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The bifurcation analysis of an epidemiological model involving two diseases in predator

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Abstract

In this paper, the local bifurcation conditions that occurrence near each of the equilibrium points of the eco-epidemiological system of one prey population apparition with two diseases in the same population of predator have been studied and analyzed, near E_1, E_2, E_3, E_4 and E_5, a transcritical bifurcation can occurred, a saddle-node bifurcation happened near E_5. Pitchfork bifurcation was occurrences at E_2, E_3, E_4 and E_5. Moreover conditions for Hopf- bifurcation was studied near both of one disease stable point E_3, E_4 and E_5 . About elucidation the status of local bifurcation the associated of the set of hypothetical of parameters with numerical results which assert our analytical results of this model.

Keywords: Eco-epidemiological model, Local bifurcation, Hopf-bifurcation, SIS disease, SI disease, Sotomayor's theorem.

1. Introduction

There is no doubt that the development of work in biological mathematics and the great merger that followed in tracing the life forms of life for various competing neighborhoods within the animal community has a great impact on understanding many facts that biologists have benefited from by modeling those working in mathematics into real models of competing neighborhoods. Many researchers have worked on a variety of environmental models with different influences and did not neglect the research on the dynamic characteristics of those models, for example, [18, 19, 1, 8, 13, 10, 14, 15, 11, 12, 23, 16]. The concept of an ecosystem is studied by representing it with a mathematical

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model using data related to that system from the environment in which the research group lives. The purpose of the above concept is to develop a reasonable conception of the dynamics of the real system that the environment represents. While the concept of epidemics has an urgent need because of the interest in studying the ways and factors of the spread of epidemics understudy that include all kinds of organisms, and thus give a logical perception about predicting what those epidemics are and the extent of their long-term impact through modeling those epidemiological models.

Systems involving epidemics have become an interesting topic, especially when they overlap with models of prey and predators. Among those interested in the ecological and epidemiological issue, which links the effect of diseases on the dynamic behavior of animals in the animal life system, was Anderson and May [2] in his work that combined the model of Lutka- Volterra, Carmack and Mackendrick, that is, they combined ecological and epidemiological models. Therefore, a new trend appeared in this study, which was later called ecological and epidemiological processes [4]. There are a lot of research studies on this same content of work as such [3, 5, 6, 21]. When studying, in general, systems of ordinary differential equations, we notice that they contain random variables in addition to a set of parameters that will have an effective effect on the nature of those equations and their solutions, as any set of parameters will have an effect on the behavior of those solutions to these systems that differs from the other group. When a small smooth made to the parameter values, the bifurcation occurred causes a sudden qualitative or topological change in its behavior. This is mean at the equilibrium, periodic orbits or other invariant sets the local, stability properties will be changed. The bifurcation theorem worthy of attention impression which appears on the dynamic systems such as local bifurcation and Hopf- bifurcation of proposed eco-epidemiological systems that be composed of numerous effects, each as stated by the model, see [17, 24, 20]. Right now, the destination is to discuss the occurrence of local bifurcation of the Eco-epidemiological model proposed by Kadhim and Majeed [9].

2. Model Formulation [9]

The following model:

$$
\begin{aligned}
\frac{dP}{dT} &= rP\left(1-\frac{P}{K_P}\right) - aPS - \frac{C_1PH}{b_1+P} - \frac{C_2PV}{b_2+P},\\
\frac{dS}{dT} &= \theta_1 aPS + \gamma H - \theta SH - \beta SV - \alpha S - d_1 S,\\
\frac{dH}{dT} &= \theta SH - \gamma H - d_2 H + \frac{e_2C_1PH}{b_1+P},\\
\frac{dV}{dT} &= \beta SV - \alpha S - d_3 V + \frac{e_3C_1PV}{b_2+P}.
\end{aligned}
\tag{2.1}
$$

Now, Table 1 explain the variables and parameters of the model.

Table 1: The parameters and positive variables by appearing in the mathematical model

P	***The prey population size at time T***
S	***The susceptible predator population size at time T***
H	***The first infected SIS predator population size at the time T***
V	***The second infected SI predator population size at the time T***
$r > 0$	***The growth rate of prey***
$K_p > 0$	***The carrying capacity***
$a > 0$	***predation rate of the susceptible predator on the prey***
$d_i, i = 1, 2, 3$	***The death rates of the susceptible, the first infected SIS and the second infected SI predator respectively***
$0 < e_i < 1, i = 1, 2, 3.$	***The rates of conversion of food to the susceptible, first infected SIS and the second infected SI predator respectively***
$\theta > 0$	***The infected rate of the first SIS disease in predator population***
$\gamma > 0$	***The recovery rate of the first SIS disease in predator population***
$c_i > 0, i = 1, 2$	***Maximum attack rate of the first infected SIS disease and the second infected SI disease in predator population respectively.***
$b_i > 0, i = 1, 2$	***half saturation rate of the first infected SIS disease and the second infected SI disease in predator population respectively.***
$\beta > 0$	***The infected rate of the second disease SI in predator population.***
$\alpha > 0$	***The external source rate of the second disease in predator population.***

System (2.1) is dimensionalized in the following system which is given in [9]:

$$\begin{aligned} \frac{dP}{dt} &= P(1-p) - u_1PS - \frac{u_2ph}{u_4+P} - \frac{u_3pv}{u_5+P} = f_1(p,s,h,v) \\ \frac{ds}{dt} &= u_6PS + u_7h - u_8sh - u_9sv - (u_{10}+u_{11})s = f_2(p,s,h,v) \\ \frac{dh}{dt} &= -u_7h + u_8sh - u_{12}h + \frac{u_{13}ph}{u_4+P} = f_3(p,s,h,v) \\ \frac{dv}{dt} &= u_{10}s + u_9sv - u_{14}v + \frac{u_{15}pv}{u_5+P} = f_4(p,s,h,v) \end{aligned} \tag{2.2}$$

where

$$u_1 = \frac{ak_p}{r}, \quad u_2 = \frac{c_1}{r}, \quad u_3 = \frac{c_2}{r}, \quad u_4 = \frac{b_1}{k_p}, \quad u_5 = \frac{b_2}{k_p}, \quad u_6 = \frac{e_1ak_p}{r}, \quad u_7 = \frac{\gamma}{r}, \quad u_8 = \frac{\theta k_p}{r}$$

$$u_9 = \frac{\beta k_p}{r}, \quad u_{10} = \frac{a}{r}, \quad u_{11} = \frac{d_1}{r}, \quad u_{12} = \frac{d_2}{r}, \quad u_{13} = \frac{e_2c_1}{r}, \quad u_{14} = \frac{d_3}{r}, \quad u_{15} = \frac{e_3c_2}{r}$$

3. The Local bifurcation analysis

In this section, the using of Sotomayor's Theorem [22] to debate local bifurcation of the system (2.2), since the necessary but not sufficient condition for the bifurcation to happen is the nonhyperbolic property of the equilibrium point.

The Jacobian matrix which is given in [22]:

$$J = [a_{\mathrm{ij}}]_{4\times4} = \begin{bmatrix} 1-2p-u_1s-\frac{u_2u_4h}{(u_4+p)^2}-\frac{u_3u_5v}{(u_5+p)^2} & -u_1p & \frac{-u_2p}{u_4+p} & \frac{-u_3p}{u_5+p} \\ u_6s & u_6p-u_8h-u_9v-(u_{10}+u_{11}) & u_7-u_8s & -u_9s \\ \frac{u_{13}u_4h}{(u_4+p)^2} & u_8h & u_8s-(u_7+u_{12})+\frac{u_{13}p}{u_4+p} & 0 \\ \frac{u_5u_{15}v}{(u_5+p)^2}, & u_9v+u_{10} & 0 & u_9s-u_{14}+\frac{u_{15}p}{u_5+p} \end{bmatrix} \tag{3.1}$$

Checking that for any non-zero vector $\Omega = (\omega_1, \omega_2, \omega_3,\ \omega_4)^T$ we have:

$$D^2 f_\mu (\aleph, \mu) (\Omega, \Omega) = [\alpha_{i1}]_{4\times 1}\,, \tag{3.2}$$

where, $\alpha_{11} = -2\omega_1 \left[\left(1 - \frac{u_2 u_4 h}{(u_4+p)^3} - \frac{u_3 u_5 v}{(u_5+p)^3}\right)\omega_1 + u_1\omega_2 + \frac{u_2 u_4}{(u_4+P)^2}\omega_3 + \frac{u_3 u_5}{(u_5+P)^2}\omega_4\right]$,

$\alpha_{21} = -2\omega_2 \left[-u_6\omega_1 + u_8\omega_3 + u_9\omega_4\right]$, $\alpha_{31} = -2\left[\frac{u_4 u_{13} h}{(u_4+p)^3}\omega_1^2 - \frac{u_4 u_{13}}{(u_4+p)^2}\omega_1\omega_3 - u_8\omega_2\omega_3\right]$,

$\alpha_{41} = -2\left[\frac{u_5 u_{15} v}{(u_4+p)^3}\omega_1^2 - \frac{u_5 u_{15}}{(u_4+p)^2}\omega_1\omega_4 - u_9\omega_2\omega_4\right]$.

$$D^3 f_\mu (\aleph, \mu) (\Omega, \Omega, \Omega) = [\beta_{i1}]_{4\times 1}\,, \tag{3.3}$$

where, $\beta_{11} = -6\omega_1^2 \left[\left(\frac{u_2 u_4 h}{(u_4+p)^4} + \frac{u_3 u_5 v}{(u_5+p)^4}\right)\omega_1 + \frac{u_2 u_4}{(u_4+P)^3}\omega_3 + \frac{u_3 u_5}{(u_5+P)^3}\omega_4\right]$, $\beta_{21} = 0$,

$\beta_{31} = -6\omega_1^2\left[\frac{-u_4 u_{13} h}{(u_4+p)^4}\omega_1 - \frac{u_4 u_{13}}{(u_4+p)^3}\omega_3\right]$, $\beta_{41} = -6\omega_1^2\left[\frac{-u_5 u_{15}}{(u_4+p)^4}\omega_1 - \frac{u_5 u_{15}}{(u_4+p)^4}\omega_4\right]$.

Theorem 3.1. *Assume that the stability conditions (3.2) and (3.3) as in [9] hold. Then system (2.2) near the equilibrium point E_1 has a transcritical bifurcation at the parameter $(u_{11}^* = u_6 - u_{10})$, under the authority of conditions:*

$$u_6 > u_{10} \tag{3.4}$$

$$Z_1 \neq Z_2 \tag{3.5}$$

where,

$$Z_1 = \frac{u_3 u_6}{u_5+1}, \qquad Z_2 = -\left(\frac{\frac{u_{15}}{u_5+1} - u_{14}}{u_{10}}\right)(u_6 + u_9).$$

Otherwise, neither saddle-node nor pitchfork bifurcation could be happened atE_1.

Proof . *using $E_1 = (1, 0, 0, 0)$ and $(u_{11}^* = u_{11})$ in the Jacobian matrix given in Eq. (3.1), thus zero eigenvalue ($\lambda_{1s} = 0$) can be appearing in the characteristic equation of J_1 .*
Contingent on condition (3.4), $u_{11}^* > 0$
Let, $\Omega^{[1]} = \left(\omega_1^{[1]}, \omega_2^{[1]}, \omega_3^{[1]}, \omega_4^{[1]}\right)^T$ eigenvector of J_1^* affiliated to the eigenvalue $\lambda_{1s} = 0$
Thus: $(J_1^* - \lambda_{1s} I)\,\Omega^{[1]} = 0$, where: $J_1^* = J(E_1, u_{11}^*)$ $\omega_1^{[1]} = -\left(\frac{u_{14} - \frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\omega_4^{[1]}$,
$\omega_2^{[1]} = \left(\frac{u_{14} - \frac{u_{15}}{u_5+1}}{u_{10}}\right)\omega_4^{[1]}$ and $\omega_4^{[1]}$ any non zero real number.
Let, $\mathcal{H}^{[1]} = \left(\hbar_1^{[1]}, \hbar_2^{[1]}, \hbar_3^{[1]}, \hbar_4^{[1]}\right)^T$ be the eigenvector of J_1^{*T} affiliated to $\lambda_{1s} = 0$, of the matrix J_1^{*T}.
Then: $\left(J_1^{*T} - \lambda_{1s} I\right)\mathcal{H}^{[1]} = 0$, Give us $\mathcal{H}^{[1]} = \left[0, \hbar_2^{[1]}, \left(\frac{u_7}{u_7 + u_{12} - \frac{u_{13}}{u_4+1}}\right)\hbar_2^{[1]}, 0\right]^T$ and $\hbar_2^{[1]}$ any nonzero real number.
Since, $\frac{\partial f}{\partial u_{11}} = f_{u_{11}}\ (\aleph, u_{11}) = \left(\frac{\partial f_1}{\partial u_{11}}, \frac{\partial f_2}{\partial u_{11}}, \frac{\partial f_3}{\partial u_{11}}, \frac{\partial f_4}{\partial u_{11}}\right) = (0, -s, 0, 0)^T$,
hence $f_{u_3}\ (E_1, u_{11}^*) = (0, 0, 0, 0)^T$. Therfore $\mathcal{H}^{[1]}\ f_{u_{11}}\ (E_1, u_{11}^*) = 0$.
The detecting of Sotomayor's Theorem [22] appears the happening of the saddle-node bifurcation cannot be occurring at E_1 . Moreover,

$$since, Df_{u_{11}}(\aleph, u_{11}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Accompanied by the vector $\aleph = (p, s, h, v)^T$, $Df_{u_{11}}(\aleph, u_{11})$ *exemplify derivative of* $f_{u_{11}}(\aleph, u_{11})$. *Therefore, we have:*

$$Df_{u_{11}}(E_1, u_{11}^*)\,\Omega^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\omega_4^{[1]} \\ \left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\ \omega_4^{[1]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\ \omega_4^{[1]} \\ 0 \\ 0 \end{bmatrix},$$

so, by condition (3.3) as in [9] we obtain that:

$\left(\mathcal{H}^{[1]}\right)^T\ \left[Df_{u_{11}}(E_1, u_{11}^*)\,\Omega^{[1]}\right] = -\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\omega_4^{[1]}\hbar_2^{[1]} \neq 0,$

Plugging $\Omega^{[1]}$ *in equation* (3.2), *we get:*

$$D^2 f_{u_{11}}(E_1, u_{11}^*)\,(\Omega^{[1]}, \Omega^{[1]}) = \begin{bmatrix} 2\left(\omega_4^{[1]}\right)^2\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\left[-\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right) - u_1\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right) + \frac{u_3 u_5}{(u_5+1)^2}\right] \\ -2\left(\omega_4^{[1]}\right)^2\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\left[u_6\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right) + u_9\right] \\ 0 \\ -2\left(\omega_4^{[1]}\right)^2\left[\ u_9\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right) + \frac{(u_5 u_{15})}{(u_5+1)^2}\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\right] \end{bmatrix}.$$

So, contingent on (3.2) and (3.3) as in [9] with condition (3.5), $(Z_2 > 0)$

Thus, $\left(\mathcal{H}^{[1]}\right)^T D^2 f_{u_{11}}(E_1, u_{11}^*)\left(\Omega^{[1]}, \Omega^{[1]}\right) = -2\left(\omega_4^{[1]}\right)^2 \hbar_2^{[1]}\left[\frac{u_7}{u_7+u_{12}-\frac{u_{13}}{u_4+1}}\right][Z_1 - Z_2] \neq 0.$

The detecting of Sotomayor's Theorem [22] appears the happening of a trancecritical bifurcation at system (2.2) *near* E_1.

Opposite of condition (3.3) as in [9] $(Z_2 < 0 \to Z_1 - Z_2 > 0)$

Either, condition (3.5) *holds. Then By Sotomayor's Theorem [22], system* (2.2) *near* E_1 *possesses a trancecritical bifurcation.*

Or, $\left(\mathcal{H}^{[1]}\right)^T D^2 f_{u_{11}}(E_1, u_{11}^*)\left(\Omega^{[1]}, \Omega^{[1]}\right) = 0,$

Plugging $\Omega^{[1]}$ *in equation* (3.3), *we get:*

$$D^3 f_{u_{11}}(E_1, u_{11}^*)\left(\Omega^{[1]}, \Omega^{[1]}, \Omega^{[1]}\ \right) = \begin{bmatrix} -6\left[\frac{u_3\ u_5}{(u_5+1)^3}\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\omega_4^3\right] \\ 0 \\ 0 \\ -6\left[\frac{-u_5\ u_{15}}{(u_5+1)^4}\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}} + \frac{u_3}{u_5+1}\right)\left(\frac{u_{14}-\frac{u_{15}}{u_5+1}}{u_{10}}\right)\omega_4^3\right] \end{bmatrix},$$

Thus, $\left(\mathcal{H}^{[1]}\right)^T D^3 f_{u_{11}}(E_1, u_{11}^*)\left(\Omega^{[1]}, \Omega^{[1]}, \Omega^{[1]}\ \right) = 0,$

The detecting of Sotomayor's Theorem [22] appears the happening of pitchfork bifurcation at system (2.2) *near* E_1 *cannot occur.* □

Theorem 3.2. *Assume that the stability condition (3.6a) as in [9] hold. Then system* (2.2) *near* $E_2 = (\widehat{p}, \widehat{s}, 0, 0)$ *possesses a transcritical bifurcation at the parameter value* $\left(\widehat{u}_{12} = u_8\widehat{s} + \frac{u_{13}\widehat{p}}{u_4+\widehat{p}} - u_7\right)$, *according to the following condition:*

$$u_7 < u_8\widehat{s} + \frac{u_{13}\widehat{p}}{u_4+\widehat{p}} \tag{3.6}$$

$$\frac{u_7}{u_8} < \widehat{s} \tag{3.7}$$

$$(-u_1\widehat{p})\left(u_9\widehat{s} - u_{14} + \frac{u_{15}\widehat{p}}{u_5+\widehat{p}}\right) > u_{10}\left(\frac{u_3\widehat{p}}{u_5+\widehat{p}}\right) \tag{3.8}$$

$$\widehat{Z}_1 \neq \widehat{Z}_2 \tag{3.9}$$

Otherwise, no saddle-node happens, but pitchfork bifurcation can occur at E_2.
where,

$$\widehat{Z}_1 = \frac{u_2u_4}{(u_4+\widehat{p})^2}\left[\frac{-\widehat{b}_{13}\left(\widehat{b}_{24}\widehat{b}_{42}-\widehat{b}_{22}\widehat{b}_{44}\right)-\widehat{b}_{14}\widehat{b}_{23}\widehat{b}_{42}}{\widehat{b}_{44}\left(\widehat{b}_{11}\widehat{b}_{23}-\widehat{b}_{13}\widehat{b}_{21}\right)}\right]+u_8,$$

$$\widehat{Z}_2 = \frac{-u_2u_4}{(u_4+\widehat{p})^2}\left[\frac{\widehat{b}_{12}\widehat{b}_{23}}{\left(\widehat{b}_{11}\widehat{b}_{23}-\widehat{b}_{13}\widehat{b}_{21}\right)}\right].$$

Proof . *By substituting $E_2 = (\widehat{p}, \widehat{s}, 0, 0)$ with $(\widehat{u}_{12} = u_{12})$ in the Jacobian matrix given in Eq.* (3.1)*, the characteristic equation of $\widehat{J}_2$, where $\widehat{J}_2 = J_2(E_2, \widehat{u}_{12})$ has zero eigenvalue ($\lambda_{2h} = 0$).*
Provided condition (3.6)*, $\widehat{u}_{12} > 0$*
Let, $\Omega^{[2]} = \left(\omega_1^{[2]}, \omega_2^{[2]}, \omega_3^{[2]}, \omega_4^{[2]}\right)^T$ be the eigenvector of $\widehat{J}_2$ affiliated to the eigenvalue$\lambda_{2h} = 0$ thus, $\left(\widehat{J}_2 - \lambda_{2h}I\right)\Omega^{[2]}$=0, which gives: $\omega_1^{[2]} = \epsilon_1\omega_2^{[2]}$, $\omega_3^{[2]} = \epsilon_2\omega_2^{[2]}, \omega_4^{[2]} = \epsilon_3\omega_2^{[2]}$ and $\omega_2^{[2]}$ any nonzero real number
where,

$$\epsilon_1 = \frac{-\widehat{b}_{23}\left(\widehat{b}_{12}\widehat{b}_{44}+\widehat{b}_{14}\widehat{b}_{42}\right)-\widehat{b}_{13}\left(\widehat{b}_{24}\widehat{b}_{42}-\widehat{b}_{22}\widehat{b}_{44}\right)}{\widehat{b}_{44}\left(\widehat{b}_{11}\widehat{b}_{23}-\widehat{b}_{13}\widehat{b}_{21}\right)}, \ \epsilon_2 = \frac{\widehat{b}_{11}\left(\widehat{b}_{24}\widehat{b}_{42}-\widehat{b}_{22}\widehat{b}_{44}\right)+\widehat{b}_{21}\left(\widehat{b}_{12}\widehat{b}_{44}+\widehat{b}_{14}\widehat{b}_{42}\right)}{\widehat{b}_{23}\widehat{b}_{44}} > 0$$

and $\epsilon_3 = \frac{-\widehat{b}_{42}}{\widehat{b}_{44}}$,
Let, $\mathcal{H}^{[2]} = \left(\hbar_1^{[2]}, \hbar_2^{[2]}, \hbar_3^{[2]}, \hbar_4^{[2]}\right)^T$ be the eigenvector of $\widehat{J}_2^{\,T}$ affiliated to $\lambda_{2h} = 0$, of the matrix $\widehat{J}_2^T$ then: $\left(\widehat{J}_2^T - \lambda_{2h}I\right)\mathcal{H}^{[2]} = 0$, Give us $\mathcal{H}^{[2]} = \left(0, 0, \hbar_3^{[2]}, 0\right)^T$ where $\hbar_3^{[2]}$ any nonzero real number.
Since, $\frac{\partial f}{\partial u_{12}} = f_{u_{12}} \ (\aleph, u_{12}) = \left(\frac{\partial f_1}{\partial u_{12}}, \frac{\partial f_2}{\partial u_{12}}, \frac{\partial f_3}{\partial u_{12}}, \frac{\partial f_4}{\partial u_{12}}\right) = (0, 0, -h, 0)^T$,
hence $f_{u_{12}} \ (E_2, \widehat{u}_{12}) = (0, 0, 0, 0)^T$, $therefore$, $\left(\mathcal{H}^{[2]}\right)^T f_{u_{12}} \ (E_2, \widehat{u}_{12}) = 0$.
Then by Sotomayor's Theorem [22], the saddle-node bifurcation cannot take place at E_2 . Moreover,

$$since, Df_{u_{12}}(\aleph, u_{12}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with the vector $\aleph = (p, s, h, v)^T$, $Df_{u_{12}}(\aleph, u_{12})$ exemplify derivative of $f_{u_{12}}(\aleph, u_{12})$.
And then we have:

$$Df_{u_{12}}(E_2, \widehat{u}_{12})\,\Omega^{[2]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1\omega_2^{[2]} \\ \omega_2^{[2]} \\ \varepsilon_2\omega_2^{[2]} \\ \varepsilon_3\omega_2^{[2]}0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -\varepsilon_2\omega_2^{[2]} \\ 0 \end{bmatrix}.$$

Under the authority of conditions, (3.6a) as in [9], with conditions (3.7) *and* (3.8), *we have*
$\left(\mathcal{H}^{[2]}\right)^T \left[Df_{u_{12}}\left(E_2,\widehat{u}_{12}\right)\Omega^{[2]}\right] = -\varepsilon_2\omega_2^{[2]}\hbar_3{}^{[2]} \neq 0, (\varepsilon_2 > 0)$
Plugging $\Omega^{[2]}$ *in Eq.* (3.2), *we get:*

$$D^2f_{u_{12}}\left(E_2,\widehat{u}_{12}\right)(\Omega^{[2]},\Omega^{[2]}) = \begin{bmatrix} -2\left(\omega_2^{[1]}\right)^2\epsilon_1\left[\epsilon_1+u_1+\frac{u_2u_4}{\left(u_4+\widehat{P}\right)^2}\epsilon_2+\frac{u_3u_5}{\left(u_5+\widehat{P}\right)^2}\epsilon_3\right] \\ -2\left(\omega_2^{[1]}\right)^2\left[-u_6\epsilon_1+u_8\epsilon_2+u_9\epsilon_3\right] \\ -2\left(\omega_2^{[1]}\right)^2\epsilon_2\left[-\frac{u_4u_{13}}{\left(u_4+\widehat{P}\right)^2}\epsilon_1-u_8\right] \\ -2\left(\omega_2^{[1]}\right)^2\epsilon_3\left[\frac{(u_5u_{15})}{\left(u_5+\widehat{P}\right)^2}\epsilon_1-u_9\right] \end{bmatrix},$$

Contingent on conditions (3.6a) as in [9] and (3.7)-(3.9), *we have*
$\left(\mathcal{H}^{[2]}\right)^T D^2f_{u_{12}}\left(E_2,u_{12}\right)\left(\Omega^{[2]},\Omega^{[2]}\right) = 2\epsilon_2\ \hbar_3{}^{[2]}\left(\omega_2^{[2]}\right)^2\left[\ \widehat{Z}_1-\widehat{Z}_1\right] \neq 0.$
The detecting of Sotomayor's Theorem [22] appear the happening of a transcritical bifurcation at system (2.2) *near* E_2 *with* $(\widehat{u}_{12} = u_{12})$.
Opposite of condition (3.9) *and plugging* $\Omega^{[2]}$ *in Eq.* (3.3), *we get:*

$$D^3f_{u_{12}}\left(E_2,\widehat{u}_{12}\right)(\Omega^{[2]},\Omega^{[2]},\Omega^{[2]}) = \begin{bmatrix} -6\epsilon_1^2\left(\omega_2^{[2]}\right)^3\left[\frac{u_2u_4}{\left(u_4+\widehat{P}\right)^2}\epsilon_2+\frac{u_3u_5}{\left(u_5+\widehat{P}\right)^3}\epsilon_3\right] \\ 0 \\ 6\left(\omega_2^{[2]}\right)^3\left[\frac{u_4u_{13}}{\left(u_4+\widehat{P}\right)^3}\epsilon_2\right] \\ 6\left(\omega_2^{[2]}\right)^3\left[\frac{(u_5u_{15})}{\left(u_5+\widehat{P}\right)^2}\epsilon_3\right] \end{bmatrix},$$

Again contingent on condition (3.6a) as in [9] with conditions (3.7) *and* (3.8), *we have*
$\left(\mathcal{H}^{[2]}\right)^T D^3f_{u_{12}}\left(E_2,\widehat{u}_{12}\right)(\Omega^{[2]},\Omega^{[2]},\Omega^{[2]}) = 6\left(\omega_2^{[1]}\right)^3\left[\frac{u_4u_{13}}{\left(u_4+\widehat{P}\right)^3}\epsilon_2\right] \neq 0.\ (\varepsilon_2 > 0)$
The detecting of Sotomayor's Theorem [22] appear the happening pitchfork bifurcation near E_2 *with* $(\widehat{u}_{12} = u_{12})$.
□

Theorem 3.3. *Assume that the local stability conditions (3.7b-3.7f) as in [9] hold. Then system* (2.2) *near the equilibrium point* $E_3 = (\overline{p},\ \overline{s},\ \overline{h},\ 0)$ *possesses a transcritical bifurcation at the parameter value* $\left[\overline{u}_{14} = u_9\overline{s}+\frac{u_{15}\overline{p}}{u_5+\overline{p}}-A\right]$, *according to the following conditions.*

$$\overline{c}_{14}\overline{c}_{23} > \overline{c}_{13}\overline{c}_{24} \tag{3.10}$$
$$\overline{Z}_1 \neq \overline{Z}_2 \tag{3.11}$$
$$\overline{Z}_3 \neq \overline{Z}_4 \tag{3.12}$$

Otherwise, no saddle-node occurred but pitchfork bifurcation can be occurred at E_3.
Where,

$$A = \frac{\overline{c}_{31}\overline{c}_{42}\left(\overline{c}_{14}\overline{c}_{23}-\overline{c}_{13}\overline{c}_{24}\right)}{\overline{c}_{23}\left(\overline{c}_{12}\overline{c}_{31}-\overline{c}_{11}\overline{c}_{32}\right)+\overline{c}_{13}\left(\overline{c}_{21}\overline{c}_{32}-\overline{c}_{12}\overline{c}_{31}\right)}, (A<0)$$
$$\overline{Z}_1 = 1+\frac{u_2u_4}{\left(u_4+\overline{P}\right)^2}\xi_2+\frac{u_4u_{13}}{\left(u_4+\overline{P}\right)^2}\xi_2-u_8\xi_1\xi_2\tau_2-u_9\xi_1\xi_2\tau_3+u_8\xi_1\xi_2\tau_1,$$
$$\overline{Z}_2 = -\left[\frac{u_2u_4\overline{h}}{\left(u_4+\overline{p}\right)^3}-u_1\xi_1-\frac{u_3u_5}{\left(u_5+\overline{p}\right)^2}\xi_3+u_6\tau_1\xi_1-u_9\xi_1\xi_3\tau_1+\frac{u_4u_{13}}{\left(u_4+\overline{p}\right)^2}\xi_2\tau_2+\frac{u_5u_{15}}{\left(u_5+\overline{p}\right)^2}\xi_3\tau_3\right],$$

$$
\begin{aligned}
\overline{Z}_3 &= \frac{u_2u_4\overline{h}}{(u_4+\overline{p})^4}+\frac{u_2u_4}{(u_4+\overline{p})^3}\xi_2,\\
\overline{Z}_4 &= -\left[\tau_2\left(\frac{u_4u_{13}\overline{h}}{(u_4+\overline{p})^4}+\frac{u_4u_{13}}{(u_4+\overline{p})^3}\xi_2\right)+\xi_3\tau_3\frac{u_5u_{15}}{(u_5+\overline{p})^4}-\frac{u_3u_5}{(u_4+\overline{p})^3}\xi_3\right],\\
\xi_1 &= \frac{-\overline{c}_{31}}{\overline{c}_{32}}<0,\\
\xi_2 &= \frac{-\left[\overline{c}_{44}\left(\overline{c}_{21}\overline{c}_{32}-\overline{c}_{22}\overline{c}_{31}\right)+\overline{c}_{24}\overline{c}_{31}\overline{c}_{42}\right]}{\overline{c}_{23}\overline{c}_{32}\overrightarrow{c}_{44}}>0
\end{aligned}
$$

and

$$
\begin{aligned}
\xi_3 &= \frac{\overline{c}_{31}\overline{c}_{42}}{\overline{c}_{32}\overline{c}_{44}}<0,\\
\tau_1 &= \frac{-\overline{c}_{13}}{\overline{c}_{23}}<0,\\
\tau_2 &= \frac{\overline{c}_{13}\overline{c}_{21}-\overline{c}_{11}\overline{c}_{23}}{\overline{c}_{23}\overline{c}_{31}}>0 \ \text{and} \ \ \tau_3=\frac{\overline{c}_{13}\overline{c}_{24}-\overline{c}_{14}\overline{c}_{23}}{\overline{c}_{23}\overline{c}_{44}}<0
\end{aligned}
$$

Proof . *via using* $E_3=\left(\overline{p},\overline{s},\overline{h},0\right)$ *in the Eq.* (3.1), *then the characteristic equation of* $\overline{J}_3$, *where* $\overline{J}_3=J_3\left(E_3,\overline{u}_{14}\right)$ *given in [9] having zero eigenvalues* ($\lambda_{3v}=0$), *if and only if* $M_4=0$ *and thus* E_3 *becomes a non-hyperbolic, whenever the parameter takes the value* $\left(\overline{u}_{14}=u_9\overline{s}+\frac{u_{15}\overline{p}}{u_5+\overline{p}}-A\right)$.
Such that $c_{ij}=\overline{c}_{ij}$, *for all* $i,j=1,2,3,4$ *except* $\overline{c}_{44}=A$.
It is clear that $\overline{u}_{14}>0$, *provided conditions (3.7a -3.7f) as in [9] with condition* (3.10).
Let, $\Omega^{[3]}=\left(\omega_1^{[3]},\omega_2^{[3]},\omega_3^{[3]},\omega_4^{[3]}\right)^T$ *be the eigenvector of* $\overline{J}_3$ *corresponding to the eigenvalue* $\lambda_{4v}=0$ *thus,* $\left(\overline{J}_3-\lambda_{3v}I\right)\Omega^{[2]}=0$, *give us:*
${\omega_2}^{[3]}=\xi_1{\omega_1}^{[3]}$, ${\omega_3}^{[3]}=\xi_2{\omega_1}^{[3]}$, ${\omega_4}^{[3]}=\xi_3{\omega_1}^{[3]}$ *and* ${\omega_1}^{[3]}$ *any nonzero real number, where,* ξ_1,ξ_2 *and* ξ_3 *are mentioned in state theorem.*
Let, $\mathcal{H}^{[3]}=\left(\hbar_1^{[3]},\hbar_2^{[3]},\hbar_3^{[3]},\hbar_4^{[3]}\right)^T$ *be the eigenvector of* $\overline{J_3}^T$ *affiliated to* $\lambda_{4v}=0$, *of the matrix* $\overline{J_3}^T$ *then:*
$\left(\overline{J_3}^T-\lambda_{4v}I\right)\mathcal{H}^{[3]}=0$, *Give us* $\mathcal{H}^{[3]}=\left(\hbar_1^{[3]},\tau_1\hbar_1^{[3]},\tau_2\hbar_1^{[3]},\tau_3\hbar_1^{[3]}\right)^T$ *where* $\hbar_1^{[3]}$*any nonzero real number.*
Where, τ_1,τ_2 *and* τ_3 *are mentioned in state theorem*
since, $\frac{\partial f}{\partial u_{14}}=f_{u_{14}}\ (\aleph,u_{14})=\left(\frac{\partial f_1}{\partial u_{14}},\frac{\partial f_2}{\partial u_{14}},\frac{\partial f_3}{\partial u_{14}},\frac{\partial f_4}{\partial u_{14}}\right)=(0,0,0,-v)^T$,
hence, $f_{u_{14}}\ (E_3,\overline{u}_{14})=(0,0,0,0)^T$. *Therefore* $\left(\Omega^{[3]}\right)^T f_{u_{14}}\ (E_3,\overline{u}_{14})=0$.
The detecting of Sotomayor's Theorem [22] appears the happening of the saddle-node bifurcation cannot be at E_3 . *Moreover,*

since, $Df_{u_{14}}\left(\aleph,\overline{u}_{14}\right)=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}$,

$$
Df_{u_{14}}\left(E_3,\overline{u}_{14}\right)\Omega^{[3]}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}\omega_1^{[3]}\\ \xi_1\omega_1^{[3]}\\ \xi_2\omega_1^{[3]}\\ \xi_3\omega_1^{[3]}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-\xi_3\omega_1^{[3]}\end{bmatrix},
$$

contingent on conditions (3.7d) and (3.7e) as in [9] with condition (3.10), *we get*
$\left(\mathcal{H}^{[3]}\right)^T\ \left[Df_{u_{14}}\left(E_3,\overline{u}_{14}\right)\Omega^{[3]}\right]=-\xi_3\tau_3\omega_1^{[3]}\hbar_1^{[3]}\neq 0$. ($\xi_3<0$ *and* $\tau_3<0$)

Plugging $\Omega^{[3]}$ *in Eq.* (3.2), *we get:*

$$D^2 f_{u_{14}}(E_3,\overline{u}_{14}) = \begin{bmatrix} -2\left(\omega_1^{[3]}\right)^2\left[\left(1-\frac{u_2u_4\overline{h}}{(u_4+\overline{p})^3}\right)+u_1\xi_1+\frac{u_2u_4}{\left(u_4+\overline{P}\right)^2}\xi_2+\frac{u_3u_5}{\left(u_5+\overline{P}\right)^2}\xi_3\right] \\ -2\xi_1\left(\omega_1^{[3]}\right)^2\left[-u_6+u_8\xi_2+u_9\xi_3\right] \\ -2\left(\omega_1^{[3]}\right)^2\left[\frac{u_4u_{13}\overline{h}}{(u_4+\overline{p})^3}-\frac{u_4u_{13}}{\left(u_4+\overline{P}\right)^2}\xi_2-u_8\xi_1\xi_2\right] \\ 2\xi_3\left(\omega_1^{[3]}\right)^2\left[\frac{(u_5u_{15})}{(u_5+\overline{p})^2}+u_9\xi_1\right] \end{bmatrix},$$

contingent on conditions (3.7b-3.7f) as in [9] with condition (3.11).

So, $\left(\mathcal{H}^{[3]}\right)^T D^2 f_{u_{14}}(E_3,\overline{u}_{14})\left(\Omega^{[3]},\Omega^{[3]}\right) = -2\left(\omega_1^{[3]}\right)^2 \hbar_1^{[3]}\left[\overline{Z}_1-\overline{Z}_2\right]\neq 0.$

The detecting of Sotomayor's Theorem [22] appears the happening a transcritical bifurcation at E_3 *with* $(\overline{u}_{14}=u_{14})$.

Opposite of condition (3.11) *and plugging* $\Omega^{[3]}$ *in eq.* (3.3), *we get:*

$$D^3 f_{u_{14}}(E_3,\overline{u}_{14}) = \begin{bmatrix} -6\left(\omega_1^{[3]}\right)^3\left[\frac{u_2u_4\overline{h}}{(u_4+\overline{p})^4}+\frac{u_2u_4}{\left(u_4+\overline{P}\right)^3}\xi_2+\frac{u_3u_5}{\left(u_5+\overline{P}\right)^3}\xi_3\right] \\ 0 \\ -6\left(\omega_1^{[3]}\right)^3\left[\frac{-u_4u_{13}\overline{h}}{(u_4+\overline{p})^4}-\frac{u_4u_{13}}{\left(u_4+\overline{P}\right)^3}\xi_2\right] \\ -6\left(\omega_1^{[3]}\right)^3\left[\frac{-u_5u_{15}}{(u_5+\overline{p})^4}\xi_3\right] \end{bmatrix},$$

Again contingent on conditions (3.7b-3.7f) as in [9] with condition (3.12).

Thus $\left(\mathcal{H}^{[3]}\right)^T D^2 f_{u_{14}}(E_3,\overline{u}_{14})\left(\Omega^{[3]},\Omega^{[3]},\Omega^{[3]}\right) = -6\left(\omega_1^{[3]}\right)^2 \hbar_1^{[3]}\left[\overline{Z}_3-\overline{Z}_4\right]\neq 0.$

The detecting of Sotomayor's Theorem [22] appears the happening of pitchfork bifurcation at E_3 *with* $(\overline{u}_{14}=u_{14})$. □

Theorem 3.4. *Assume that the local stability conditions (3.9b-3.9d) as in [9] hold. Then system* (2.2) *near the equilibrium point* $E_4=(\bar{\bar{p}},\bar{\bar{s}},0,\bar{\bar{v}})$ *possesses a transcritical bifurcation at the parameter value* $\bar{\bar{u}}_{12}=u_8\bar{\bar{s}}-u_7+\frac{u_{13}\bar{\bar{p}}}{u_4+\bar{\bar{p}}}$ *according to the following conditions:*

$$u_7 < u_8\bar{\bar{s}}+\frac{u_{13}\bar{\bar{p}}}{u_4+\bar{\bar{p}}} \tag{3.13}$$

$$\bar{\bar{s}} > \frac{u_7}{u_8} \tag{3.14}$$

$$\bar{\bar{d}}_{13}\bar{\bar{d}}_{24}\bar{\bar{d}}_{42} > -\bar{\bar{d}}_{23}\left(\bar{\bar{d}}_{12}\bar{\bar{d}}_{44}-\bar{\bar{d}}_{14}\bar{\bar{d}}_{42}\right), \tag{3.15}$$

$$\bar{\bar{d}}_{24}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{21}\bar{\bar{d}}_{44} > -\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}m_1, \tag{3.16}$$

Otherwise, no saddle-node but pitchfork bifurcation can be occurred under the following condition

$$\frac{u_4u_{13}}{\left(u_4+\bar{\bar{P}}\right)^2} = -u_8m_1, \tag{3.17}$$

Where, $m_1=\frac{\bar{\bar{d}}_{23}\left(\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{11}\bar{\bar{d}}_{44}\right)-\bar{\bar{d}}_{13}\left(\bar{\bar{d}}_{24}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{21}\bar{\bar{d}}_{44}\right)}{\bar{\bar{d}}_{13}\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}+\bar{\bar{d}}_{23}\left(\bar{\bar{d}}_{12}\bar{\bar{d}}_{44}-\bar{\bar{d}}_{14}\bar{\bar{d}}_{42}\right)}$ $(m_1>0)$

Proof . *By substituting* $E_4=(\bar{\bar{p}},\bar{\bar{s}},0,\bar{\bar{v}})$ *in the Eq.* (3.1), *then the characteristic equation of* $\bar{\bar{J}}_4$, *where* $\bar{\bar{J}}_4=J_4(E_4,\bar{\bar{u}}_{12})$ *given in [9] having zero eigenvalues* ($\lambda_{4h}=0$),

Contingent on condition (3.13), $\bar{\bar{u}}_{12}>0$.

Let, $\Omega^{[4]} = \left(\omega_1^{[4]}, \omega_2^{[4]}, \omega_3^{[4]}, \omega_4^{[4]}\right)^T$ *be the eigenvector of* $\bar{\bar{J}}_4$ *corresponding to the eigenvalue* $\lambda_{4h} = 0$
Thus, $\left(\bar{\bar{J}}_4 - \lambda_{4h} I\right)\Omega^{[4]}=0$*, which gives:*
$w_2{}^{[4]} = m_1\omega_1{}^{[4]},\ \omega_3{}^{[4]} = m_2\omega_1{}^{[4]},\ \omega_4{}^{[4]} = m_3\omega_1{}^{[4]}$ *and* $\omega_1{}^{[4]}$ *any nonzero real number.*
Where, m_1 *, mentioned in state theorem,* $m_2 = \frac{(\bar{\bar{d}}_{24}\bar{\bar{d}}_{41} - \bar{\bar{d}}_{21}\bar{\bar{d}}_{44}) + \bar{\bar{d}}_{24}\bar{\bar{d}}_{42}m_1}{\bar{\bar{d}}_{23}\bar{\bar{d}}_{44}}$ $(m_2 > 0)$ *and* $m_3 = \frac{-(\bar{\bar{d}}_{41} + \bar{\bar{d}}_{42}m_1)}{\bar{\bar{d}}_{44}}$.
Let, $\mathcal{H}^{[4]} = \left(\hbar_1^{[4]}, \hbar_2^{[4]}, \hbar_3^{[4]}, \hbar_4^{[4]}\right)^T$ *be the eigenvector of* $\bar{\bar{J}}_4^{\,T}$ *affiliated to* $\lambda_{4h} = 0$*, of the matrix* $\bar{\bar{J}}_4^{\,T}$
then: $\left(\bar{\bar{J}}_4^{\,T} - \lambda_{4h} I\right)\mathcal{H}^{[4]} = 0$*, Give us:* $\mathcal{H}^{[4]} = \left(0, 0, \hbar_3^{[4]}, 0\right)^T$ *where* $\hbar_3^{[4]}$*any nonzero real number.*
since, $\frac{\partial f}{\partial u_{12}} = f_{u_{12}}\ (\aleph, u_{12}) = \left(\frac{\partial f_1}{\partial u_{12}}, \frac{\partial f_2}{\partial u_{12}}, \frac{\partial f_3}{\partial u_{12}}, \frac{\partial f_4}{\partial u_{12}}\right) = (0, 0, -h, 0)^T$,
hence, $f_{u_{12}}\ (E_4, \bar{\bar{u}}_{12}) = (0,0,0,0)^T$*. Therefore* $\left(\mathcal{H}^{[4]}\right)^T f_{u_{12}}\ (E_4, \bar{\bar{u}}_{12}) = 0.$
The detecting of Sotomayor's Theorem [22] appears the happening of the saddle-node bifurcation cannot be at E_4*. Moreover,*

$$\text{since, } Df_{u_{12}}(\aleph, \bar{\bar{u}}_{12}) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$Df_{u_{12}}(E_4, \bar{\bar{u}}_{12})\,\Omega^{[4]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \omega_1^{[4]} \\ m_1\omega_1^{[4]} \\ m_2\omega_1^{[4]} \\ m_3\omega_4^{[4]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -m_2\omega_1^{[4]} \\ 0 \end{bmatrix},$$

contingent on conditions (3.9b-3.9d) as in [9] with conditions (3.14), (3.15) *and* (3.16).
so, $\left(\mathcal{H}^{[4]}\right)^T\ \left[Df_{u_{12}}(E_4, \bar{\bar{u}}_{12})\,\Omega^{[4]}\right] = -m_2\omega_1{}^{[4]}\hbar_3{}^{[4]} \neq 0,$
Plugging $\Omega^{[4]}$ *in Eq.* (3.2)*, we get:*

$$D^2 f_{u_{12}}(E_4, \bar{\bar{u}}_{12}) = \begin{bmatrix} -2\left(\omega_1^{[4]}\right)^2\left[\left(1 - \frac{u_3u_5\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^3}\right) + u_1m_1 + \frac{u_2u_4}{\left(u_4+\bar{\bar{P}}\right)^2}m_2 + \frac{u_3u_5}{\left(u_5+\bar{\bar{P}}\right)^2}m_3\right] \\ -2m_1\left(\omega_1^{[4]}\right)^2[-u_6 + u_8m_2 + u_9m_3] \\ 2m_2\left(\omega_1^{[4]}\right)^2\left[\frac{u_4u_{13}}{\left(u_4+\bar{\bar{P}}\right)^2} + u_8m_1\right] \\ -2\left(\omega_1^{[4]}\right)^2\left[\frac{(u_5u_{15})\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^3} - \frac{(u_5u_{15})}{(u_5+\bar{\bar{p}})^2}m_3 - u_9m_1m_3\right] \end{bmatrix},$$

contingent on conditions (3.9b-3.9d) as in [9] with conditions (3.14) *and* (3.15). (*since* $m_1 > 0$)
So, $\left(\mathcal{H}^{[4]}\right)^T D^2 f_{u_{12}}(E_4, \bar{\bar{u}}_{12})\left(\Omega^{[4]}, \Omega^{[4]}\right) = 2m_1\left(\omega_1^{[4]}\right)^2\hbar_3^{[4]}\left[\frac{u_4u_{13}}{\left(u_4+\bar{\bar{P}}\right)^2} + u_8m_1\right] \neq 0.$
The detecting of Sotomayor's Theorem [22] appears the happening of a transcritical bifurcation at system (2.2) *near* E_4 *with* $(\bar{\bar{u}}_{12} = u_{12})$.
Opposite both of conditions (3.9d) in [9] and (3.14), $(m_1 < 0)$
Again by conditions (3.9b-3.9c) as in [9] and condition (3.17)*, then plugging* $\Omega^{[4]}$ *in Eq.* (3.3)*, we get:*

$$D^3 f_{u_{12}}(E_4, \bar{\bar{u}}_{12}) = \begin{bmatrix} -6\left(\omega_1^{[4]}\right)^3\left[\frac{u_3u_5\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^4} + \frac{u_2u_4}{(u_4+\bar{\bar{p}})^3}m_2 + \frac{u_3u_5}{(u_5+\bar{\bar{p}})^3}m_3\right] \\ 0 \\ 6\left(\omega_1^{[4]}\right)^3\left[\frac{u_4u_{13}}{(u_4+\bar{\bar{p}})^3}m_2\right] \\ -6\left(\omega_1^{[4]}\right)^3\left[\frac{-u_5u_{15}\bar{\bar{v}}}{(u_5+\bar{\bar{p}})^4} - \frac{u_5u_{15}}{(u_5+\bar{\bar{p}})^4}m_3\right] \end{bmatrix},$$

contingent on conditions (3.9b-3.9c) as in [9] with condition (3.16) $(m_2 > 0)$
So, $\left(\mathcal{H}^{[4]}\right)^T D^3 f_{u_{12}}\left(E_4, \bar{\bar{u}}_{12}\right)\left(\Omega^{[4]}, \Omega^{[4]}, \Omega^{[4]}\right) = 6\left(\omega_1^{[4]}\right)^3 \hbar_3^{[4]}[\frac{u_4 u_{13}}{(u_4+\bar{p})^3} m_2] \neq 0,$
The detecting of Sotomayor's Theorem [22] appears the happening of pitchfork bifurcation at system (2.2) *near* E_4 *with* $(\bar{\bar{u}}_{12} = u_{12})$. □

Theorem 3.5. *Assume that the local stability conditions (3.11a-3.11g) as in [9]. Then system* (2.2) *near* $E_5 = \left(\widetilde{p}, \widetilde{s}, \widetilde{h}, \widetilde{v}\right)$, *possesses saddle-node bifurcation at the parameter value* $\widetilde{u}_7 = u_7 = \mathcal{R} + u_8\widetilde{s}$, *according to the following conditions hold:*

$$\widetilde{e}_{31}\left(\widetilde{e}_{22}\widetilde{e}_{44} - \widetilde{e}_{24}\widetilde{e}_{42}\right) < -\widetilde{e}_{32}\left(\widetilde{e}_{24}\widetilde{e}_{41} - \widetilde{e}_{21}\widetilde{e}_{44}\right) \tag{3.18}$$

$$\widetilde{e}_{31}\left(\widetilde{e}_{12}\widetilde{e}_{44} - \widetilde{e}_{14}\widetilde{e}_{42}\right) > -\widetilde{e}_{32}\left(\widetilde{e}_{14}\widetilde{e}_{41} - \widetilde{e}_{11}\widetilde{e}_{44}\right) \tag{3.19}$$

$$\widetilde{Z}_1 \neq \widetilde{Z}_2 \tag{3.20}$$

Otherwise, a transcritical bifurcation and pitchfork bifurcation can be occurred at E_5, *according to the following conditions hold:*

$$l_0 \neq l_1 \tag{3.21}$$

$$\widetilde{Z}_3 \neq \widetilde{Z}_4 \tag{3.22}$$

$$\widetilde{Z}_5 \neq \widetilde{Z}_6 \tag{3.23}$$

Where

$$\mathcal{R} = \frac{\widetilde{e}_{13}\left[\widetilde{e}_{31}\left(\widetilde{e}_{22}\widetilde{e}_{44} - \widetilde{e}_{24}\widetilde{e}_{42}\right) + \widetilde{e}_{32}\left(\widetilde{e}_{24}\widetilde{e}_{41} - \widetilde{e}_{21}\widetilde{e}_{44}\right)\right]}{\widetilde{e}_{31}\left(\widetilde{e}_{12}\widetilde{e}_{44} - \widetilde{e}_{14}\widetilde{e}_{42}\right) + \widetilde{e}_{32}\left(\widetilde{e}_{14}\widetilde{e}_{41} - \widetilde{e}_{11}\widetilde{e}_{44}\right)},$$

$$\widetilde{Z}_1 = 1 - \frac{u_3 u_5}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} - u_9 n_0 l_0 \frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} + l_1\left(\frac{u_4 u_{13}\widetilde{h}}{(u_4+\widetilde{p})^3} - \frac{u_4 u_{13}}{(u_4+\widetilde{p})^2}n_1\right) +$$
$$\frac{\widetilde{e}_{13}\widetilde{e}_{24}}{\widetilde{e}_{23}\widetilde{e}_{44}}\left[\frac{u_5 u_{15}\widetilde{v}}{(u_5+\widetilde{p})^3} - \frac{u_5 u_{15}}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}} + u_9 n_0 \frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}\right] - \frac{\widetilde{e}_{14}}{\widetilde{e}_{44}}\left[\frac{u_5 u_{15}}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} - u_9 n_0 \frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right]$$

$$\widetilde{Z}_2 = -\left\{\frac{u_2 u_4\widetilde{h}}{(u_4+\widetilde{p})^3} + \frac{u_3 u_5\widetilde{v}}{(u_5+\widetilde{p})^3} - u_1 n_0 - \frac{u_2 u_4}{(u_4+\widetilde{p})^2}n_1 - \frac{u_3 u_5}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}} - n_0 l_0\left[-u_6 + u_8 n_1 + u_9\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right]\right.$$
$$\left.+u_8 n_0 n_1 l_1 - \frac{\widetilde{e}_{13}\widetilde{e}_{24}}{\widetilde{e}_{23}\widetilde{e}_{44}}\left[\frac{u_5 u_{15}}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} - u_9 n_0\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right] + \frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}\left[\frac{u_5 u_{15}\widetilde{v}}{(u_5+\widetilde{p})^3} - \frac{u_5 u_{15}}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}} + u_9 n_0\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}\right]\right\},$$

$$\widetilde{Z}_3 = 1 + \frac{u_2 u_4}{(u_4+\widetilde{p})^2}n_1 - \frac{u_3 u_5}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} + n_0 l_0\left[-u_6 + u_9\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right] + l_1\left[\frac{u_2 u_4\widetilde{h}}{(u_4+\widetilde{p})^3} - u_8 n_0 n_1\right] +$$
$$l_2\left[\frac{u_5 u_{15}}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}} - u_9 n_0\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right],$$

$$\widetilde{Z}_4 = -\left\{\frac{u_2 u_4\widetilde{h}}{(u_4+\widetilde{p})^3} + \frac{u_3 u_5\widetilde{v}}{(u_5+\widetilde{p})^3} - u_1 n_0 - \frac{u_3 u_5}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}} - n_0 l_0\left[u_8 n_1 - u_9\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}\right] + \frac{u_2 u_4}{(u_4+\widetilde{p})^2}n_1 l_1 -\right.$$
$$\left.l_2\left[\frac{u_3 u_5\widetilde{v}}{(u_5+\widetilde{p})^3} - \frac{u_3 u_5}{(u_5+\widetilde{p})^2}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}} + u_9 n_0\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}\right]\right\},$$

$$\widetilde{Z}_5=\frac{u_2u_4\widetilde{h}}{(u_4+\widetilde{p})^4}+\frac{u_3u_5\widetilde{v}}{(u_5+\widetilde{p})^4}+\frac{u_2u_4}{(u_4+\widetilde{p})^3}n_1-\frac{u_3u_5}{(u_5+\widetilde{p})^3}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}-l_2\left[\frac{u_3u_5\widetilde{v}}{(u_5+\widetilde{p})^4}-\frac{u_5u_{15}}{(u_5+\widetilde{p})^3}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right],$$
$$\widetilde{Z}_6=-\left\{l_1\left(\frac{u_4u_{13}\widetilde{h}}{(u_4+\widetilde{p})^4}+\frac{u_4u_{13}}{(u_4+\widetilde{p})^3}n_1\right)+l_2\frac{u_5u_{15}}{(u_5+\widetilde{p})^3}\frac{\widetilde{e}_{41}}{\widetilde{e}_{44}}-\frac{u_3u_5}{(u_5+\widetilde{p})^3}\frac{\widetilde{e}_{31}\widetilde{e}_{42}}{\widetilde{e}_{32}\widetilde{e}_{44}}\right\},$$

$n_0=\frac{-\widetilde{e}_{31}}{\widetilde{e}_{32}}$, $n_1=\frac{\widetilde{e}_{44}(\widetilde{e}_{22}\widetilde{e}_{31}-\widetilde{e}_{21}\widetilde{e}_{32})-\widetilde{e}_{24}(\widetilde{e}_{31}\widetilde{e}_{42}-\widetilde{e}_{32}\widetilde{e}_{41})}{\widetilde{e}_{23}\widetilde{e}_{32}\widetilde{e}_{44}}$ *and* $n_2=\frac{\widetilde{e}_{31}\widetilde{e}_{42}-\widetilde{e}_{32}\widetilde{e}_{41}}{\widetilde{e}_{32}\widetilde{e}_{44}}$.
$l_0=\frac{-\widetilde{e}_{13}}{\widetilde{e}_{23}}<0$, $l_1=\frac{\widetilde{e}_{23}(\widetilde{e}_{14}\widetilde{e}_{41}-\widetilde{e}_{11}\widetilde{e}_{44})-\widetilde{e}_{13}(\widetilde{e}_{24}\widetilde{e}_{41}-\widetilde{e}_{21}\widetilde{e}_{44})}{\widetilde{e}_{23}\widetilde{e}_{31}\widetilde{e}_{44}}>0$ *and* $l_2=\frac{\widetilde{e}_{13}\widetilde{e}_{24}-\widetilde{e}_{14}\widetilde{e}_{23}}{\widetilde{e}_{32}\widetilde{e}_{44}}$.

Proof . *By substituting* $E_5=\left(\widetilde{p},\widetilde{s},\widetilde{h},\widetilde{v}\right)$ *in the Eq.*(3.1)*, then the characteristic equation of* $\widetilde{J}_5$*, where* $\widetilde{J}_5=J_5\left(E_5,\widetilde{u}_7\right)$ *which is given in [9] having zero eigenvalues (say* $\lambda_{5h}=0$*), if and only if* $K_4=0$ *and then* E_5 *becomes a non-hyperbolic, whenever the parameter takes the value* (*say* $\widetilde{u}_7=u_7$)*.*
At E_5*, jacobian matrix of system (2.2) becomes:*
$\widetilde{J}_5=J\left(E_5,\widetilde{u}_7\right)=[\widetilde{e}_{ij}]_{4\times4}$, *where* $\widetilde{e}_{ij}=e_{ij}$ *for all* $i,j=1,2,3,4$ *given in [9] except* $\widetilde{e}_{23}=\mathcal{R}$,
$\widetilde{u}_7>0$, *contingent on conditions (3.11a-3.11e) as in [9] with conditions* (3.18) *and* (3.19) *are hold.*
Let, $\Omega^{[5]}=\left(\omega_1^{[5]},\omega_2^{[5]},\omega_3^{[5]},\omega_4^{[5]}\right)^T$ *be the eigenvector of* $\widetilde{J}_5$ *affiliated to* $\lambda_{5h}=0$*. Thus,* $\left(\widetilde{J}_5-\lambda_{5h}I\right)\Omega^{[5]}=0$*, which gives:* ${\omega_2}^{[5]}=n_0{\omega_1}^{[5]}$, ${\omega_3}^{[5]}=n_1{\omega_1}^{[5]}$, ${\omega_4}^{[5]}=n_2{\omega_1}^{[5]}$ *and* ${\omega_1}^{[5]}$ *any nonzero real number.*
Where, n_0*,* n_1 *and* n_2 *are mentioned in state theorem.*
Let, $\mathcal{H}^{[5]}=\left(\hbar_1^{[5]},\hbar_2^{[5]},\hbar_3^{[5]},\hbar_4^{[5]}\right)^T$ *be the eigenvector of* $\widetilde{J}_5^{\,T}$ *affiliated to* $\lambda_{5h}=0$*, of the matrix* $\widetilde{J}_5^{\,T}$ *then:* $\left(\widetilde{J}_5^{\,T}-\lambda_{5h}I\right)\mathcal{H}^{[5]}=0$*, Give us:* $\mathcal{H}^{[5]}=\left(\hbar_1^{[5]},l_0\hbar_1^{[5]},l_1\hbar_1^{[5]},l_2\hbar_1^{[5]}\right)^T$ *, where* $\hbar_1^{[5]}$ *any non-zero real number.* l_0*,* l_1 *and* l_2 *are mentioned in state theorem,*
since, $\frac{\partial f}{\partial u_7}=f_{u_7}\ (\aleph,u_7)=\left(\frac{\partial f_1}{\partial u_7},\frac{\partial f_2}{\partial u_7},\frac{\partial f_3}{\partial u_7},\frac{\partial f_4}{\partial u_7}\right)=(0,-h,h,0)^T$*, hence,* $f_{u_7}\ (E_5,\widetilde{u}_7)=\left(0,-\widetilde{h},\widetilde{h},0\right)^T$*.*
Therefore $\left(\mathcal{H}^{[5]}\right)^T f_{u_7}\ (E_5,\widetilde{u}_7)=-\hbar_1\widetilde{h}\left[l_0-l_1\right]\neq0$ *.*
Contingent on conditions (3.11a), (3.11c), (3.11d), (3.11e) and (3.11g). ($l_0<0$ *and* $l_1>0\rightarrow l_0-l_1<0$)
Plugging $\Omega^{[5]}$ *in Eq.* (3.2)*, we get:*

$$D^2f_{u_7}\left(E_5,\widetilde{u}_7\right)=\begin{bmatrix}-2\left(\omega_1^{[5]}\right)^2\left[\left(1-\frac{u_2u_4\widetilde{h}}{(u_4+\widetilde{p})^3}-\frac{u_3u_5\widetilde{v}}{(u_5+\widetilde{p})^3}\right)+u_1n_0+\frac{u_2u_4}{\left(u_4+\widetilde{P}\right)^2}n_1+\frac{u_3u_5}{\left(u_5+\widetilde{P}\right)^2}n_2\right]\\ -2n_0\left(\omega_1^{[5]}\right)^2\left[-u_6+u_8n_1+u_9n_2\right]\\ -2\left(\omega_1^{[5]}\right)^2\left[\frac{u_4u_{13}\widetilde{h}}{(u_4+\widetilde{p})^2}-\frac{u_4u_{13}}{(u_4+\widetilde{p})^2}n_1-u_8n_0n_1\right]\\ -2\left(\omega_1^{[5]}\right)^2\left[\frac{(u_5u_{15})\widetilde{v}}{(u_5+\widetilde{p})^3}-\frac{(u_5u_{15})}{(u_5+\widetilde{p})^2}n_2-u_9n_0n_2\right]\end{bmatrix},$$

contingent on conditions (3.11a-3.11g) as in [9] with condition (3.22)*.*
So, $\left(\mathcal{H}^{[5]}\right)^T D^2f_{u_7}\left(E_5,\widetilde{u}_7\right)\left(\Omega^{[5]},\Omega^{[5]}\right)=-2\left(\omega_1^{[5]}\right)^2\hbar_1\left[\widetilde{Z}_1-\widetilde{Z}_2\right]\neq0.$
The detecting of Sotomayor's Theorem [22] appear the happening of a saddle-node bifurcation at system (2.2) *near* E_5 *with* ($\widetilde{u}_7=u_7$)*, moreover neither transcritical bifurcation nor pitchfork bifurcation can be occurring at* E_5*.*
Opposite of condition (3.11d), we get ($\widetilde{e}_{23}>0$)*,* $\left(\mathcal{H}^{[5]}\right)^T f_{u_7}\ (E_5,\widetilde{u}_7)=-\hbar_1\widetilde{h}\left[l_0-l_1\right]$ ($l_0>0$ *and* $l_1>0$)*,*
Either, condition (3.21) *hold*
contingent on conditions (3.11b), (3.11c) and (3.11e) as in [9] with condition (3.18)
so, $\left(\mathcal{H}^{[5]}\right)^T\left[Df_{u_7}\left(E_5,\widetilde{u}_7\right)\Omega^{[5]}\right]=n_1l_1{\omega_1}^{[5]}{\hbar_1}^{[5]}\neq0\left(n_1>0\text{ and }l_1>0\rightarrow n_1l_1>0\right),$

Plugging $\Omega^{[5]}$ *in Eq.* (3.2), *we get:*

$$D^2 f_{u_7}(E_5, \widetilde{u}_7) = \begin{bmatrix} -2\left(\omega_1^{[5]}\right)^2 \left[\left(1 - \frac{u_2 u_4 \widetilde{h}}{(u_4+\widetilde{p})^3} - \frac{u_3 u_5 \widetilde{v}}{(u_5+\widetilde{p})^3}\right) + u_1 n_0 + \frac{u_2 u_4}{\left(u_4+\widetilde{P}\right)^2} n_1 + \frac{u_3 u_5}{\left(u_5+\widetilde{P}\right)^2} n_2\right] \\ -2 n_0 \left(\omega_1^{[5]}\right)^2 \left[-u_6 + u_8 n_1 + u_9 n_2\right] \\ -2\left(\omega_1^{[5]}\right)^2 \left[\frac{u_4 u_{13} \widetilde{h}}{(u_4+\widetilde{p})^2} - \frac{u_4 u_{13}}{(u_4+\widetilde{p})^2} n_1 - u_8 n_0 n_1\right] \\ -2\left(\omega_1^{[5]}\right)^2 \left[\frac{(u_5 u_{15})\widetilde{v}}{(u_5+\widetilde{p})^3} - \frac{(u_5 u_{15})}{(u_5+\widetilde{p})^2} n_2 - u_9 n_0 n_2\right] \end{bmatrix}$$

Under authority of conditions (3.11a), (3.11b), (3.11e) and (3.11g) as in [9] with condition (3.22).
so, $\left(\mathcal{H}^{[5]}\right)^T D^2 f_{u_7}(E_5, \widetilde{u}_7)\left(\Omega^{[5]}, \Omega^{[5]}\right) = -2\left(\omega_1^{[5]}\right)^2 \hbar_1{}^{[5]}\left[\widetilde{Z}_3 - \widetilde{Z}_4\right] \neq 0.$
The detecting of Sotomayor's Theorem [22] appear the happening of a saddle-node bifurcation at system (2.2) *near* E_5 *with* $(\widetilde{u}_7 = u_7)$.
Or opposite of condition (3.21), *by Sotomayor's theorem saddle-node bifurcation cannot be occurred at system* (2.2) *near* E_5 *with* $(\widetilde{u}_7 = u_7)$.
Moreover,

$$since,\ Df_{u_7}(\aleph, \widetilde{u}_7) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\ Df_{u_7}(E_5, \widetilde{u}_7)\,\Omega^{[5]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_1^{[5]} \\ n_0 \omega_1^{[5]} \\ n_1 \omega_1^{[5]} \\ n_2 \omega_4^{[5]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ n_1 \omega_1^{[5]} \\ 0 \end{bmatrix},$$

Under authority of conditions (3.11a), (3.11b), (3.11e) and (3.11g) as in [9] with condition (3.22).
So, $\left(\mathcal{H}^{[5]}\right)^T D^2 f_{u_7}(E_5, \widetilde{u}_7)\left(\Omega^{[5]}, \Omega^{[5]}\right) = -2\left(\omega_1^{[5]}\right)^2 \hbar_1{}^{[5]}\left[\widetilde{Z}_3 - \widetilde{Z}_4\right] \neq 0.$
The detecting of Sotomayor's Theorem [22] appear the happening of a transcritical bifurcation at system (2.2) *near* E_5 *with* $(\widetilde{u}_7 = u_7)$.
Now, opposite of condition (3.22) *and plugging* $\Omega^{[5]}$ *in Eq.* (3.3) *we get:*

$$D^3 f_{u_7}(E_5, \widetilde{u}_7) = \begin{bmatrix} -6\left(\omega_1^{[5]}\right)^3 \left[\frac{u_2 u_4 \widetilde{h}}{(u_4+\widetilde{p})^4} + \frac{u_3 u_5 \widetilde{v}}{(u_5+\widetilde{p})^4} + \frac{u_2 u_4}{(u_4+\widetilde{p})^3} n_1 + \frac{u_3 u_5}{(u_5+\widetilde{p})^3} n_2\right] \\ 0 \\ -6\left(\omega_1^{[5]}\right)^3 \left[\frac{-u_4 u_{13} \widetilde{h}}{(u_4+\widetilde{p})^4} - \frac{u_4 u_{13}}{(u_4+\widetilde{p})^3} n_1\right] \\ -6\left(\omega_1^{[4]}\right)^3 \left[\frac{-u_5 u_{15} \widetilde{v}}{(u_5+\widetilde{p})^4} - \frac{u_5 u_{15}}{(u_5+\widetilde{p})^3} n_2\right] \end{bmatrix},$$

Again, according to conditions (3.11a), (3.11b), (3.11e) and (3.11g) as in [9] with condition (3.23).
So, $\left(\mathcal{H}^{[5]}\right)^T D^3 f_{u_7}(E_5, \widetilde{u}_7)\left(\Omega^{[5]}, \Omega^{[5]}, \Omega^{[5]}\right) = -6\left(\omega_1^{[5]}\right)^3 \hbar_1^{[5]}\left[\widetilde{Z}_5 - \widetilde{Z}_6\right] \neq 0,$
The detecting of Sotomayor's Theorem [22] appears the happening of pitchfork bifurcation at system (2.2) *near* E_4 *with* $(\widetilde{u}_7 = u_7)$. □

4. The Hopf-bifurcation analysis

In this section, the investigation of Hopf-bifurcation near the equilibrium points of system (2.2) in the opinion of Hague and Venturino methods [7] as below.

Theorem 4.1. *Suppose that the locally conditions (3.7b-3.7f) as in [9] and the following conditions* *hold:*

$$M_1M_2 > M_3, \tag{4.1}$$
$$0 < \Delta_1(\overline{u}_2) < \frac{M_1^3(\overline{u}_2)}{4}, \tag{4.2}$$
$$r_1 < r_2, \tag{4.3}$$
$$r_3 < r_4, \tag{4.4}$$
$$r_5 > r_6, \tag{4.5}$$
$$\overline{\alpha}_1 \neq \overline{\alpha}_2 \tag{4.6}$$

Where,

$$\begin{aligned}
r_1 &= \overline{c}_{11}\overline{c}_{31}\left[\overline{c}_{31}\left(\overline{c}_{11}+\overline{c}_{22}\right)-\overline{c}_{21}\overline{c}_{32}\right],\\
r_2 &= -\left\{\overline{c}_{21}\overline{c}_{32}\left[\overline{c}_{21}\overline{c}_{32}-\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)\right]\right\},\\
r_3 &= \overline{c}_{14}\overline{c}_{21}\overline{c}_{42}\left[\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)-2\overline{c}_{21}\overline{c}_{32}\right],\\
r_4 &= -\Big\{\overline{c}_{31}\overline{c}_{44}^2\left(\overline{c}_{11}^2+\overline{c}_{22}^2\right)+2\overline{c}_{11}\overline{c}_{13}\overline{c}_{44}\left(\overline{c}_{22}\overline{c}_{44}-\overline{c}_{12}\overline{c}_{21}\right)+\overline{c}_{12}\overline{c}_{21}\overline{c}_{31}\left(\overline{c}_{23}\overline{c}_{32}-\overline{c}_{11}\overline{c}_{22}\right)-\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)\\
&\quad\left[\overline{c}_{12}\overline{c}_{23}\overline{c}_{31}+\overline{c}_{24}\overline{c}_{42}\overline{c}_{44}\right]+\overline{c}_{31}\overline{c}_{44}\left(\overline{c}_{11}+\overline{c}_{22}\right)\left[\overline{c}_{44}\left(\overline{c}_{22}+\overline{c}_{44}\right)+3\overline{c}_{11}\overline{c}_{22}\right]-\overline{c}_{22}\left[\overline{c}_{21}\overline{c}_{22}\overline{c}_{32}\overline{c}_{44}-\overline{c}_{11}\overline{c}_{31}\left(\overline{c}_{22}^2+\overline{c}_{44}^2\right)\right]\\
&\quad-\overline{c}_{22}\left(\overline{c}_{12}\overline{c}_{21}+\overline{c}_{23}\overline{c}_{32}+\overline{c}_{24}\overline{c}_{42}\right)\left[\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)-\overline{c}_{21}\overline{c}_{32}\right]-\overline{c}_{11}\overline{c}_{31}\overline{c}_{44}\left(\overline{c}_{12}\overline{c}_{21}+\overline{c}_{23}\overline{c}_{32}\right)+\overline{c}_{22}^2\left(\overline{c}_{11}^2\overline{c}_{31}+\overline{c}_{22}\overline{c}_{44}\right)\\
&\quad-\overline{c}_{11}\overline{c}_{21}\overline{c}_{32}\left[\overline{c}_{22}\left(\overline{c}_{11}+\overline{c}_{44}\right)-\overline{c}_{12}\overline{c}_{32}\right]-\overline{c}_{11}\overline{c}_{21}\overline{c}_{32}\left(\overline{c}_{22}^2+\overline{c}_{44}^2\right)-\overline{c}_{21}\overline{c}_{32}\overline{c}_{44}\left(\overline{c}_{22}\overline{c}_{44}-\overline{c}_{12}\overline{c}_{21}\right)\\
&\quad-\overline{c}_{11}^2\overline{c}_{31}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{24}\overline{c}_{42}\right)+\overline{c}_{21}\overline{c}_{31}\left[\overline{c}_{12}\overline{c}_{23}\overline{c}_{32}+\overline{c}_{11}\overline{c}_{14}\overline{c}_{42}\right]\Big\},\\
r_5 &= \overline{c}_{11}\left(\overline{c}_{22}+\overline{c}_{44}\right)\left[-\overline{c}_{11}\overline{c}_{44}\left(\overline{c}_{14}\overline{c}_{21}\overline{c}_{42}-\overline{c}_{11}\overline{c}_{23}\overline{c}_{32}\right)-\overline{c}_{11}^2\overline{c}_{24}\overline{c}_{42}\right]-\overline{c}_{23}\left[\overline{c}_{11}\overline{c}_{22}\overline{c}_{32}\overline{c}_{44}^2+\overline{c}_{12}\overline{c}_{14}\overline{c}_{21}\overline{c}_{31}\overline{c}_{42}\right]\\
&\quad+\left(\overline{c}_{11}+\overline{c}_{22}\right)\Big[-\overline{c}_{11}\overline{c}_{44}^2\left(\overline{c}_{12}\overline{c}_{21}+\overline{c}_{23}\overline{c}_{32}\right)-\overline{c}_{11}^2\overline{c}_{23}\overline{c}_{32}\left(\overline{c}_{22}+2\overline{c}_{44}\right)+\overline{c}_{11}\overline{c}_{42}\overline{c}_{44}\left(\overline{c}_{14}\overline{c}_{21}-\overline{c}_{22}\overline{c}_{24}\right)+\\
&\quad\overline{c}_{23}\overline{c}_{31}\left(\overline{c}_{14}\overline{c}_{42}\overline{c}_{44}-\overline{c}_{12}\overline{c}_{44}^2\right)\Big]+\Big[\overline{c}_{12}\overline{c}_{21}\left(\overline{c}_{11}+\overline{c}_{44}\right)+\overline{c}_{12}\overline{c}_{23}\overline{c}_{31}+\overline{c}_{22}\left(\overline{c}_{23}\overline{c}_{32}-\overline{c}_{11}\overline{c}_{44}\right)+\left(\overline{c}_{22}+\overline{c}_{44}\right)\\
&\quad\left(\overline{c}_{24}\overline{c}_{42}-\overline{c}_{22}\overline{c}_{44}\right)-\overline{c}_{11}\left(\overline{c}_{22}^2+\overline{c}_{44}^2\right)\Big]\left[\overline{c}_{42}\left(\overline{c}_{11}\overline{c}_{24}-\overline{c}_{14}\overline{c}_{21}\right)+\overline{c}_{44}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{12}\overline{c}_{21}-\overline{c}_{11}\overline{c}_{44}\right)\right]+\overline{c}_{11}\overline{c}_{22}^2\overline{c}_{44}\\
&\quad\left(\overline{c}_{11}^2+\overline{c}_{22}\overline{c}_{44}\right)+\overline{c}_{11}^2\overline{c}_{23}\overline{c}_{32}\left(\overline{c}_{12}\overline{c}_{21}-\overline{c}_{22}\overline{c}_{44}\right)+\overline{c}_{11}\overline{c}_{22}\overline{c}_{23}\overline{c}_{32}\left[-\overline{c}_{22}\left(\overline{c}_{11}+\overline{c}_{22}\right)+\overline{c}_{23}\overline{c}_{32}+\overline{c}_{12}\overline{c}_{21}+\overline{c}_{24}\overline{c}_{42}\right]+\\
&\quad\overline{c}_{23}\overline{c}_{32}\left[\overline{c}_{11}\overline{c}_{21}\overline{c}_{23}\overline{c}_{32}\right]+\overline{c}_{23}\overline{c}_{32}\left[\overline{c}_{11}\overline{c}_{24}\overline{c}_{42}\overline{c}_{44}+\overline{c}_{12}\overline{c}_{31}\left(\overline{c}_{11}\overline{c}_{23}-\overline{c}_{14}\overline{c}_{21}\overline{c}_{42}\right)\right]+\overline{c}_{14}\overline{c}_{21}\overline{c}_{42}\overline{c}_{44}\left(\overline{c}_{22}\overline{c}_{44}-\overline{c}_{24}\overline{c}_{42}\right)+\\
&\quad\overline{c}_{23}\overline{c}_{31}\left(\overline{c}_{11}^2+\overline{c}_{22}^2+\overline{c}_{44}^2\right)\left(\overline{c}_{14}\overline{c}_{42}-\overline{c}_{12}\overline{c}_{44}\right)+2\overline{c}_{11}\overline{c}_{22}\overline{c}_{23}\overline{c}_{31}\left(\overline{c}_{14}\overline{c}_{42}-\overline{c}_{12}\overline{c}_{44}\right),\\
r_6 &= -\Big\{-\overline{c}_{12}\overline{c}_{23}\overline{c}_{31}\left[\overline{c}_{11}\overline{c}_{44}\left(\overline{c}_{11}+\overline{c}_{44}\right)+\left(\overline{c}_{11}+\overline{c}_{22}\right)\left(\overline{c}_{11}\overline{c}_{22}-\overline{c}_{12}\overline{c}_{21}\right)+\left(\overline{c}_{22}+\overline{c}_{44}\right)\left(\overline{c}_{22}\overline{c}_{44}-\overline{c}_{24}\overline{c}_{42}\right)-\right.\\
&\quad\left.\overline{c}_{23}\left(\overline{c}_{22}\overline{c}_{32}+\overline{c}_{12}\overline{c}_{31}\right)-\overline{c}_{11}\overline{c}_{22}\left(\overline{c}_{42}+2\overline{c}_{44}\right)\right]-\overline{c}_{14}\overline{c}_{21}\overline{c}_{42}\left[\overline{c}_{44}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{12}\overline{c}_{21}-\overline{c}_{11}\overline{c}_{22}\right)+\overline{c}_{11}\overline{c}_{23}\overline{c}_{32}+\right.\\
&\quad\left.\overline{c}_{42}\left(\overline{c}_{11}\overline{c}_{24}-\overline{c}_{14}\overline{c}_{21}\right)\right]-\overline{c}_{11}\overline{c}_{23}\overline{c}_{32}\overline{c}_{44}\left[\overline{c}_{11}^2+\overline{c}_{22}^2+\overline{c}_{44}^2+2\overline{c}_{11}\overline{c}_{22}+2\overline{c}_{44}\left(\overline{c}_{11}+\overline{c}_{22}\right)\right]\Big\},
\end{aligned}$$

$$\begin{aligned}
\overline{\alpha}_1 &= 2\frac{M_3}{M_1}\left\{M_3\left[\frac{\overline{p}}{u_4+\overline{p}}\overline{c}_{31}-\frac{u_4\overline{h}}{(u_4+\overline{p})^2}\left(\overline{c}_{22}+\overline{c}_{44}\right)\right]-M_2\left[\frac{\overline{p}}{(u_4+\overline{p})^2}\left(\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)-\overline{c}_{21}\overline{c}_{32}\right)+\right.\right.\\
&\quad\left.\left.\frac{u_4\overline{h}}{(u_4+\overline{p})^2}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{24}\overline{c}_{42}-\overline{c}_{22}\overline{c}_{44}\right)\right]+2\left(\frac{M_3}{M_1}\right)^2\frac{u_4\overline{h}}{(u_4+\overline{p})^2}\right\}\\
\overline{\alpha}_2 &= -2\left\{M_3\left[\frac{u_4\overline{h}}{(u_4+\overline{p})^2}\overline{c}_{23}\overline{c}_{32}\overline{c}_{44}-\frac{\overline{p}}{u_4+\overline{p}}\left(\overline{c}_{44}\left(\overline{c}_{21}\overline{c}_{32}-\overline{c}_{22}\overline{c}_{31}\right)+\overline{c}_{24}\overline{c}_{31}\overline{c}_{42}\right)\right]\right\}-\left(\frac{M_3}{M_1}\right)^2\\
&\quad\left[\frac{u_4\overline{h}}{(u_4+\overline{p})^2}M_2-2\left[\frac{\overline{p}}{(u_4+\overline{p})^2}\left(\overline{c}_{31}\left(\overline{c}_{22}+\overline{c}_{44}\right)-\overline{c}_{21}\overline{c}_{32}\right)+\frac{u_4\overline{h}}{(u_4+\overline{p})^2}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{24}\overline{c}_{42}-\overline{c}_{22}\overline{c}_{44}\right)\right]\right]
\end{aligned}$$

Such that $\overline{c}_{ij}$ $(i,j=1,2,3,4)$ *it was mentioned in [9].*
Then at the parameter $(\overline{u}_2 = u_2)$ *, system* (2.2) *has a Hopf- bifurcation near* E_3.

Proof . *The characteristic equation of system* (2.2) *at* E_3 *mentioned in local stability in [9].*

$$\lambda^4 + M_1\lambda^3 + M_2\lambda^2 + M_3\lambda + M_4 = 0 \tag{4.7}$$

We requirement to find the parameter ($\overline{u}_2$) *so to check the necessary and sufficient conditions for Hopf bifurcation to happen which accept*

$$M_i\ (\overline{u}_2) > 0,\ \Delta_1(\overline{u}_2) = M_1M_2 - M_3 > 0\ \ ,\ M_1^3(\overline{u}_2) - 4\Delta_1(\overline{u}_2) > 0$$
$$and\ \ \Delta_2 = (M_1M_2 - M_3)\, M_3 - M_1^2M_4 = 0$$

Now, Contingent on conditions (3.7b-3.7f) as in [9] with conditions (4.1) *and* (4.2), *we get* $M_i\ (\overline{u}_2) > 0, \quad (i = 1, 3, 4)\ , \Delta_1(\overline{u}_2) = M_1M_2 - M_3 > 0$ *and* $M_1^3(\overline{u}_2) - 4\Delta_1(\overline{u}_2) > 0$,
Notes that $\Delta_2 = 0$, *gives:*

$$\Gamma_1\overline{p}^2{\overline{u}_2}^2 + \Gamma_2\overline{pu}_2 + \Gamma_3 = 0, \tag{4.8}$$

where, $\Gamma_1 = r_1 - r_2,\ \ \Gamma_2 = r_3 - r_4$, *and* $\Gamma_3 = r_5 - r_6$. *Where;* r_i , $(\ i = 1 - 9)$ *are above-mentioned in the theorem.* $\Gamma_1 < 0,\ \Gamma_2 < 0$ *and* $\Gamma_3 > 0$, *contingent on conditions (3.7b-3.7f) as in [9] with conditions* (4.3)-(4.5).
Via to using Descartes rule of sign, equation (4.8) *has a unique positive root*
Now, at($\overline{u}_2 = u_2$), *the characteristic equation* (4.7) *can be rewritten as:*

$$P_4(\lambda) = \left(\lambda^2 + \frac{M_3}{M_1}\right)\left(\lambda^2 + M_1\lambda + \frac{\Delta_1}{M_1}\right) = 0,$$

Which; have four roots:

$$\lambda_{h,v} = \left[-M_1 \mp \sqrt{M_1^2 - 4\frac{\Delta_1}{M_1}}\right] \quad and\ \lambda_{p,s} = \pm i\sqrt{\frac{M_3}{M_1}},$$

At $(u_2 = \overline{u}_2)$, *there are two pure imaginary eigenvalues* $(\lambda_{p,s})$ *and two eigenvalue* $(\lambda_{h,v})$, *which is real and negative.*
Now, for all values of u_2 *in the neighborhood of* $\overline{u}_2$, *the roots in general of the following form:* $\lambda_{p,s} = \delta_1(u_2) \pm i\delta_2(u_2)$ *and* $\lambda_{h,v} = \left[-M_1 \mp \sqrt{M_1^2 - 4\frac{\Delta_1}{M_1}}\right]$
Thus, $Re\,(\lambda_{p,s}(u_2))\,|_{u_2=\overline{u}_2} = \delta_1(\overline{u}_2) = 0$, *that implies the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at* $(\overline{u}_2 = u_2)$
Now, to check the transversally condition we ought to prove that:

$$\overline{\Psi}(\overline{u}_2)\,\overline{\Theta}(\overline{u}_2) + \overline{\Gamma}(\overline{u}_2)\,\overline{\Phi}(\overline{u}_2) \neq 0.$$

Note that for $(\overline{u}_2 = u_2)$ *we have:*
$\delta_1(\overline{u}_2) = 0$ *and* $\delta_2(\overline{u}_2) = \sqrt{\frac{M_3}{M_1}}$, *changing to the value of* δ_2 *yield the following:*

$$\overline{\Psi}(\overline{u}_2) = -2M_3;\ \overline{\Phi}(\overline{u}_2) = 2\ \sqrt{\frac{M_3}{M_1}}\left[M_2 - 2\frac{M_3}{M_1}\right],$$

$$\overline{\Theta}(\overline{u}_2) = \frac{u_4\overline{h}}{(u_4+\overline{p})^2}\overline{c}_{23}\overline{c}_{32}\overline{c}_{44} - \frac{\overline{p}}{u_4+\overline{p}}\left[\overline{c}_{44}(\overline{c}_{21}\overline{c}_{32} - \overline{c}_{22}\overline{c}_{31}) - \overline{c}_{24}\overline{c}_{31}\overline{c}_{42}\right] - \frac{M_3}{M_1}$$
$$\left[\frac{\overline{p}}{u_4+\overline{p}}\overline{c}_{31} - \frac{u_4\overline{h}}{(u_4+\overline{p})^2}(\overline{c}_{22} + \overline{c}_{44})\right]$$

and

$$\overline{\Gamma}(\overline{u}_2) = -\sqrt{\frac{M_3}{M_1}}\left[\frac{\overline{p}}{u_4+\overline{p}}\left[\overline{c}_{31}\left(\overline{c}_{22}+\ \overline{c}_{44}\right)-\overline{c}_{21}\overline{c}_{32}\right]+\frac{u_4\overline{h}}{\left(u_4+\overline{p}\right)^2}\left(\overline{c}_{23}\overline{c}_{32}+\overline{c}_{24}\overline{c}_{42}-\overline{c}_{22}\overline{c}_{44}\right)+\frac{u_4\overline{h}}{\left(u_4+\overline{p}\right)^2}\frac{M_3}{M_1}\right],$$

Thus, under authority of conditions (3.7c), (3.7d) and (3.7e) in [9] with condition (4.6)*, give us:*

$$\Psi(\overline{u}_2)\,\Theta(\overline{u}_2)+\Gamma(\overline{u}_2)\,\Phi(\overline{u}_2)=\overline{\alpha}_1-\overline{\alpha}_2\neq 0,$$

Therefore system (2.2) *at* E_3 *with the parameter* $\overline{u}_2$ *has a Hopf-bifurcation.* □

Theorem 4.2. *Suppose that the locally conditions (3.9a-3.9d) as in [9] and the following conditions hold:*

$$1<2\bar{\bar{p}}+\frac{u_3u_5\bar{\bar{v}}}{\left(u_5+\bar{\bar{p}}\right)^2}, \tag{4.9}$$

$$\bar{\bar{d}}_{14}\bar{\bar{d}}_{21}\bar{\bar{d}}_{42}>\bar{\bar{d}}_{44}\left(\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}+\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}\right)+\bar{\bar{d}}_{12}\bar{\bar{d}}_{24}\bar{\bar{d}}_{41}, \tag{4.10}$$

$$j^2\bar{\bar{d}}_{44}<j\left(\bar{\bar{d}}_{44}^2-\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{12}\bar{\bar{d}}_{21}\right)+\left(-\bar{\bar{d}}_{44}\left(\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}+\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}\right)-\bar{\bar{d}}_{12}\bar{\bar{d}}_{24}d_{41}-\bar{\bar{d}}_{14}\bar{\bar{d}}_{21}\bar{\bar{d}}_{42}\right) \tag{4.11}$$

$$\bar{\bar{\alpha}}_1\neq\bar{\bar{\alpha}}_2, \tag{4.12}$$

Where,

$$\begin{aligned}
j&=1-2\bar{\bar{p}}-\frac{u_3u_5\bar{\bar{v}}}{\left(u_5+\bar{\bar{p}}\right)^2},\\
\bar{\bar{\alpha}}_1&=2N_2\left[\bar{\bar{s}}N_2-\bar{\bar{d}}_{24}\bar{\bar{d}}_{41}\bar{\bar{p}}\right]+2N_1N_2\left[\bar{\bar{d}}_{21}\bar{\bar{p}}-\bar{\bar{d}}_{44}\bar{\bar{s}}\right]\\
\bar{\bar{\alpha}}_2&=-2N_2\left[-\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}\bar{\bar{s}}-\bar{\bar{d}}_{21}\bar{\bar{d}}_{44}\bar{\bar{p}}\right]
\end{aligned}$$

Such that $\bar{\bar{d}}_{ij}$ *and* N_i, $(\ i,j=1,2,3,4)$ *it was mentioned in [9]*
Then at the parameter $(\bar{\bar{u}}_1=u_1)$, *system* (2.2) *has a Hopf- bifurcation near* E_4.
Proof . *The characteristic equation of system* (2.2) *at* E_4 *mentioned in local stability in [9].*

$$\left(\bar{\bar{d}}_{33}-\lambda\right)\left[\lambda^3+N_1\lambda^2+N_2\lambda+N_3\right]=0 \tag{4.13}$$

The requirement to find the parameter $(\bar{\bar{u}}_1)$ *for checking the necessary and sufficient conditions for Hopf bifurcation to crop up that satisfy,* $N_i\ (\bar{\bar{u}}_1)>0,\quad i=1,2$. *and* $\Delta_1(\bar{\bar{u}}_1)=N_1N_2-N_3=0$
Under authority of (3.9b-3.9d) in [9], $N_i\ (\bar{\bar{u}}_1)>0,\ (i=1,2)\,,\ \Delta(\bar{\bar{u}}_1)=0$, *gives:*

$$R_1\bar{\bar{u}}_1^2\bar{\bar{s}}^2+R_2\bar{\bar{u}}_1\bar{\bar{s}}+R_3=0, \tag{4.14}$$

where, $R_1=\bar{\bar{d}}_{44}$, $R_2=\left(\bar{\bar{d}}_{44}^2-\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{12}\bar{\bar{d}}_{21}\right)-2j\bar{\bar{d}}_{44}$, *and* $R_3=j^2\bar{\bar{d}}_{44}-j\left(\bar{\bar{d}}_{44}^2-\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}-\bar{\bar{d}}_{12}\bar{\bar{d}}_{21}\right)+\left(\bar{\bar{d}}_{44}\left(\bar{\bar{d}}_{14}\bar{\bar{d}}_{41}+\bar{\bar{d}}_{24}\bar{\bar{d}}_{42}\right)+\bar{\bar{d}}_{12}\bar{\bar{d}}_{24}d_{41}+\bar{\bar{d}}_{14}\bar{\bar{d}}_{21}\bar{\bar{d}}_{42}\right)$.
$R_1<0$, *under the authority of condition (3.9c) as in [9] and* $R_3>0$, *under the authority of condition (3.9c) as in [9] with conditions* (4.9)-(4.12) *are hold.*
Via to using Descartes rule of sign, equation (4.14) *has a unique positive root.*
Now, at$(u_1=\bar{\bar{u}}_1)$, *the characteristic equation* (4.13) *can be rewritten as:*

$$P_4(\lambda)=\left(\bar{\bar{d}}_{44}-\lambda_{4h}\right)\left(\lambda_{41}+N_1\right)\left(\lambda_{42}^2+N_2\right)=0, \tag{4.15}$$

Which; have four roots:
$\lambda_{4h} = \bar{\bar{d}}_{44},\ \lambda_{41} = -N_1,\ \lambda_{42} = i\sqrt{N_2},\ \lambda_{43} = -i\sqrt{N_2}$
At $(u_1 = \bar{\bar{u}}_1)$, *there are two pure imaginary eigenvalues* $(\lambda_{42}$ *and* $\lambda_{43})$ *and two eigenvalue* $(\lambda_{4h}$ *and* $\lambda_{41})$, *which is real and negative according condition (3.9a-3.9d) as in [9].*
Now for all values of u_1 *in the neighborhood of* $\bar{\bar{u}}_1$, *the roots in general of the following form:* $\lambda_{2,3} = \delta_1(u_1) \pm i\delta_2(u_1)$ *and* $\lambda_{4h} = \bar{\bar{d}}_{44},\ \lambda_{41} = -N_1$
Clearly, $Re(\lambda_{2,3}(u_1))|_{u_1=\bar{\bar{u}}_1} = \delta_1(\bar{\bar{u}}_1) = 0$, *that means the necessary and sufficient conditions for Hopf-bifurcation is satisfied at* $(\bar{\bar{u}}_1 = u_1)$.
Now, to chuck the transversally condition, we must prove that:

$$\bar{\bar{\Psi}}(\bar{\bar{u}}_1)\,\bar{\bar{\Theta}}(\bar{\bar{u}}_1) + \bar{\bar{\Gamma}}(\bar{\bar{u}}_1)\,\bar{\bar{\Phi}}(\bar{\bar{u}}_1) \neq 0$$

Note that for $(\bar{\bar{u}}_1 = u_1)$ *we have:* $\delta_1(\bar{\bar{u}}_1) = 0$ *and* $\delta_2(\bar{\bar{u}}_1) = \sqrt{N_2}$
substituting the value of δ_2 *gives the following simplifications:*

$$\begin{aligned}
\bar{\bar{\Psi}}(\bar{\bar{u}}_1) &= -2N_2;\ \bar{\bar{\Phi}}(\bar{\bar{u}}_1) = 2\ N_1\sqrt{N_2},\\
\bar{\bar{\Theta}}(\bar{\bar{u}}_1) &= -\bar{\bar{s}}\left[\bar{\bar{d}}_{24}\bar{\bar{d}}_{42} - N_2\right] - \bar{\bar{p}}\left(\bar{\bar{d}}_{21}\bar{\bar{d}}_{44} - \bar{\bar{d}}_{24}\bar{\bar{d}}_{41}\right),
\end{aligned}$$

and

$$\bar{\bar{\Gamma}}(\bar{\bar{u}}_1) = \sqrt{N_2}\left[\bar{\bar{d}}_{21}\bar{\bar{p}} - \bar{\bar{s}}\bar{\bar{d}}_{44}\right],$$

Under the authority of conditions (3.9c) in [9] and condition (4.12), *give us:*

$$\bar{\bar{\Psi}}(\bar{\bar{u}}_1)\,\bar{\bar{\Theta}}(\bar{\bar{u}}_1) + \bar{\bar{\Gamma}}(\bar{\bar{u}}_1)\,\bar{\bar{\Phi}}(\bar{\bar{u}}_1) = \bar{\bar{\alpha}}_1 - \bar{\bar{\alpha}}_2 \neq 0,$$

Therefore system (2.2) *at* E_4 *with the parameter* $\bar{\bar{u}}_1$ *has a Hopf-bifurcation.* □

Theorem 4.3. *Suppose that the locally conditions (14a-14g) as in [9] and the following conditions hold:*

$$K_1K_2 > K_3, \tag{4.16}$$

$$0 < \Delta_1(\widetilde{u}_9) < \frac{K_1^3(\widetilde{u}_9)}{4}, \tag{4.17}$$

$$t_1 < t_2, \tag{4.18}$$

$$t_3 > t_4, \tag{4.19}$$

$$t_5 > t_6, \tag{4.20}$$

$$t_7 > r_8, \tag{4.21}$$

$$3(t_1 - t_2) < -(t_3 - t_4), \tag{4.22}$$

$$(t_7 - r_8) > -3(t_1 - t_2)l^2 - 2(t_3 - t_4)l, \tag{4.23}$$

$$(t_1 - t_2)l^3 + (t_7 - r_8) > -(t_3 - t_4)l^2 - (t_4 - t_5)l, \tag{4.24}$$

$$\widetilde{\alpha}_1 \neq \widetilde{\alpha}_2 \tag{4.25}$$

Where,

$$t_1 = \widetilde{e}_{21}\left(\widetilde{e}_{11}+\widetilde{e}_{22}\right)\left(\widetilde{e}_{21}\widetilde{e}_{44}-\widetilde{e}_{31}\widetilde{e}_{32}\right),$$

$$t_2 = -\left\{\widetilde{e}_{13}\widetilde{e}_{32}\left[\widetilde{e}_{13}\widetilde{e}_{32}-\widetilde{e}_{21}\widetilde{e}_{44}\right]\right\},$$

$$\begin{aligned} t_3 =& \left(\widetilde{e}_{13}\widetilde{e}_{31}\widetilde{e}_{32}\right)\left[\widetilde{e}_{31}\left(\widetilde{e}_{22}+2\widetilde{e}_{44}\right)+\widetilde{e}_{11}\widetilde{e}_{44}^{2}-\widetilde{e}_{21}\widetilde{e}_{32}\right]+\widetilde{e}_{31}\widetilde{e}_{32}\left[\widetilde{e}_{32}\left(\widetilde{e}_{11}\widetilde{e}_{23}-\widetilde{e}_{13}\widetilde{e}_{21}\right)+\widetilde{e}_{44}\left(\widetilde{e}_{23}\widetilde{e}_{32}-\widetilde{e}_{11}\widetilde{e}_{22}\right)\right.\\ &\left.+\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{41}+\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{42}\right]+\widetilde{e}_{32}\left[\widetilde{e}_{13}\widetilde{e}_{21}^{2}\widetilde{e}_{44}+\widetilde{e}_{31}\left(\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{41}+\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{42}\right)\right], \end{aligned}$$

$$t_4 = -\left\{-\left(\widetilde{e}_{11}+\widetilde{e}_{22}\right)\left[\widetilde{e}_{13}\widetilde{e}_{21}\widetilde{e}_{31}\left(\widetilde{e}_{22}+\widetilde{e}_{44}\right)+\widetilde{e}_{21}\widetilde{e}_{44}\left(\widetilde{e}_{23}\widetilde{e}_{23}-\widetilde{e}_{22}^{2}\right)+\widetilde{e}_{21}\left(\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{41}+\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{42}\right)\right]\right\},$$

$$\begin{aligned} t_5 =& \left(\widetilde{e}_{11}+\widetilde{e}_{22}\right)\left[\widetilde{e}_{13}\widetilde{e}_{31}\left(\widetilde{e}_{22}\left(\widetilde{e}_{11}+\widetilde{e}_{22}\right)-2\widetilde{e}_{24}\widetilde{e}_{42}\right)+\widetilde{e}_{23}\widetilde{e}_{32}\left(\widetilde{e}_{11}^{2}-2\widetilde{e}_{14}\widetilde{e}_{41}\right)-\widetilde{e}_{11}\widetilde{e}_{13}\widetilde{e}_{21}\widetilde{e}_{32}\right]+\widetilde{e}_{13}\widetilde{e}_{31}\\ &\left[\widetilde{e}_{32}\left(2\widetilde{e}_{12}\widetilde{e}_{31}-\widetilde{e}_{13}\widetilde{e}_{21}\right)-2\widetilde{e}_{11}\widetilde{e}_{22}\right]+\widetilde{e}_{23}\widetilde{e}_{32}\left[\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{31}-\widetilde{e}_{13}\widetilde{e}_{21}\right)-\widetilde{e}_{11}\left(\widetilde{e}_{11}\widetilde{e}_{22}-\widetilde{e}_{12}\widetilde{e}_{31}-\widetilde{e}_{13}\widetilde{e}_{21}\right)\right]+\\ &\widetilde{e}_{11}\left[\widetilde{e}_{24}\widetilde{e}_{42}\left(\widetilde{e}_{14}\widetilde{e}_{24}\widetilde{e}_{42}-2\widetilde{e}_{13}\widetilde{e}_{31}\right)-\widetilde{e}_{13}\widetilde{e}_{21}\widetilde{e}_{22}\widetilde{e}_{32}\right]+\widetilde{e}_{12}\widetilde{e}_{21}\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{31}+\widetilde{e}_{13}\widetilde{e}_{21}\right)-\widetilde{e}_{14}\widetilde{e}_{41}, \end{aligned}$$

$$\begin{aligned} t_6 =& -\{-(\widetilde{e}_{11}+\widetilde{e}_{22})\left[\widetilde{e}_{13}\widetilde{e}_{31}\left(\widetilde{e}_{22}(\widetilde{e}_{11}+\widetilde{e}_{22})+\widetilde{e}_{12}\widetilde{e}_{21}-\widetilde{e}_{11}\widetilde{e}_{12}\right)+\widetilde{e}_{11}\widetilde{e}_{32}(2\widetilde{e}_{12}\widetilde{e}_{31}+\widetilde{e}_{13}\widetilde{e}_{21}-\widetilde{e}_{14}(\widetilde{e}_{22}+2\widetilde{e}_{22}))\right.\\ &\left.-\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{41}(\widetilde{e}_{11}+\widetilde{e}_{22})+\widetilde{e}_{14}(\widetilde{e}_{24}\widetilde{e}_{41}\widetilde{e}_{42})-\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{42}(\widetilde{e}_{11}-\widetilde{e}_{22})-\widetilde{e}_{12}\widetilde{e}_{23}\widetilde{e}_{31}(\widetilde{e}_{11}+\widetilde{e}_{22})\right]\\ &-\widetilde{e}_{13}\widetilde{e}_{31}\left[\widetilde{e}_{22}(2\widetilde{e}_{24}\widetilde{e}_{42}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{41}\widetilde{e}_{41})-\widetilde{e}_{11}(\widetilde{e}_{21}\widetilde{e}_{12}+\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}+\widetilde{e}_{24}\widetilde{e}_{42})\right]\\ &-\widetilde{e}_{11}\left[\widetilde{e}_{14}\widetilde{e}_{24}^{2}\widetilde{e}_{42}^{2}-\widetilde{e}_{11}\widetilde{e}_{22}\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}\right.\right.\\ &\left.\left.+2\widetilde{e}_{12}\widetilde{e}_{21}-\widetilde{e}_{22}^{2}\right)+\widetilde{e}_{22}^{2}\left(\widetilde{e}_{22}+\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{24}\widetilde{e}_{42}\right)+2\widetilde{e}_{13}\widetilde{e}_{24}\widetilde{e}_{32}\widetilde{e}_{41}+2\widetilde{e}_{14}\widetilde{e}_{23}\widetilde{e}_{31}\widetilde{e}_{42}\right]\\ &-\widetilde{e}_{22}\left[\widetilde{e}_{23}\left(\widetilde{e}_{11}\widetilde{e}_{32}+2\widetilde{e}_{14}\widetilde{e}_{31}\widetilde{e}_{42}\right)+\widetilde{e}_{13}\widetilde{e}_{32}\ \ (2\widetilde{e}_{24}\widetilde{e}_{41}-\widetilde{e}_{21})\right]\}, \end{aligned}$$

$$\begin{aligned} t_7 =& \widetilde{e}_{11}\left(\widetilde{e}_{11}+\widetilde{e}_{22}\right)\left[-\widetilde{e}_{22}\left(\widetilde{e}_{14}\widetilde{e}_{41}+\widetilde{e}_{13}\widetilde{e}_{31}\right)+\widetilde{e}_{23}\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{21}-\widetilde{e}_{11}\widetilde{e}_{22}\right)+\widetilde{e}_{42}\left(\widetilde{e}_{14}\widetilde{e}_{23}\widetilde{e}_{31}-e_{22}e_{24}\right)\right]+\\ &\widetilde{e}_{13}\widetilde{e}_{22}\widetilde{e}_{31}\left[\widetilde{e}_{22}\left(\widetilde{e}_{24}\widetilde{e}_{42}+\widetilde{e}_{12}\widetilde{e}_{21}\right)+\widetilde{e}_{11}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}+\widetilde{e}_{22}\widetilde{e}_{23}e_{31}\right)\right]+\widetilde{e}_{12}\widetilde{e}_{31}\left[\widetilde{e}_{22}^{2}\left(\widetilde{e}_{13}\widetilde{e}_{21}+\right.\right.\\ &\left.\left.\widetilde{e}_{11}\widetilde{e}_{32}\right)+\widetilde{e}_{32}\left[\widetilde{e}_{11}\left(\widetilde{e}_{11}\widetilde{e}_{22}-\widetilde{e}_{13}\widetilde{e}_{31}-\widetilde{e}_{12}\widetilde{e}_{21}-\widetilde{e}_{14}\widetilde{e}_{41}\right)-\widetilde{e}_{22}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)\right]\right]+\widetilde{e}_{13}\widetilde{e}_{21}\widetilde{e}_{32}\\ &\left[\widetilde{e}_{11}\left(\widetilde{e}_{11}\widetilde{e}_{22}-\widetilde{e}_{13}\widetilde{e}_{31}-\widetilde{e}_{14}\widetilde{e}_{41}-\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{22}^{2}\right)-\widetilde{e}_{22}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)\right]\\ &+\widetilde{e}_{11}\widetilde{e}_{23}\widetilde{e}_{32}\left[\widetilde{e}_{11}\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}\right)+\widetilde{e}_{22}\left(\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)\right]+\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{41}\left[\widetilde{e}_{11}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}\right)+\right.\\ &\left.3\widetilde{e}_{22}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)\right]+\widetilde{e}_{11}^{2}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}\right)+\widetilde{e}_{24}\widetilde{e}_{42}\left[\widetilde{e}_{22}\left[\widetilde{e}_{11}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)\right.\right.\\ &\left.\left.-\widetilde{e}_{13}\widetilde{e}_{31}\right]\right]+\widetilde{e}_{22}^{2}\left(\widetilde{e}_{12}\widetilde{e}_{21}+\widetilde{e}_{23}\widetilde{e}_{32}+\widetilde{e}_{24}\widetilde{e}_{42}\right)+\widetilde{e}_{14}\widetilde{e}_{22}\widetilde{e}_{23}\left[\widetilde{e}_{22}\widetilde{e}_{42}-\widetilde{e}_{32}\widetilde{e}_{41}+\widetilde{e}_{11}\widetilde{e}_{31}\widetilde{e}_{42}\right], \end{aligned}$$

$$\begin{aligned} t_8 =& -\left\{-e_{31}e_{32}\left[\widetilde{e}_{13}\widetilde{e}_{22}\left(\widetilde{e}_{12}+\widetilde{e}_{13}\widetilde{e}_{21}\right)-\widetilde{e}_{21}\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{31}+\widetilde{e}_{12}\widetilde{e}_{13}\right)-\widetilde{e}_{12}\widetilde{e}_{13}\widetilde{e}_{21}\right]-\widetilde{e}_{13}\widetilde{e}_{21}\widetilde{e}_{32}^{2}\left[\widetilde{e}_{11}\widetilde{e}_{23}-\widetilde{e}_{13}\widetilde{e}_{21}\right]-\right.\\ &\widetilde{e}_{14}\widetilde{e}_{41}\left[\widetilde{e}_{22}\widetilde{e}_{24}\widetilde{e}_{42}+\widetilde{e}_{22}\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{31}+\widetilde{e}_{13}\widetilde{e}_{21}\right)-\widetilde{e}_{11}\widetilde{e}_{23}\widetilde{e}_{32}\left(\widetilde{e}_{11}+2\widetilde{e}_{22}\right)\right]-\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{42}\left[\widetilde{e}_{32}\left(\widetilde{e}_{12}\widetilde{e}_{31}+\widetilde{e}_{13}\widetilde{e}_{21}\right)-\right.\\ &\left.\left.\widetilde{e}_{11}\widetilde{e}_{13}\widetilde{e}_{31}\right]+\widetilde{e}_{22}\left[\widetilde{e}_{13}\widetilde{e}_{22}\widetilde{e}_{24}\left(\widetilde{e}_{32}\widetilde{e}_{41}-\widetilde{e}_{31}\widetilde{e}_{42}\right)+2\widetilde{e}_{11}\widetilde{e}_{24}\widetilde{e}_{32}\widetilde{e}_{41}\right]\right\}, \end{aligned}$$

$$\begin{aligned} \widetilde{\alpha}_1 =& 2\left\{\widetilde{s}\left[\left(\frac{k_3}{k_1}\right)^2\left(k_2-2\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{23}\widetilde{e}_{32}\right)\right)-k_3\left(\widetilde{e}_{13}\widetilde{e}_{32}\left(\widetilde{e}_{21}+\widetilde{e}_{41}\right)-\widetilde{e}_{11}\widetilde{e}_{23}\widetilde{e}_{32}\right)\right]+\right.\\ &\left.\widetilde{v}\left[k_3\left(\widetilde{e}_{14}\widetilde{e}_{23}\widetilde{e}_{31}-\widetilde{e}_{11}\frac{k_3}{k_1}\right)-k_2\frac{k_3}{k_1}\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{41}\right)-2\left(\frac{k_3}{k_1}\right)^3\right]\right\}, \end{aligned}$$

$$\begin{aligned} \widetilde{\alpha}_2 =& -2\left\{\widetilde{s}\left[k_3\left(\widetilde{e}_{12}\widetilde{e}_{23}\widetilde{e}_{31}-\widetilde{e}_{11}\frac{k_3}{k_1}\right)-k_2\frac{k_3}{k_1}\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{23}\widetilde{e}_{32}\right)+2\left(\frac{k_3}{k_1}\right)^3\right]+\right.\\ &\left.\left(\frac{k_3}{k_1}\right)^2\widetilde{v}\left[k_2-2\left(\widetilde{e}_{13}\widetilde{e}_{31}+\widetilde{e}_{14}\widetilde{e}_{44}\right)\right]\right\}, \end{aligned}$$

Such that $\widetilde{e}_{ij}$ $(\ i,j=1,2,3,4)$ *it was mentioned in [9]*
Then at the parameter $(\widetilde{u}_9 = u_9)$, *system* (2.2) *has a Hopf- bifurcation near* E_5.
Proof . *The characteristic equation of system* (2.2) *at* E_5 *mentioned in local stability in [9].*

$$\lambda^4 + K_1\lambda^3 + K_2\lambda^2 + K_3\lambda + M_4 = 0 \tag{4.26}$$

The requirement to find the parameter $(\widetilde{u}_9)$ *for checking the necessary and sufficient conditions for Hopf bifurcation to crop up that satisfy;* $K_i\ (\widetilde{u}_9) > 0, \quad (i = 1, 3, 4)\ , \quad \Delta_1 (\widetilde{u}_9) = K_1K_2 - K_3 > 0,$ $K_1^3 (\widetilde{u}_9) - 4\Delta_1 (\widetilde{u}_9) > 0,$ *and* $\Delta_2 (\widetilde{u}_9) = 0$ *.*
Under the authority of conditions (3.11a-3.11e) as in [9] with condition (4.16) *and* (4.17)*, implies that* $K_i\ (\widetilde{u}_9) > 0,\ (i = 1, 3,\ 4)\ ,\ \Delta_1 (\widetilde{u}_9) > 0,\ K_1^3 (\widetilde{u}_9) - 4\Delta_1 (\widetilde{u}_9) > 0$
Notes that, $\Delta_2 = 0$ *gives:*

$$\Lambda_1 \overline{s}^3 \widetilde{u}_9^3 + \Lambda_2 \overline{s}^2 \widetilde{u}_9^2 + \Lambda_3 \widetilde{s} \widetilde{u}_9 + \Lambda_4 = 0 \tag{4.27}$$

where, $\Lambda_1 = t_1 - t_2,\ \Lambda_2 = 3(t_1 - t_2) + (t_3 - t_4),\ \Lambda_3 = 3(t_1 - t_2) l^2 + 2(t_3 - t_4) l + (t_5 - t_6),\ \Lambda_4 = (t_1 - t_2) l^3 + (t_3 - t_4) l^2 + (t_5 - t_6) l + (t_7 - t_8),$ *and*
$l = \frac{u_{15}\widetilde{p}}{u_5 + \widetilde{p}} - u_{14} < 0$ *, under authority of condition (3.11c) in [9].*
Note that $\Lambda_1 < 0\ ,\ \Lambda_2 < 0\ ,\ \Lambda_3 > 0$ *and* $\Lambda_4 > 0$ *under authority of conditions (3.11a), (3.11b) and (3.11d) in [9] with conditions* (4.18)-(4.24)*. Where* $t_i\ ,\ (\ i = 1 - 8)$ *are mentioned in the state theorem. Via to using Descartes rule of sign, equation* (4.27) *has a unique positive root.*
At $(\widetilde{u}_9 = u_9)$*, the characteristic equation* (4.26) *can be rewritten as:*
$P_4(\lambda) = \left(\lambda^2 + \frac{k_3}{k_1}\right)\left(\lambda^2 + k_1\lambda + \frac{\Delta_1}{k_1}\right) = 0,$
Which; have four roots:
$\lambda_{h,v} = \left[-k_1 \mp \sqrt{k_1^2 - 4\frac{\Delta_1}{k_1}}\right]$ *and* $\lambda_{p,s} = \pm i\sqrt{\frac{k_3}{k_1}},$
Observe that at$(\widetilde{u}_9 = u_9)$*, there are two pure imaginary eigenvalues* $(\lambda_{p,s})$ *and two eigenvalue* $(\lambda_{h,v})$*, which is real and negative.*
Now, for all values of u_9 *in the neighborhood of* $\widetilde{u}_9$*, the roots in general of the following form:*

$$\lambda_{p,s} = \delta_1 (u_9) \pm i\delta_2 (u_9) \quad \text{and} \quad \lambda_{h,v} = \left[-k_1 \mp \sqrt{k_1^2 - 4\frac{\Delta_1}{k_1}}\right]$$

Clearly, $Re\left(\lambda_{p,s}(u_6)\right)|_{u_9 = \widetilde{u}_9} = \delta_1 (\widetilde{u}_9) = 0$*, implies that the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at* $(\widetilde{u}_9 = u_9)$
Now, to check the transversally condition, we must prove that:
$\widetilde{\Psi}(\widetilde{u}_9)\,\widetilde{\Theta}(\widetilde{u}_9) + \widetilde{\Gamma}(\widetilde{u}_9)\,\widetilde{\Phi}(\widetilde{u}_9) \neq 0.$
Note that for $(\widetilde{u}_9 = u_9)$ *we have:*
$\delta_1 (\widetilde{u}_9) = 0$ *and* $\delta_2 (\widetilde{u}_9) = \sqrt{\frac{k_3}{k_1}}$*, substituting the value of* δ_2 *gives the following simplifications:*

$$\widetilde{\Psi}(\widetilde{u}_9) = -2k_3;\ \widetilde{\Phi}(\widetilde{u}_9) = 2\ \sqrt{\frac{k_3}{k_1}}\left[k_2(\widetilde{u}_9) - 2\frac{k_3}{k_1}\right],$$
$$\widetilde{\Theta}(\widetilde{u}_9) = \left\{\widetilde{s}\left[\widetilde{e}_{12}\widetilde{e}_{23}\widetilde{e}_{31} + \widetilde{e}_{13}\widetilde{e}_{32}(\widetilde{e}_{21} + \widetilde{e}_{41}) - \widetilde{e}_{11}\left(\widetilde{e}_{23}\widetilde{e}_{32} + \frac{k_3}{k_1}\right)\right] - \widetilde{v}\left(\widetilde{e}_{11}\frac{k_3}{k_1} + \widetilde{e}_{14}\widetilde{e}_{23}\widetilde{e}_{31}\right)\right\}$$

and

$$\widetilde{\Gamma}(\widetilde{u}_9) = \sqrt{\frac{k_3}{k_1}}\left[\widetilde{s}\left(\widetilde{e}_{13}\widetilde{e}_{31} + \widetilde{e}_{23}\widetilde{e}_{32} + \frac{k_3}{k_1}\right) - \widetilde{v}\left(\widetilde{e}_{13}\widetilde{e}_{31} + \widetilde{e}_{14}\widetilde{e}_{44} + \frac{k_3}{k_1}\right)\right],$$

Thus, under the authority of conditions (14a), (14b), (14d) and (14e) in [9] with condition (4.25)*, give us:*

$$\widetilde{\Psi}(\widetilde{u}_9)\,\widetilde{\Theta}(\widetilde{u}_9) + \widetilde{\Gamma}(\widetilde{u}_9)\,\widetilde{\Phi}(\widetilde{u}_9) = \widetilde{\alpha}_1 - \widetilde{\alpha}_2 \neq 0,$$

Therefore system (2.2) *at* E_5 *with the parameter* $\widetilde{u}_9$ *has a Hopf-bifurcation.* □

5. The Numerical Results

In this section, the benefit of this numerical simulations, which give us a clear explanation about the influence of alter the parameters values of system (2.2), as well as to assert analytical results. Now according to following parameters below

$$\begin{aligned} &u_1 = 0.2, \quad u_2 = 0.4, \quad u_3 = 0.3, \quad u_4 = 0.3, \quad u_5 = 0.1, \quad u_6 = 0.001, \\ &u_7 = 0.095, \quad u_8 = 0.95, \quad u_9 = 0.001, \quad u_{10} = 0.0001, \quad u_{11} = 0.00001, \\ &u_{12} = 0.0001, \quad u_{13} = 0.0001, \quad u_{14} = 0.001, \quad u_{15} = 0.0008 \end{aligned} \tag{5.1}$$

From Eq. (5.1) which represent the set of data starting from various initial values, it is observed the solution of system (2.2) approaches asymptotically to a positive equilibrium point $E_5 = (0.732,\ 0.1,\ 0.401,\ 0.381)$, which illustrated in Figure 1(a-d):

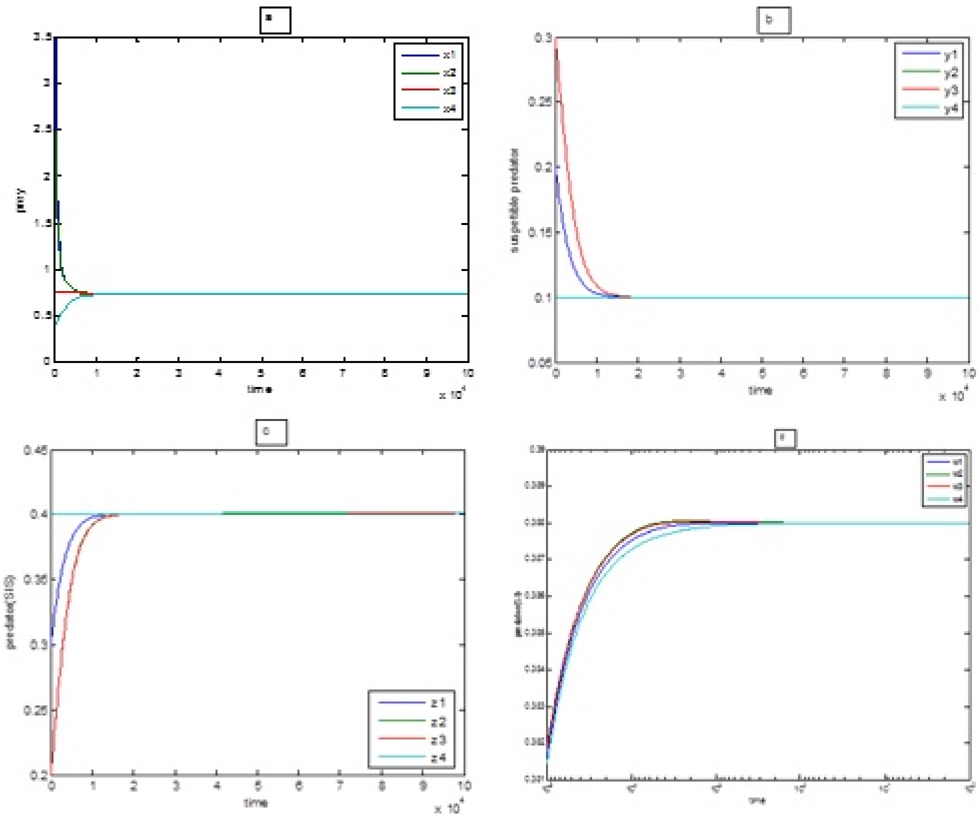

Figure 1: The time series of system (2.2) beginning with different initial points (3.5, 0.2, 0.3, 0.388),(2.5, 0.3, 0.2, 0.388) , (0.75, 0.3, 0.2, 0.388) and (0.4, 0.1, 0.4, 0.388), for the data given in Eq. (5.1). The solution approaches asymptotically to the positive equilibrium point $E_5 = (0.732, \ 0.1, \ 0.401, \ 0.388)$, (a) trajectory of (p) as a function of time, (b) trajectory of (s) as a function of time, (c) trajectory of (h) as a function of time, (d) trajectory of (v) as a function of time.

Varying the parameter $0.2 \le u_1 \le 2$, the solution still approaches to a positive equilibrium point E_5, as shown in Fig.2 for typical value $(u_1 = 0.3)$

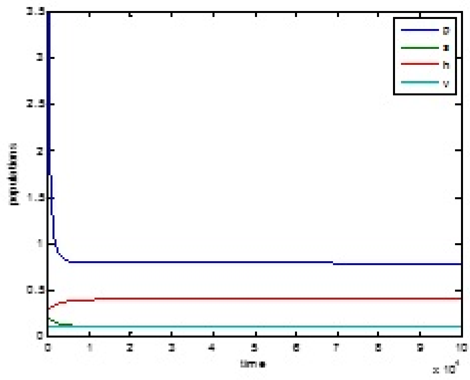

Figure 2: Time series of the solution of system (2.2) for the data given in Eq. (5.1) which approach to $E_5 = (0.787, 0.1, 0.405, 0.099)$.

Varying (u_7) in the range , $0.01 \leq u_7 < 0.495$ keeping the rest parameters fixed in Eq. (5.1), it is observed that the solution of system (2.2) still approach to the positive equilibrium point E_5, as shown in Fig.(3a), for typical value $u_7 = 0.3$, while $0.495 \leq u_7 < 1$ the solution of system (2.2) approach to E_4, as shown in Fig.(3b), for typical value $u_7 = 0.9$, thus $(u_7 = 0.495)$ is bifurcation point.

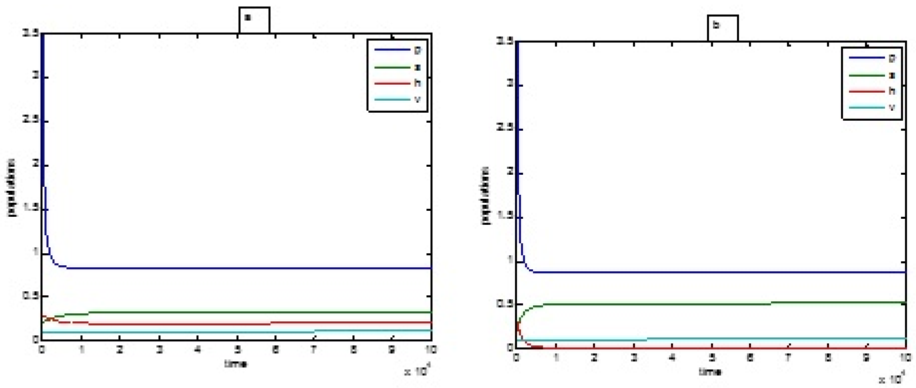

Figure 3: (a) Time series of the solution of system (2.2) for the data given in eq. (53) with $u_7 = 0.3$ which approach to $E_5 = (0.831,\ \ 0.316,\ \ 0.202,\ \ 0.103)$. (b) Time series of the solution of system (2.2) for the data given in Eq. (62) with $u_7 = 0.9$ which approach to $E_4 = (0.859,\ \ 0.533,\ \ 0,\ \ 0.107)$.

In system (2.2) as stated above, the same performance for the rest of parameters. Thus finally the following table make results summarized.

Table 2: shows the values that consist of BB box of the faces whose parts are to be revealed.

Parameters varied in system(2)	*Numerical behavior of system* (2.2)	*bifurcation point*
$0.2 \leq u_1 \leq 2$	***Approach to*** E_5	
$0.4 \leq u_2 < 0.93$	***Approach to*** E_5	
$0.3 \leq u_3 < 1.79$	***Approach to*** E_5	
$0.00001 \leq u_4 \leq 2$	***Approach to*** E_5	
$0.00001 \leq u_5 \leq 2$	***Approach to*** E_5	
$0.00001 \leq u_6 < 0.094$	***Approach to*** E_5	
$0.01 \leq u_7 < 0.495$	***Approach to*** E_5	$u_7 = 0.495$
$0.495 \leq u_7 < 1$	***Approach to*** E_4	
$0.00001 \leq u_8 < 0.137$	***Approach to*** E_4	$u_8 = 0.137$
$0.137 \leq u_8 \leq 2$	***Approach to*** E_5	
$0.00001 \leq u_9 < 0.28$	***Approach to*** E_5	
$0.00001 \leq u_{10} < 0.076$	***Approach to*** E_5	
$0.000001 \leq u_{11} < 0.0737$	***Approach to*** E_5	$u_{11} = 0.0737$
$0.0737 \leq u_{11} < 0.0999$	***Approach to*** E_4	
$0.000001 \leq u_{12} < 0.0385$	***Approach to*** E_5	$u_{12} = 0.0.0385$
$0.0385 \leq u_{12} \leq 1$	***Approach to*** E_4	
$0.000001 \leq u_{13} < 0.0158$	***Approach to*** E_5	
$0.00001 \leq u_{14} < 0.025$	***Approach to*** E_5	$u_{14} = 0.025$
$0.025 \leq u_{14} \leq 1$	***Approach to*** E_3	
$0.00001 \leq u_{15} < 0.0259$	***Approach to*** E_5	

6. Conclusion and Discussion

In this work, the occurrence of local bifurcation and Hopf-bifurcation are discussed with a suitable conditions of an eco-epidemiological of prey population and two different diseases (SIS and SI) in the predator population only, the transcritical bifurcation take place near E_1, E_2, E_3 and E_4, a saddle-node bifurcation take place near E_5, at E_2, E_3 and E_4 pitchfork bifurcation take place near all of these equilibrium points. Moreover fulfillment for the Hopf-bifurcation near E_3, E_4 and E_5 was done. Finally, numerical simulations are used to clarification the manifestation of local bifurcation of this system. And the following impressions are listed below:

1. According to data given in Eq. (5.1) used in system (2.2), the solution remain accession to positive stable point E_5 whatever changing the parameters u_i, $i = 1, 2, 3, 4, 5, 6, 9, 10, 13, 15$, therefore these parameters don't have any influence on the dynamical behavior.
2. Varying the parameters u_i, $i = 7, 8, 11, 12, 14$, keeping other parameters in system (2.2) with set of date in Eq. (5.1), these parameters played substantial role in dynamics behavior in terms of the local bifurcation and Hopf-bifurcation.

References

[1] Z.K. Alabacy and A.A. Majeed, *The fear effect on a food chain prey-predator model incorporating a prey refuge and harvesting*, J. Phys. Conf. Ser. 1804 (2021) 012077.

[2] R.M. Anderson and R.M. May, *The invasion persistence and spread of infectious diseases within animal and plant communities*, Philos, Trans. R. Soc. Lond. Bio. Sci. 314 (1982) 533–570.

[3] V. Andreasen and A. Sasaki, *Shaping the phylogenetic tree of influenza by cross-immunity*, Theore. Popul. Bio. 70 (2006) 164–173.

[4] N. Bairagi, P.K. Roy and J. Chattopadhyay, *Role of infection on the stability of a predator-prey system with several response functions A comparative study*, J. Theore. Bio. 248 (2007) 10–25.

[5] D.J. Earn, D.J. Dushoff and S.A. Levin, *Ecology and evolution of the flu*, Trends Ecol. Evolut. 17 (2002) 334–340.

[6] N.M. Ferguson, A.P. Galvani and R.M. Bush, *Ecological and immunological determinants of influenza evolution*, Nature 422 (2003) 428–433

[7] M. Haque and E. Venturino, *Increase of the prey may decrease the healthy predator population in presence of disease in the predator*, HERMIS 7 (2006) 38–59.

[8] A.J. Kadhim and A.A. Majeed, *The impact of toxicant on the food chain ecological model*, AIP Conf. Proc. 2292 (2020) 0030690.

[9] A.J. Kadhim and A.A. Majeed, *Epidemiological model involving two diseases in predator population with Holling type-II functional response*, Int. J. Nonlinear Anal. Appl. 12 (2021) 2085–2107.

[10] E.M. Kafi and A.A. Majeed, *The dynamics and analysis of stage-structured predator-prey model involving disease and refuge in prey population*, J. Phys. Conf. Ser. 1530 (2020) 012036.

[11] H.S. Kareem and A.A. Majeed, *Qualitative study of an eco-toxicant model with anti-predator behavior*, Int. J. Anal. Appl. 12 (2021) 1861–1882.

[12] H.S. Kareem and A.A. Majeed, *The bifurcation analysis of an eco-toxicant model with anti-predator behavior*, Int. J. Anal. Appl. 13 (2021) 1785–1801.

[13] M.A. Lafta, A.A. Majeed, *The food web prey-predator model with toxin*, AIP Conf. Proc. 2292 (2020) 0030935.

[14] A.A. Majeed and M.A. Ismaeel, *The bifurcation analysis of prey-predator model in the presence of stage-structured with harvesting and toxicity*, J. Phys. Conf. Ser. 1362(1) (2019) 012155.

[15] A.A. Majeed and E.M. Kafi, *The role of mathematical models in responding to pavement failures and distresses in taxas*, J. Phys. Conf. Ser. 1362 (2019) 012149.

[16] A.A. Majeed and Z.Kh. Alabacy, *The bifurcation analysis of an ecological model involving SIS disease with refuge*, Int. J. Anal. Appl. 13 (2022) 901–919.

[17] A.A. Majeed, *The bifurcation of an ecological model involving sage structures in both populations with toxin*, Itlian J. Pure Math. 46 (2021) 171-183.

[18] A.A. Majeed and M.A. Lafta, *The bifurcation analysis of food web prey-predator model with toxin*, J. Phys. Conf. Ser. 1897 (2021) 012080.

[19] A.A. Majeed and A.J. Kadhim, *The bifurcation analysis and persistence of the food chain ecological model with toxicant*, J. Phys. Conf. Ser. 1818 (2021) 012191.

[20] A.A. Majeed and S.G. Haider, *The bifurcation of the dynamics of prey–predator model with harvesting involving diseases in both populations*, Sci.Int. (Lahour) 30 (2018) 793–805.

[21] R. Omori, B. Adams and A. Sasaki, *Coexistence conditions for strains of influenza with immune cross-reaction*, J. Theor. Bio. 262 (2010) 48–57.

[22] L. Perko, *Differential Equation and Dynamical Systems*, Third Edition, New York, Springer-Verlag Inc., 2001.

[23] T.A. Radie and A.A. Majeed, *Qualitative study of an eco-toxicant model with migration*, Int. J. Anal. Appl. 12 (2022) 1883–1902.

[24] S. Wang and H. Yu, *Stability and bifurcation analysis of the Bazykin's predator-prey ecosystem with Holling type II functional response*, Math. Biosci. Engin. 18 (2021) 7877–7918.