative.

Theorem 2.4. *Let* $\alpha > 0$, $m = [\alpha] + 1$ *and* $f : [a, b] \to \mathbb{R}$ *be a function, then*

$$ {}_{t_0}I_t^{\alpha}[{}_{t_0}^{C}D_t^{\alpha} f(t)] = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k, \tag{2.7} $$

$$ {}_{t}I_{t_f}^{\alpha}[{}_{t}^{C}D_{t_f}^{\alpha} f(t)] = f(t) - \sum_{k=0}^{m-1} \frac{(-1)^k f^{(k)}(t_f)}{k!}(t_f-t)^k, \tag{2.8} $$

for $0 < \alpha < 1$ *we have the following properties:*

$$ {}_{t_0}I_t^{\alpha}[{}_{t_0}^{C}D_t^{\alpha} f(t)] = f(t) - f(t_0), \tag{2.9} $$

$$ {}_{t}I_{t_f}^{\alpha}[{}_{t}^{C}D_{t_f}^{\alpha} f(t)] = f(t) - f(t_f). \tag{2.10} $$

In [45, 36, 33, 34] a good approximation is obtained without the requirement of such higher-order smoothness on the admissible functions. The method can be explained for left Riemann-Liouville fractional derivatives in the following way

$$ {}_{t_0}D_t^{\alpha} x(t) = A(\alpha)(t-t_0)^{-\alpha} x(t) + B(\alpha)(t-t_0)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{\infty} C(\alpha,p)(t-t_0)^{1-p-\alpha} v_p(t), \tag{2.11} $$

where $v_p(t)$ is defined as the solution of the system

$$ \begin{cases} \dot{v}_p(t) = (1-p)(t-t_0)^{p-2} x(t), \\ v_p(t_0) = 0, \end{cases} $$

for $p = 2, 3, \cdots$, and $A(\alpha), B(\alpha)$ and $C(\alpha, p)$ are given by

$$ \begin{aligned} A(\alpha) &= \frac{1}{\Gamma(1-\alpha)} \left[1 + \sum_{p=2}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!} \right], \\ B(\alpha) &= \frac{1}{\Gamma(2-\alpha)} \left[1 + \sum_{p=1}^{\infty} \frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!} \right], \\ C(\alpha,p) &= \frac{\Gamma(p-1+\alpha)}{\Gamma(2-\alpha)\Gamma(\alpha-1)(p-1)!}. \end{aligned} $$

For computational purposes, we truncate the sum and consider the finite expansion:

$$ {}_{t_0}D_t^{\alpha} x(t) = A(\alpha,N)(t-t_0)^{-\alpha} x(t) + B(\alpha,N)(t-t_0)^{1-\alpha}\dot{x}(t) - \sum_{p=2}^{N} C(\alpha,p)(t-t_0)^{1-p-\alpha} v_p(t), \tag{2.12} $$

where $A(\alpha, N)$ and $B(\alpha, N)$ are now defined by

$$A(\alpha, N) = \frac{1}{\Gamma(1-\alpha)}\left[1 + \sum_{p=2}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha)(p-1)!}\right],$$
$$B(\alpha, N) = \frac{1}{\Gamma(2-\alpha)}\left[1 + \sum_{p=1}^{N}\frac{\Gamma(p-1+\alpha)}{\Gamma(\alpha-1)p!}\right].$$

See [33, 34] for proofs and other details.

3. Problem statement

Consider the linear fractional time-varying multi-delay system:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = A(t)x(t) + A_1(t)x(t-\tau_x) + B(t)u(t) + B_1(t)u(t-\tau_u), \\ x(t) = \phi(t), \qquad t_0 - \tau_x \leqslant t \leqslant t_0, \\ u(t) = \psi(t), \qquad t_0 - \tau_u \leqslant t \leqslant t_0, \end{cases} \tag{3.1}$$

with the following cost functional

$$J = \frac{1}{2}x^T(t_f)H(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right)dt, \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, are the state and control vectors respectively; $A(t)$, $A_1(t)$, $B(t)$ and $B_1(t)$ are real, piecewise continuous matrices of appropriate dimensions defined on the appropriate intervals; $\phi(t)$ and $\psi(t)$ are specified initial functions; τ_x and τ_u are constant positive scalars; the matrix $H(t_f) \in \mathbb{R}^{n\times n}$ is symmetric positive semi-definite, $Q(t) \in \mathbb{R}^{n\times n}$ and $R(t) \in \mathbb{R}^{m\times m}$ are chosen to be positive semi-definite and positive definite matrices respectively.

Here, it is assumed that the system (3.1) is controllable and assume that $\tau_u < \tau_x$. The aim is to find a control signal $u(t)$ which the cost functional (3.2) is minimized while the dynamic equality constraint (3.1) is satisfied.
Let us now use the approximation given by (2.12). The system in (3.1) becomes:

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha}x(t) = A(\alpha, N)(t-t_0)^{-\alpha}x(t) + B(\alpha, N)(t-t_0)^{1-\alpha}\dot{x}(t) \\ \qquad\qquad - \sum\limits_{p=2}^{N} C(\alpha, p)(t-t_0)^{1-p-\alpha}v_p(t) - \frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ \qquad\qquad = A(t)x(t) + A_1(t)x(t-\tau_x) + B(t)u(t) + B_1(t)u(t-\tau_u), \\ \dot{v}_p(t) = (1-p)(t-t_0)^{p-2}x(t), \qquad p = 2, 3, \cdots, N, \\ x(t) = \phi(t), \qquad t_0 - \tau_x \leqslant t \leqslant t_0, \\ u(t) = \psi(t), \qquad t_0 - \tau_u \leqslant t \leqslant t_0, \\ v_p(t_0) = 0, \qquad p = 2, 3, \cdots, N. \end{cases} \tag{3.3}$$

By sorting the dynamic system (3.3), we have

$$\begin{cases} \dot{x}(t) = \left[\frac{A(t)-A(\alpha,N)(t-t_0)^{-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}}\right] x(t) + \left[\frac{A_1(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}}\right] x(t-\tau_x) \\ \qquad + \left[\frac{B(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}}\right] u(t) + \left[\frac{B_1(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}}\right] u(t-\tau_u) \\ \qquad + \sum_{p=2}^{N} \left[\frac{C(\alpha,p)(t-t_0)^{1-p-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}}\right] v_p(t) + \frac{x(t_0)(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)B(\alpha,N)(t-t_0)^{1-\alpha}}, \\ \dot{v}_p(t) = (1-p)(t-t_0)^{p-2}x(t), \quad p = 2,3,\cdots,N, \\ x(t) = \phi(t), \quad t_0-\tau_x \leqslant t \leqslant t_0, \\ u(t) = \psi(t), \quad t_0-\tau_u \leqslant t \leqslant t_0, \\ v_p(t_0) = 0, \quad p = 2,3,\cdots,N. \end{cases} \tag{3.4}$$

Below, as a result, we get the initial value of $v_p(t) = \phi_p(t)$ in the interval $t \in [t_0 - \tau_x, t_0]$.

Corollary 3.1. *If $v_p(t) = \phi_p(t)$ in $t \in [t_0 - \tau_x, t_0]$ then it can be shown that*

$$\phi_p(t) = \frac{v_p(t_0 - \tau_x)}{\tau_x}(t_0 - t). \tag{3.5}$$

Proof . Let's assume first

$$v_p(t) = \phi_p(t); \quad t_0 - \tau_x \leqslant t \leqslant t_0,$$

Because the following relationships are established:

$$\begin{aligned} x(t) &= \phi(t); \quad t_0 - \tau_x \leqslant t \leqslant t_0, \\ \dot{v}(t) &= (1-p)(t-t_0)^{p-2}x(t); \quad p = 2,3,\cdots,N, \\ v_p(t_0) &= 0; \quad p = 2,3,\cdots,N. \end{aligned}$$

So, we have

$$\begin{aligned} \int_{t_0}^{t} \dot{v}(s)ds = v_p(t) - v_p(t_0) &= \int_{t_0}^{t} (1-p)(s-t_0)^{p-2}x(s)ds \\ \Longrightarrow v_p(t) &= \int_{t_0}^{t} (1-p)(s-t_0)^{p-2}x(s)ds. \end{aligned} \tag{3.6}$$

Hence, it can be written

$$\begin{aligned} v_p(t_0 - \tau_x) &= \int_{t_0}^{t_0-\tau_x} (1-p)(s-t_0)^{p-2}x(s)ds \\ &= -\int_{t_0-\tau_x}^{t_0} (1-p)(s-t_0)^{p-2}x(s)ds \\ &= -\int_{t_0-\tau_x}^{t_0} (1-p)(s-t_0)^{p-2}\phi(s)ds, \end{aligned}$$

On the other hand, because $v_p(t_0)$ and $v_p(t_0 - \tau_x)$ are known, so $\phi_p(t)$ can be considered as follows:

$$\phi_p(t) = \frac{v_p(t_0 - \tau_x)}{\tau_x}(t_0 - t).$$

□ In the following, according to the dynamic system (3.4), we construct a new time-varying multi-delayed optimal control problems as follows:

$$minimize \quad \bar{J} = \frac{1}{2}X^T(t_f)\bar{H}(t_f)X(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(X^T(t)\bar{Q}(t)X(t) + u^T(t)R(t)u(t)\right) dt, \tag{3.7}$$

subject to:

$$\begin{cases} \dot{X}(t) = \bar{A}(t)X(t) + \bar{A}_1(t)X(t-\tau_x) + \bar{B}(t)u(t) + \bar{B}_1(t)u(t-\tau_u) + \Xi(t), \\ X(t) = \bar{\phi}(t); \quad t_0 - \tau_x \leqslant t \leqslant t_0, \\ u(t) = \psi(t); \quad t_0 - \tau_u \leqslant t \leqslant t_0, \end{cases} \tag{3.8}$$

where

$$\bar{A}(t) = \begin{bmatrix} \frac{A(t)-A(\alpha,N)(t-t_0)^{-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} & \frac{C(\alpha,2)(t-t_0)^{1-2-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} & \cdots & \frac{C(\alpha,N)(t-t_0)^{1-N-\alpha}}{B(\alpha,N)(t-t_0)^{1-\alpha}} \\ (1-2)(t-t_0)^{2-2} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (1-N)(t-t_0)^{2-N} & 0 & \cdots & 0 \end{bmatrix},$$

$$\bar{A}_1(t) = \begin{bmatrix} \frac{A_1(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{B}(t) = \begin{bmatrix} \frac{B(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \bar{B}_1(t) = \begin{bmatrix} \frac{B_1(t)}{B(\alpha,N)(t-t_0)^{1-\alpha}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$\Xi(t) = \begin{bmatrix} \frac{x(t_0)}{\Gamma(1-\alpha)B(\alpha,N)(t-t_0)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad X(t) = \begin{bmatrix} x(t) \\ v_2(t) \\ \vdots \\ v_N(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ v_p(t) \end{bmatrix}, \quad \bar{\phi}(t) = \begin{bmatrix} \phi(t) \\ \phi_p(t) \end{bmatrix},$$

$$\bar{H}(t) = \begin{bmatrix} H(t_f) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \quad \bar{Q}(t) = \begin{bmatrix} Q(t) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{3.9}$$

In order to find the optimal control, a Hamiltonian function for the problem (3.7)-(3.9) is given by

$$\begin{aligned} \mathcal{H}(X,u,\lambda,t) &= \frac{1}{2}X^T(t)\bar{Q}(t)X(t) + \frac{1}{2}u^T(t)R(t)u(t) \qquad (3.10) \\ &+ \lambda^T(t)\left[\bar{A}(t)X(t) + \bar{A}_1(t)X(t-\tau_x) + \bar{B}(t)u(t) + \bar{B}_1(t)u(t-\tau_u) + \Xi(t)\right], \end{aligned}$$

where $\lambda(t) \in \mathbb{R}^n$ is the vector of the Lagrange multiplier. According to the necessary conditions for optimality, we can obtain the following nonlinear two point boundary value problem (TPBVP) [14, 29]:

$$\dot{X}(t) = \begin{cases} \bar{A}(t)X(t) + \bar{A}_1(t)X(t-\tau_x) - (S_1(t)+S_2(t))\lambda(t) \\ -S_3(t)\lambda(t+\tau_u) - S_4(t)\lambda(t-\tau_u) + \Xi(t), & t_0 \leqslant t < t_f - \tau_u, \\ \bar{A}(t)X(t) + \bar{A}_1(t)X(t-\tau_x) - S_1(t)\lambda(t) \\ -S_4(t)\lambda(t-\tau_u) + \Xi(t), & t_f - \tau_u \leqslant t \leqslant t_f, \end{cases} \tag{3.11}$$

and

$$\dot{\lambda}(t)=\begin{cases}-\bar{Q}(t)X(t)-\bar{A}^T(t)\lambda(t)-\bar{A}_1^T(t+\tau_x)\lambda(t+\tau_x), & t_0\leqslant t<t_f-\tau_x,\\ -\bar{Q}(t)X(t)-\bar{A}^T(t)\lambda(t), & t_f-\tau_x\leqslant t\leqslant t_f,\end{cases}\tag{3.12}$$

with initial conditions

$$\begin{cases}X(t)=\bar{\phi}(t), & t_0-\tau_x\leqslant t\leqslant t_0,\\ u(t)=\psi(t), & t_0-\tau_u\leqslant t\leqslant t_0,\\ \lambda(t_f)=\bar{H}(t_f)X(t_f),\end{cases}\tag{3.13}$$

where

$$\begin{aligned}S_1(t)&=\bar{B}(t)R^{-1}(t)\bar{B}^T(t),\\ S_2(t)&=\bar{B}_1(t)R^{-1}(t-\tau_u)\bar{B}_1^T(t),\\ S_3(t)&=\bar{B}(t)R^{-1}(t)\bar{B}_1^T(t+\tau_u),\\ S_4(t)&=\bar{B}_1(t)R^{-1}(t-\tau_u)\bar{B}^T(t-\tau_u).\end{aligned}$$

Hence, the optimal control law is obtained by:

$$u(t)=\begin{cases}-R^{-1}(t)\bar{B}^T(t)\lambda(t)-R^{-1}(t)\bar{B}_1^T(t+\tau_u)\lambda(t+\tau_u), & t_0\leqslant t<t_f-\tau_u,\\ -R^{-1}(t)\bar{B}^T(t)\lambda(t), & t_f-\tau_u\leqslant t\leqslant t_f.\end{cases}\tag{3.14}$$

Note that, relations (3.11)-(3.13) form a nonlinear TPBVP with time-varying coefficient involving both delay and advance terms. The exact solution of this problem is, in general, extremely difficult, if not impossible. Therefore, it is necessary to find approximation approaches for solving the optimal control problem. We propose an sensitivity approach in this paper.

4. Design of suboptimal control

First, we introduce a sensitivity parameter ε [12, 19, 38] in TPBVP (3.11)-(3.13) and obtain the following TPBVP including sensitivity coefficient ε

$$\begin{cases}\dot{X}(t,\varepsilon)=\begin{cases}\bar{A}(t)X(t,\varepsilon)+\varepsilon\bar{A}_1(t)X(t-\tau_x,\varepsilon)-(S_1(t)+\varepsilon^2S_2(t))\lambda(t,\varepsilon)\\ -\varepsilon S_3(t)\lambda(t+\tau_u,\varepsilon)-\varepsilon S_4(t)\lambda(t-\tau_u,\varepsilon)+\Xi(t), & t_0\leqslant t<t_f-\tau_u,\\ \bar{A}(t)X(t,\varepsilon)+\varepsilon\bar{A}_1(t)X(t-\tau_x,\varepsilon)-S_1(t)\lambda(t,\varepsilon)\\ -\varepsilon S_4(t)\lambda(t-\tau_u,\varepsilon)+\Xi(t), & t_f-\tau_u\leqslant t\leqslant t_f,\end{cases}\\ \dot{\lambda}(t,\varepsilon)=\begin{cases}-\bar{Q}(t)X(t,\varepsilon)-\bar{A}^T(t)\lambda(t,\varepsilon)-\varepsilon\bar{A}_1^T(t+\tau_x)\lambda(t+\tau_x,\varepsilon), & t_0\leqslant t<t_f-\tau_x,\\ -\bar{Q}(t)X(t,\varepsilon)-\bar{A}^T(t)\lambda(t,\varepsilon), & t_f-\tau_x\leqslant t\leqslant t_f,\end{cases}\\ X(t,\varepsilon)=\bar{\phi}(t), \qquad t_0-\tau_x\leqslant t\leq t_0,\\ \lambda(t_f,\varepsilon)=\bar{H}(t_f)X(t_f,\varepsilon).\end{cases}\tag{4.1}$$

The control law $u(t)$ concluding sensitivity coefficient ε can be expressed as

$$u(t,\varepsilon)=\begin{cases}-R^{-1}(t)\bar{B}^T(t)\lambda(t,\varepsilon)-\varepsilon R^{-1}(t)\bar{B}_1^T(t+\tau_u)\lambda(t+\tau_u,\varepsilon), & t_0\leqslant t<t_f-\tau_u,\\ -R^{-1}(t)\bar{B}^T(t)\lambda(t,\varepsilon), & t_f-\tau_u\leqslant t\leqslant t_f.\end{cases}\tag{4.2}$$

where $0 \leq \varepsilon \leq 1$ is a scalar. In the following discussion, we always assume that the solution of TPBVP (4.1) is uniquely existed, and $u(t,\varepsilon)$, $x(t,\varepsilon)$ and $\lambda(t,\varepsilon)$ with parameter ε are infinitely differentiable with respect to the ε around $\varepsilon = 0$, and their Maclaurin series expansions are convergent at $\varepsilon = 1$; obviously when $\varepsilon = 1$, TPBVP (4.1) and control law (4.2) are equivalent to original problem in (3.11) and (3.14), respectively.
According to this assumption, we can write:

$$\begin{cases} u(t,\varepsilon) = \sum\limits_{i=0}^{\infty} \frac{\varepsilon^i u^{(i)}(t)}{i!}, \\ \\ X(t,\varepsilon) = \sum\limits_{i=0}^{\infty} \frac{\varepsilon^i X^{(i)}(t)}{i!}, \\ \\ \lambda(t,\varepsilon) = \sum\limits_{i=0}^{\infty} \frac{\varepsilon^i \lambda^{(i)}(t)}{i!}, \end{cases} \tag{4.3}$$

where the superscript i denotes ith-order differentiation with respect to ε evaluated at $\varepsilon = 0$. Hence, by substituting (4.3) into (4.1) and equating terms with the same order of ε on each side we have:

$$\begin{cases} \dot{X}^{(0)}(t) = \bar{A}(t)X^{(0)}(t) - S_1(t)\lambda^{(0)}(t) + \Xi(t), & t_0 \leqslant t < t_f, \\ \dot{\lambda}^{(0)}(t) = -\bar{Q}X^{(0)}(t) - \bar{A}^T(t)\lambda^{(0)}(t), & t_0 \leq t \leq\leq t_f, \\ X^{(0)}(t) = \bar{\phi}(t), \\ \lambda^{(0)}(t_f) = \bar{H}(t_f)X(t_f), \end{cases} \tag{4.4}$$

and

$$\begin{cases} \dot{X}^{(i)}(t) = \begin{cases} \bar{A}(t)X^{(i)}(t) + \bar{A}_1(t)X^{(i-1)}(t-\tau_x) - S_1(t)\lambda^{(i)}(t) \\ -S_3(t)\lambda^{(i-1)}(t+\tau_u) - S_4(t)\lambda^{(i-1)}(t-\tau_u) + \Xi(t), & t_0 \leqslant t < t_f - \tau_u, \\ \bar{A}(t)X^{(i)}(t) + \bar{A}_1(t)X^{(i-1)}(t-\tau_x) - S_1(t)\lambda^{(i)}(t) \\ -S_4(t)\lambda^{(i-1)}(t-\tau_u) + \Xi(t), & t_f - \tau_u \leqslant t \leqslant t_f, \end{cases} \\ \dot{\lambda}^{(i)}(t) = \begin{cases} -\bar{Q}(t)X^{(i)}(t) - \bar{A}^T(t)\lambda^{(i)}(t) - \bar{A}_1^T(t+\tau_x)\lambda^{(i-1)}(t+\tau_x), t_0 \leqslant t < t_f - \tau_x, \\ -\bar{Q}(t)X^{(i)}(t) - \bar{A}^T(t)\lambda^{(i)}(t), & t_f - \tau_x \leqslant t \leqslant t_f, \end{cases} \\ X^{(i)}(t) = [0,0,\cdots,0]^T, \\ \lambda^{(i)}(t_f) = [0,0,\cdots,0]^T, \end{cases} \tag{4.5}$$

for $i = 1,2,\cdots$. Also, from (4.2) of $u(t,\varepsilon)$ and the infinite series of $u(t)$ in (4.3), we obtain the i-th-order control terms of the optimal control

$$u^{(i)}(t) = \begin{cases} -R^{-1}(t)\bar{B}^T(t)\lambda^{(i)}(t) - R^{-1}(t)\bar{B}_1^T(t+\tau_u)\lambda^{(i-1)}(t+\tau_u), & t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)\bar{B}^T(t)\lambda^{(i)}(t), & t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{4.6}$$

It should be noted that in (4.5) $X^{(i-1)}(t-\tau_x)$ and $\lambda^{(i-1)}(t+\tau_x)$ are known from previous iteration so (4.4) and (4.5) is a sequence of inhomogeneous linear TPBVPs without time-delay and time-advance terms in each iteration. After determining $X^{(i)}(t)$ and $\lambda^{(i)}(t)$ for $i \geq 0$, $X(t,\varepsilon)$ and $\lambda(t,\varepsilon)$ can be

determined as the solution of TPBVP (4.1) by using (4.2).
If we stop the procedure at this step, by using (3.14) and (4.3) we can find the optimal control law which is an open loop control; but for obtaining a close loop control in the form of state feedback, we continue our discussion by assuming that ith-order terms of $\lambda(t)$ in (4.5) be

$$\lambda^{(i)}(t) = P(t)X^{(i)}(t) + g_i(t), \tag{4.7}$$

where $P(t) \in \mathbb{R}^{N\times N}$ is unknown positive-semidefinite function matrix, $g_i(t) \in \mathbb{R}^N$ is the adjoint vector.
Computing the derivatives to the both sides with respect to t of equation (4.7), we have

$$\begin{aligned}\dot{\lambda}^{(i)}(t) &= \dot{P}(t)X^{(i)}(t) + P(t)\dot{X}^{(i)}(t) + \dot{g}_i(t), \quad t_0 \leqslant t \leqslant t_f\\ &= \left[\dot{P}(t) + P(t)\bar{A}(t) - P(t)S_1(t)P(t)\right] X^{(i)}(t) - P(t)S_1(t)g_i(t)\\ &+ P(t)\bar{A}_1(t)X^{(i-1)}(t-\tau_x) + P(t)F_i(t) + \dot{g}_i(t),\end{aligned} \tag{4.8}$$

where

$$F_i(t) = \begin{cases} -S_3(t)\left[P(t+\tau_u)X^{(i-1)}(t+\tau_u) + g_{i-1}(t+\tau_u)\right] \\ -S_4(t)\left[P(t-\tau_u)X^{(i-1)}(t-\tau_u) + g_{i-1}(t-\tau_u)\right] + \Xi(t), & t_0 \leqslant t < t_f - \tau_u, \\ -S_4(t)\left[P(t-\tau_u)X^{(i-1)}(t-\tau_u) + g_{i-1}(t-\tau_u)\right] + \Xi(t), & t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{4.9}$$

Putting (4.7) into equation (4.5), we get

$$\dot{\lambda}^{(i)}(t) = \begin{cases} -\bar{Q}(t)X^{(i)}(t) - \bar{A}^T(t)P(t)X^{(i)}(t) - \bar{A}^T(t)g_i(t) \\ -\bar{A}_1^T(t+\tau_x)[P(t+\tau_x)X^{(i-1)}(t+\tau_x) + g_{i-1}(t+\tau_x)], & t_0 \leqslant t < t_f - \tau_x, \\ -\bar{Q}(t)X^{(i)}(t) - \bar{A}^T(t)P(t)X^{(i)}(t) - \bar{A}^T(t)g_i(t), & t_f - \tau_x \leqslant t \leqslant t_f. \end{cases} \tag{4.10}$$

Thus, from (4.8) and (4.10), we can obtain the following Riccati matrix differential equation:

$$-\dot{P}(t) = P(t)\bar{A}(t) + \bar{A}^T(t)P(t) - P(t)S_1(t)P(t) + \bar{Q}(t), \quad P(t_f) = \bar{H}(t_f), \tag{4.11}$$

and adjoint vector differential equation the following:

$$\dot{g}_i(t) = -\left[\bar{A}(t) - S_1(t)P(t)\right]^T g_i(t) - P(t)\bar{A}_1(t)X^{(i-1)}(t-\tau_x) + G_i(t), \quad g_i(t_f) = 0, \tag{4.12}$$

where

$$G_i(t) = \begin{cases} -\bar{A}_1^T(t+\tau_x)\left[P(t+\tau_x)X^{(i-1)}(t+\tau_x) + g_{i-1}(t+\tau_x)\right] \\ +P(t)S_3(t)\left[P(t+\tau_u)X^{(i-1)}(t+\tau_u) + g_{i-1}(t+\tau_u)\right] \\ +P(t)S_4(t)\left[P(t-\tau_u)X^{(i-1)}(t-\tau_u) + g_{i-1}(t-\tau_u)\right], & t_0 \leqslant t < t_f - \tau_x, \\ P(t)S_3(t)\left[P(t+\tau_u)X^{(i-1)}(t+\tau_u) + g_{i-1}(t+\tau_u)\right] \\ +P(t)S_4(t)\left[P(t-\tau_u)X^{(i-1)}(t-\tau_u) + g_{i-1}(t-\tau_u)\right], & t_f - \tau_x \leqslant t < t_f - \tau_u, \\ P(t)S_4(t)\left[P(t-\tau_u)X^{(i-1)}(t-\tau_u) + g_{i-1}(t-\tau_u)\right], & t_f - \tau_u \leqslant t \leqslant t_f, \end{cases} \tag{4.13}$$

and $g_0(t) = 0$ for $t_0 - \tau_x \leq t < t_0$. Substituting (4.7) into (4.4) and (4.5) yields:

$$\begin{cases} \dot{X}^{(0)}(t) = [\bar{A}(t) - S_1(t)P(t)]X^{(0)}(t) + \Xi(t), & t_0 \leqslant t \leq t_f, \\ X^{(0)}(t) = \bar{\phi}(t), \end{cases} \tag{4.14}$$

and

$$\begin{cases} \dot{X}^{(i)}(t) = \left[\bar{A}(t) - S_1(t)P(t)\right] X^{(i)}(t) - S_1(t)g(t) + A_1(t)X^{(i-1)}(t-\tau_x) + F_i(t), \\ X^{(i)}(t) = [0, 0, \cdots, 0]^T, \end{cases} \tag{4.15}$$

where $F_i(t)$ is defined in relation (4.9). Solving problems (4.12) and (4.15), we can obtain the i-th-order terms $X^{(i)}(t)$ and $g_i(t)$. Substituting $X^{(i)}(t)$ and $g_i(t)$ into (4.6) and letting $\varepsilon = 1$, we can get $u(t) = u(t, 1)$,

$$u(t) = \begin{cases} -R^{-1}(t)\bar{B}^T(t)\sum\limits_{i=0}^{\infty}\frac{1}{i!}\left[P(t)X(t) + g_i(t)\right] \\ -R^{-1}(t)\bar{B}_1^T(t+\tau_u)\sum\limits_{i=0}^{\infty}\frac{1}{i!}\left[P(t+\tau_u)X(t+\tau_u) + g_i(t+\tau_u)\right], t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)\bar{B}^T(t)\sum\limits_{i=0}^{\infty}\frac{1}{i!}\left[P(t)X(t) + g_i(t+\tau_u)\right], \qquad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{4.16}$$

Summarizing the above, we obtain the following theorem.

Theorem 4.1. *Consider the problem of minimizing the cost functional* (3.2) *subject to system* (3.1)*; the control law*

$$u^*(t) = \begin{cases} -R^{-1}(t)\bar{B}^T(t)\left[P(t)X(t) + \sum\limits_{i=1}^{\infty}\frac{g_i(t)}{i!}\right] \\ -R^{-1}(t)\bar{B}_1^T(t+\tau_u)\left[P(t+\tau_u)X(t+\tau_u) + \sum\limits_{i=1}^{\infty}\frac{g_i(t+\tau_u)}{i!}\right], t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)\bar{B}^T(t)\left[P(t)X(t) + \sum\limits_{i=1}^{\infty}\frac{g_i(t+\tau_u)}{i!}\right], \qquad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{4.17}$$

is optimal, where $P(t)$ and $g_i(t)$ are solved by (4.11) *and* (4.12)*, respectively.*

It should be noted that state feedback term in (4.17) is the exact solution. In the following, we present an iterative procedure for finding the suboptimal control law.

Remark 4.2. *In practice, calculating infinite terms of series in* (4.17) *is almost impossible, intercepting M terms of the series, we obtain a suboptimal solution*

$$u_M(t) = \begin{cases} -R^{-1}(t)\bar{B}^T(t)\left[P(t)\sum\limits_{i=0}^{M}\frac{X^{(i)}(t)}{i!} + \sum\limits_{i=1}^{M}\frac{g_i(t)}{i!}\right] \\ -R^{-1}(t)\bar{B}_1^T(t+\tau_u)\left[P(t+\tau_u)\sum\limits_{i=0}^{M}\frac{X^{(i)}(t+\tau_u)}{i!} + \sum\limits_{i=1}^{M}\frac{g_i(t+\tau_u)}{i!}\right], t_0 \leqslant t < t_f - \tau_u, \\ -R^{-1}(t)\bar{B}^T(t)\left[P(t)\sum\limits_{i=0}^{M}\frac{X^{(i)}(t)}{i!} + \sum\limits_{i=1}^{M}\frac{g_i(t+\tau_u)}{i!}\right], \qquad t_f - \tau_u \leqslant t \leqslant t_f. \end{cases} \tag{4.18}$$

The integer Mth in (4.18) *is generally determined according to a concrete control precision. Then, from* (3.7) *the following cost functional can be calculated:*

$$\bar{J}_M = \frac{1}{2}X^T(t_f)\bar{H}(t_f)X(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(X^T(t)Q(t)X(t) + u_M^T(t)R(t)u_M(t)\right)dt, \tag{4.19}$$

The Mth order in (4.19) has the desirable accuracy, if for given positive constants $\epsilon > 0$, the following condition hold jointly:

$$\left|\frac{J_M - J_{M-1}}{J_M}\right| < \epsilon, \tag{4.20}$$

If the tolerance error bound be chosen small enough, the Mth order suboptimal control law will be very close to $u^*(t)$, and thus, the value of cost functional in (4.19) and its optimal value J^* will be almost identical.

Algorithm: Suboptimal control law of system (3.8):

Step 1: Obtain $P(t)$ from (4.11).
Step 1: Obtain $X^{(0)}(t)$ from (4.14). And set $i = 1$.

Step 2: Compute $X^{(i)}(t)$ and $g_i(t)$ from (4.15) and (4.12).

Step 3: Let $M = i$ and obtain $u_M(t)$ from (4.18).

Step 4: Calculate J_M according to (4.19). If $\left|\frac{J_M - J_{M-1}}{J_M}\right| < \epsilon$, then stop and output $u_M(t)$; else, replace k by $i + 1$ and go to step 2.

5. Simulation results

In this section, the proposed method is illustrated by some test problems. The calculations are performed using the Matlab software.

Example 5.1. *Consider the following linear time-varying multi-delay systems [22]:*

$$\begin{cases} {}_0^C D_t^\alpha x(t) = x(t - \frac{1}{2}) + tx(t - \frac{3}{4}) + u(t), & 0 \leqslant t \leqslant 1, \\ x(t) = t + 1, \quad -\frac{3}{4} \leqslant t \leqslant 0, \end{cases} \tag{5.1}$$

with the cost functional

$$J = \frac{3}{2}x^2(1) + \frac{1}{2}\int_0^1 u^2(t)dt. \tag{5.2}$$

When $\alpha = 1$, the exact solutions of $x(t)$ and $u(t)$ are given by

$$x(t) = \begin{cases} \frac{239075}{420332}t^3 + \frac{3129081}{1681328}t^2 - \frac{1178769}{611392}t + 1, & 0 \leqslant t < \frac{1}{4}, \\ \frac{1}{3}t^3 + \frac{1119201}{840664}t^2 - \frac{680513}{420332}t + \frac{7039811}{7336704}, & \frac{1}{4} \leqslant t < \frac{1}{2}, \\ \frac{239075}{1681328}t^4 + \frac{3375959}{5043984}t^3 - \frac{20932555}{13450624}t^2 + \frac{1156163}{1222784}t + \frac{56216927}{161407488}, & \frac{1}{2} \leqslant t < \frac{3}{4}, \\ \frac{47815}{420332}t^5 + \frac{2306773}{10087968}t^4 - \frac{2461871}{2521992}t^3 + \frac{14865377}{53802496}t^2 + \frac{17419475}{26901248}t & \\ + \frac{2652913}{14673408}, & \frac{3}{4} \leqslant t \leqslant 1, \end{cases}$$

and

$$u(t) = \begin{cases} \frac{296893}{420332}t^2 + \frac{2078251}{840664}t - \frac{1484465}{611392}, & 0 \leqslant t < \frac{1}{4}, \\ \frac{296893}{210166}t - \frac{890679}{420332}, & \frac{1}{4} \leqslant t < \frac{1}{2}, \\ -\frac{296893}{210166}, & \frac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Furthermore, when $\alpha = 1$, exact value of cost functional is available and equal to $J = 1.70648554$. We solve the problem by the proposed method with $N = 2$ and $M = 10$. The values of the cost functional J and central processing unit (CPU) times for $\alpha = 1, 0.9, 0.8$ are reported in Table 1. The results compare well with the exact solution. In Figures 1 and 2, the obtained numerical solutions (for $N = 2$ and $M = 10$) for both state and control functions are also depicted. The exact and the approximate solutions for control function with $\alpha = 1$ are plotted in Figure 3. This confirms that the proposed method yields excellent results.

Table 1: Approximate values of J in Example 5.1.

method	Value of α	Value of J	CPU time
The present method	$\alpha = 1$	1.70648351	8.951
(N=2 and M=10)	$\alpha = 0.9$	1.93215424	9.322
	$\alpha = 0.8$	1.43254178	9.891
Exact solution	$\alpha = 1$	1.70648554	-

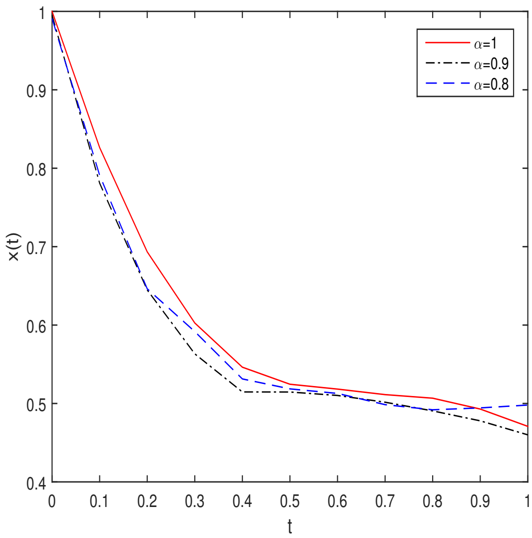

Figure 1: Approximate solutions of $x(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 10$ in Example 5.1.

Example 5.2. *Consider the following linear time-varying multi-delay systems [7, 31, 15, 11, 28]:*

$$\begin{cases} {}_0^C D_t^\alpha x(t) = -x(t) + x(t - \frac{1}{3}) + u(t) - \frac{1}{2}u(t - \frac{2}{3}), & 0 \leqslant t \leqslant 1, \\ x(t) = 1, & -\frac{1}{3} \leqslant t \leqslant 0, \\ u(t) = 0, & -\frac{2}{3} \leqslant t \leqslant 0, \end{cases} \tag{5.3}$$

with the cost functional

$$J = \frac{1}{2}\int_0^1 \left(x^2(t) + \frac{1}{2}u^2(t) \right)(t)dt. \tag{5.4}$$

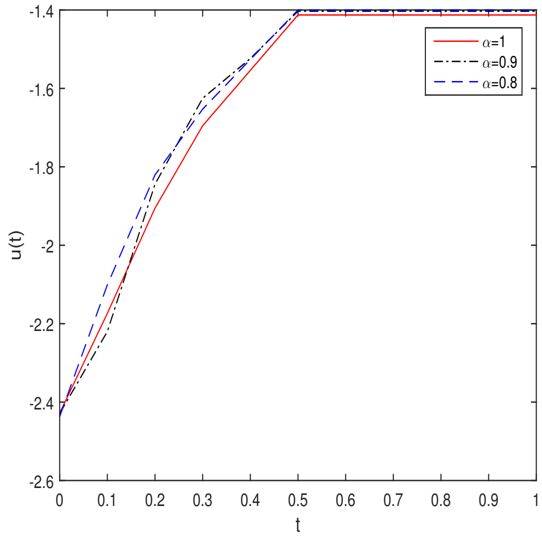

Figure 2: Approximate solutions of $u(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 10$ in Example 5.1.

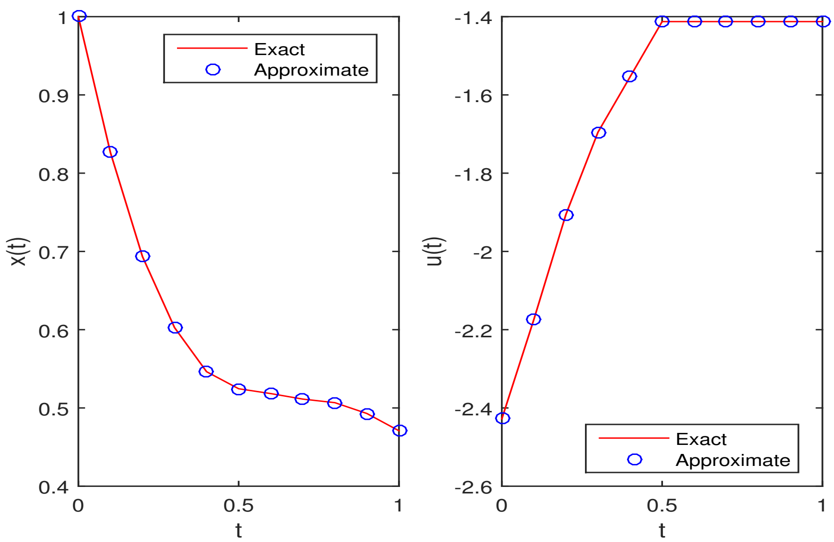

Figure 3: Approximate and exact solutions of $x(t)$ and $u(t)$ for $\alpha = 1$ in for Example 5.1.

According to system (3.1)*, we have* $A = -1, A_1 = B = 1, B_1 = -\frac{1}{2}, Q = 1, R = \frac{1}{2}$.
We solve the problem by the proposed method with $N = 2$ *and* $M = 17$*. The obtained cost functional values and central processing unit (CPU) times, for different values of* $\alpha = 1, 0.9, 0.8$*, are shown in Table 2. In Table2, we compare the value* J *obtained using the proposed method with the value of* J *reported in [7] by formulation of Euler-Lagrange equations, in [31] by using numerical approach based on the Chelyshkov wavelets, in [15] by Muntz-Legendre neural network construction and in [11] by using a Muntz-Legendre spectral colocations method. Also, For different values of* α*, the optimal state and control functions* $x(t)$ *and* $u(t)$ *are also depicted in Figures 4 and 5, respectively. From these figures, it is clear that when* α *approaches to 1, the numerical solutions for both the state and the control variables approach the analytical solutions for* $\alpha = 1$.

Table 2: Approximate values of J in Example 5.2.

method	Value of α	Value of J	CPU time
The present method	$\alpha = 1$	0.37052341	17.355
(N=2 and M=17)	$\alpha = 0.9$	0.30125489	18.896
	$\alpha = 0.8$	0.21125367	18.211
Effati et al. [7]	$\alpha = 1$	0.37314692	-
Moradi et al. [31]	$\alpha = 1$	0.37311264	-
Kheyrinataj and Nazemi [15]	$\alpha = 1$	0.3656	-
Hosseinpour et al. [11]	$\alpha = 1$	0.3677	-

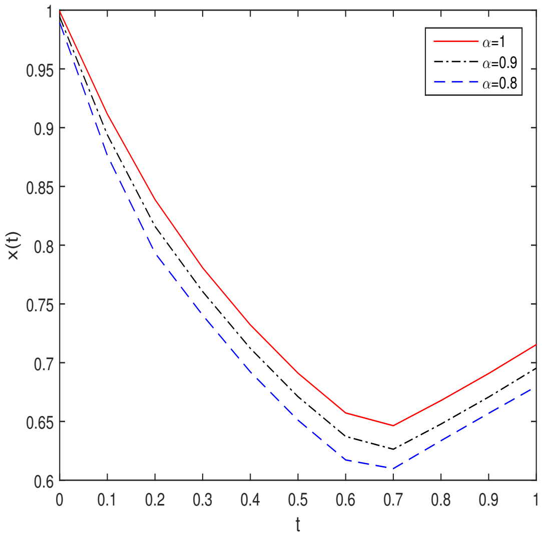

Figure 4: Approximate solutions of $x(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 17$ in Example 5.2.

Example 5.3. *We now consider the following nonlinear time-varying multi-delay systems:*

$$\begin{cases} {}_0^C D_t^\alpha x(t) = x(t-1)x(t-2)u(t-2), \qquad 0 \leqslant t \leqslant 3, \\ x(t) = 1, \qquad -2 \leqslant t \leqslant 0, \\ u(t) = 0, \qquad -2 \leqslant t \leqslant 0, \end{cases} \tag{5.5}$$

with the cost functional

$$J = \int_0^3 (x^2(t) + u^2(t))dt. \tag{5.6}$$

This optimal control is adopted from [40, 41, 9]. In order to obtain an accurate enough suboptimal control law, we applied the proposed algorithm with $N = 2$ and $M = 15$. The obtained cost functional values and central processing unit (CPU) times, for different values of $\alpha = 1, 0.9, 0.8$, are shown in Table 3. In Table 3, we compare the value J obtained using the proposed method with the value of J reported in [40] by using an interior-point algorithm, in [41] by an interior-point filter line-search algorithm and in [9] by pontryagin's maximum principle. Also, For different values of α, the optimal

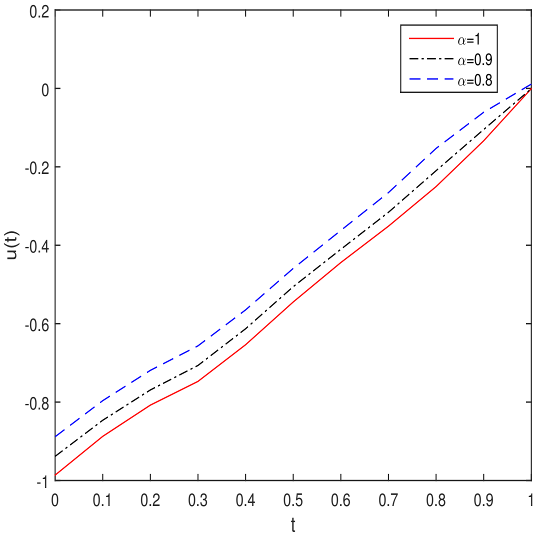

Figure 5: Approximate solutions of $u(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 17$ in Example 5.2.

state and control functions $x(t)$ and $u(t)$ are also depicted in Figures 6 and 7, respectively. Therefore, in view of the results, the present method is quite effective.

Table 3: Approximate values of J in Example 5.3.

method	Value of α	Value of J	CPU time
The present method	$\alpha = 1$	2.761574	15.452
(N=2 and M=15)	$\alpha = 0.9$	3.125476	16.562
	$\alpha = 0.8$	2.305341	17.893
Vanderbei et al. [40]	$\alpha = 1$	2.763044	-
Wachter et al. [41]	$\alpha = 1$	2.763044	-
Gollmann et al. [9]	$\alpha = 1$	2.761591012	-

6. Conclusions

In this paper, we developed an approximate solution method to solve fractional order optimal control problems with delay argument, where the dynamic control system depends on Caputo fractional derivatives. It is important to notice, that, in the proposed algorithm, only a few iteration steps are required to get the suboptimal control law. Numerical methods to solve the problems are presented, and some computational simulations are discussed detail. The study show that the method is effective techniques to solve time-delayed optimal control problems, and the method is easy to implement and computationally very attractive without sacrificing the accuracy of the solution.

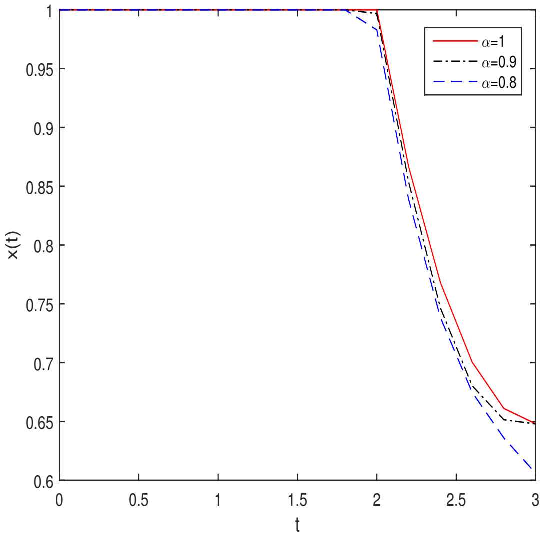

Figure 6: Approximate solutions of $x(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 15$ in Example 5.3.

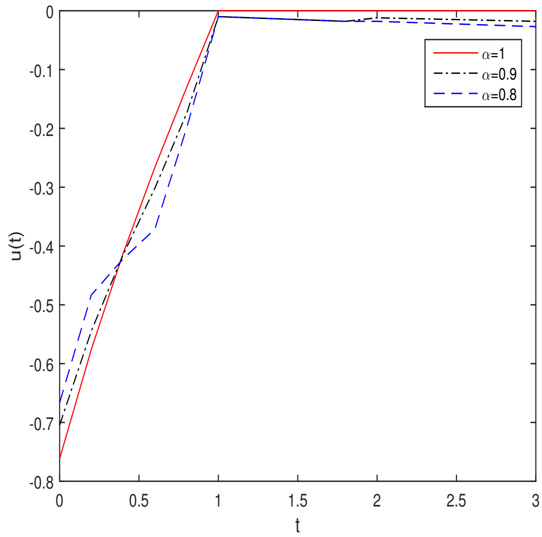

Figure 7: Approximate solutions of $u(t)$ for $\alpha = 1, 0.9, 0.8$ and $M = 15$ in Example 5.3.

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