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Refinements of Hermite-Hadamard inequality for F_h-convex functions on time scales

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Abstract

In this paper, new improvements, refinements and extensions to show that an F_h-convex function on time scales satisfies Hermite-Hadamard inequality is given in several directions. Examples and applications are as well provided to further support the results obtained.

Keywords: F_h-convex, Hermite-Hadamard, Time scales, Dynamic model.
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1. Introduction and Preliminaries

Let $I \subseteq \mathbb{R}$. A real-valued function $f: I \to \mathbb{R}$ is said to be convex on the classical closed interval $[a, b]$ if $\forall\ x, y \in I$ and $\lambda \in [0, 1]$, we have

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda) f(x) + \lambda f(y).$$

Hermite-Hadamard inequality has remarkable importance in mathematics, particularly in difference and differential equations. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}, \qquad a, b \in \mathbb{R},\ a < b, \tag{1.1}$$

holds for any convex function f defined on $\mathbb{R}$. This classical inequality (1.1) estimates the mean value of a convex function from both extremes [12]. Several authors have extended, developed, generalized

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and improved the fundamental inequality (1.1) for uni- and multi-variate convex functions, as well as other classes of convex functions with extensions to time scales [2]-[7], [10, 11], [14]-[17], [19].

Recently, new developments of the theory of time scales were introduced [13], to unify and extend the classical difference and differential calculus with accuracy, and also introduce the delta (Δ) and nabla (∇) time scales calculi (see [1], [8], [13]). Thus, the diamond-α $(\diamond_\alpha)$ dynamic calculus on time scales, which is essentially a linear combination of the delta and nabla calculi, was developed [18]. The $\diamond_\alpha$ dynamic derivative reduces to the standard Δ derivative for $\alpha = 1$ and to the standard ∇ derivative for $\alpha = 0$. On the other hand, it represents a weighted dynamic derivative on any uniformly discrete time scale when $\alpha \in (0,1)$. From literature, inequality (1.1) has been further extended and improved to time scales via the delta, nabla and diamond-α dynamics. For a detailed introduction to the theory of the calculus on time scales, see [1, 4, 6, 7, 10, 19].

The following useful concepts on time scales have been introduced [3].

Definition 1.1. *[1] A function $f:\mathbb{T} \to \mathbb{R}$ is called convex on $I_\mathbb{T}$, if*

$$f\left(\lambda t + (1-\lambda)s\right) \leq \lambda f(t) + (1-\lambda) f(s), \tag{1.2}$$

for all $t, s \in I_\mathbb{T}$ and $\lambda \in [0,1]$ such that $\lambda t + (1-\lambda)s \in I_\mathbb{T}$.

Definition 1.2. *[3] A function $f:\mathbb{T} \to \mathbb{R}$ is called rd-continuous, if it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits are finite at all left-dense points in $\mathbb{T}$. C_{rd} denotes the set of all rd-continuous functions.*

A function $f:\mathbb{T} \to \mathbb{R}$ is called ld-continuous, if it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits are finite at all right-dense points in $\mathbb{T}$. C_{ld} denotes the set of all rd-continuous functions.

The set of continuous functions on $\mathbb{T}$ contains both C_{rd} and C_{ld}.

It is worthy to note that every rd-continuous function is continuous and every ld-continuous function is continuous but the converse need not be true.

Definition 1.3. *[1] A function $F:\mathbb{T} \to \mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, the delta integral of f is defined as*

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s) \tag{1.3}$$

for all $s, t \in \mathbb{T}$; A function $G:\mathbb{T} \to \mathbb{R}$ is called a nabla antiderivative of $g:\mathbb{T} \to \mathbb{R}$ if $G^\nabla(t) = g(t)$ for all $t \in \mathbb{T}_k$. In this case, the nabla integral of g is defined as

$$\int_s^t g(\tau)\nabla\tau = G(t) - G(s) \tag{1.4}$$

for all $s, t \in \mathbb{T}$.

The importance of *rd*-continuous and *ld*-continuous functions is revealed by the following result.

Theorem 1.4. *[1] Every rd-continuous function has a delta antiderivative and every ld-continuous function has a nabla antiderivative. Thus;*

(i) If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then $\int_t^\sigma tf(s)\Delta s = \mu(t)f(t)$.

(ii) If $g \in C_{ld}$ and $t \in \mathbb{T}_k$, then $\int_\rho^t tg(s)\nabla s = \nu(t)g(t)$.

Definition 1.1 and the diamond-α calculus [18] were employed, to establish a full variant of the classical Hermite-Hadamard inequality (1.1) on time scales [4], by proving the following result.

Theorem 1.5. *[4] Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous convex function. Then*

$$f(x_\alpha) \leq \frac{1}{b-a}\int_a^b f(t)\diamond_\alpha t \leq \frac{b-x_\alpha}{b-a} f(a) + \frac{x_\alpha - a}{b-a} f(b), \tag{1.5}$$

where $x_\alpha = \frac{1}{b-a}\int_a^b t\diamond_\alpha t$.

The following corollary was obtained as a middle-point variant of the Hermite-Hadamard inequality (1.5) on time scales.

Corollary 1.6. *[4] Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}} t \leq \frac{f(a)+f(b)}{2}. \tag{1.6}$$

A refinement of (1.5) was therefore obtained [4] as follows.

Theorem 1.7. *[4] Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a, b] \to \mathbb{R}$ be a continuous convex function. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left(f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right) \leq \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}} t \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.7}$$

Recently, some Hermite-Hadamard's inequalities were established for (1.2) via the delta calculus, with the concept of rd-continuity in definition 1.2 on time scales [1, 13, 19]. A result of [19] is stated as follows.

Theorem 1.8. *[19] Let $f \in C([a, b], R)$ be convex. If $h \in C_{rd}([a, b], R)$ is symmetric with respect to $t = \frac{a+b}{2}$ and $\int_a^b |h(t)|\Delta t > 0$, then,*

$$f\left(\frac{a+b}{2}\right)\int_a^b |h(t)|\,\Delta t \leq \int_a^b |h(t)|\, f(t)\,\Delta t \leq \frac{f(a)+f(b)}{2}\int_a^b |h(t)|\,\Delta t. \tag{1.8}$$

The following can be deduced from Theorem 1.8 above if we set $h(t) = 1$.

Corollary 1.9. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f \in C_{rd}([a, b], R)$ be rd-continuous and convex on $[a, b]$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,\Delta t \leq \frac{f(a)+f(b)}{2}. \tag{1.9}$$

Observe that Corollary 1.9 can also be obtained from Theorem 1.5 of [4] above if $\alpha = 1$.

More recently, a more generalized class of convex functions called F_h-convex functions, among others, was introduced on time scales [8] as follows:

Definition 1.10. *[8] Let h:$\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a nonzero non negative function with the property that $h(t) > 0$ for all $t = 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. A mapping f:$I_{\mathbb{T}} \to \mathbb{R}$ is said to be F_h-convex on time scales if*

$$f(\lambda x + (1-\lambda) y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^s f(x) \ + \ \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f(y), \tag{1.10}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and $x, y \in I_{\mathbb{T}}$.

Remark 1.11. *We observe that:*

(i) If s= 1 and $h(\lambda) = 1$, then $f \in SX(I_{\mathbb{T}})$, i.e, f satisfies (1.2) (see [3])
(ii) If s= 1, $h(\lambda) = 1$ and λ=$\frac{1}{2}$, then $f \in J(I_{\mathbb{T}})$ is mid-point convex on time scales (see [8]).
(iii) When s= 0, $f \in P(I_{\mathbb{T}})$ is P-convex on time scales (see [8]).
(iv) Choosing $h(\lambda)$ =$\lambda^{\frac{s}{s+1}}$ gives h-convexity on time scales, that is, $f \in SX(h, I_{\mathbb{T}})$ (see [8]).
(v) If s= 1 and $h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$, then $f \in MT(I_{\mathbb{T}})$ is MT-convex on time scales (see [8]).
(vi) If s= 1, $h(\lambda) = 1$ and $\mathbb{T}$=$\mathbb{R}$, then f is convex in the classical sense (see [8]).

Note that F_h-convex functions is continuous on $\mathbb{T}$, that is, contains both C_{rd} and C_{ld}, see [8].

Definition 1.12. *[8] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function, with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. The diamond-F_h integral of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;*

$$\int_a^b f(t) \diamond_{F_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \quad s \in [0,1], \quad 0 \leq \lambda \leq 1, \tag{1.11}$$

provided that f has a delta and nabla integral on $[a, b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Obviously, each continuous function has a diamond-F_h integral. The combined derivative $\diamond_{F_h}$ is not a dynamic derivative, since we do not have a $\diamond_{F_h}$ antiderivative.

A more general and combined dynamic calculus on time scales, referred to as the diamond-F_h calculus, has been introduced and employed to establish Hermite-Hadamard integral inequality for the class of F_h-convex functions (1.10), see [7, 8, 10].

Theorem 1.13. *[10] Let h:$\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero, non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$ and f:$I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$, with a<b, $s \in [0,1]$. Then*

$$2^s \left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \diamond_{F_h} t \quad \leq f(a) \int_0^1 \left(\frac{\lambda}{h(\lambda)}\right)^s \Delta\lambda + f(b) \int_0^1 \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \nabla\lambda. \tag{1.12}$$

Note that the inequality (1.12) can be rewritten as follows.

$$2^s \left(h(\frac{1}{2})\right)^s f(t_\alpha) \leq \frac{1}{b-a} \int_a^b f(t) \diamond_{F_h} t \leq \left(\frac{\frac{b-t_\alpha}{b-a}}{h\left(\frac{b-t_\alpha}{b-a}\right)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{\frac{t_\alpha-a}{b-a}}{h\left(\frac{t_\alpha-a}{b-a}\right)}\right)^s \int_a^b f(t)\nabla t, \tag{1.13}$$

where $t_\alpha = \int_a^b x \diamond_\alpha x$.

The following corollary is obtained as a middle-point variant of the Hermite-Hadamard inequality (1.13) for F_h-convex functions on time scales $\mathbb{T}$ when α=$\frac{1}{2}$ and F_h=$\frac{1}{2}$.

Proposition 1.14. *Let $h{:}\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero, non negative function with the property that $h(\frac{1}{2}) \neq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$ and $f{:}I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$, with $a<b$, $s \in [0,1]$. Then*

$$2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}}t \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s\left[\int_a^b f(t)\,\Delta t + \int_a^b f(t)\,\nabla t\right]. \tag{1.14}$$

For more details on the diamond-F_h time scales calculus, see [6]-[10].

The next section establishes various estimations of Hermite-Hadamard integral inequality, which better improve inequalities (1.1), (1.5)-(1.14).

2. Main results

Our main result is presented as follows.

Theorem 2.1. *Let $h{:}\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero, non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$ and $f{:}I_{\mathbb{T}} \to \mathbb{R}$ be a continuous F_h-convex function, $a, b, t \in I_{\mathbb{T}}$, with $a<b$ and $s \in [0,1]$. Then for real numbers $l_f(\lambda)$ and $L_f(\lambda)$; the lower and upper bounds of f respectively, we have*

$$\begin{aligned}
2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) &\leq l_f(\lambda) \leq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right)a}\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\diamond_{\frac{1}{2}}t \\
&\quad + \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right)b - \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\diamond_{\frac{1}{2}}t \\
&\leq L_f(\lambda) \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s\left[\int_a^b f(t)\,\Delta t + \int_a^b f(t)\,\nabla t\right],
\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}
l_f(\lambda) &= 2^s\left(h(\frac{1}{2})\right)^s \frac{1}{2}\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s + 1\right)a\right)\right] \\
&\quad + 2^s\left(h(\frac{1}{2})\right)^s \frac{1}{2}\left[\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\left(\left(\frac{\lambda}{h(\lambda)}\right)^s + 1\right)b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)\right]
\end{aligned}$$

and

$$\begin{aligned}
L_f(\lambda) &= \left(\frac{\lambda}{h(\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t + \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right] \\
&\quad + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\Delta t + \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\nabla t\right].
\end{aligned}$$

Proof . Since f is F_h-convex on $I_{\mathbb{T}}$, it is well-known that if $f : I_{\mathbb{T}} \to \mathbb{R}$ is continuous on subinterval $\mathbb{T} \cap \left[a, \left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right]$, then

$$2^s \left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \diamond_{\frac{1}{2}} t \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s \left[\int_a^b f(t)\, \Delta t + \int_a^b f(t)\, \nabla t\right],$$

with $h(t) \neq 0$ for all $t \geq 0$, we have

$$\begin{aligned}
&2^s \left(h(\frac{1}{2})\right)^s \frac{1}{2} f\left[\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s + 1\right) a\right] \\
&\leq \frac{1}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t \\
&\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t + \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right].
\end{aligned} \tag{2.2}$$

Multiply (2.2) by $\left(\frac{\lambda}{h(\lambda)}\right)^s$ to get

$$\begin{aligned}
&\left(\frac{\lambda}{h(\lambda)}\right)^s 2^s \left(h(\frac{1}{2})\right)^s \frac{1}{2} f\left[\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s + 1\right) a\right] \\
&\leq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s - 1\right) a} \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t) \diamond_{\frac{1}{2}} t \\
&\leq \left(\frac{\lambda}{h(\lambda)}\right)^s \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t + \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right].
\end{aligned} \tag{2.3}$$

Also, using

$$2^s \left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t) \diamond_{\frac{1}{2}} t \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s \left[\int_a^b f(t)\, \Delta t + \int_a^b f(t)\, \nabla t\right],$$

with $h(t) \neq 0$ for all $t \geq 0$ on subinterval $\mathbb{T} \cap \left[\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a, b\right]$ gives

$$\begin{aligned}
&2^s \left(h(\frac{1}{2})\right)^s \frac{1}{2} f\left[\left(\left(\frac{\lambda}{h(\lambda)}\right)^s + 1\right) b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right] \\
&\leq \frac{1}{\left(1 - \left(\frac{\lambda}{h(\lambda)}\right)^s\right) b - \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t) \diamond_{\frac{1}{2}} t \\
&\leq \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\Delta t + \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\nabla t\right].
\end{aligned} \tag{2.4}$$

Multiplying (2.4) by $\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s$, we get

$$\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s 2^s \left(h(\frac{1}{2})\right)^s \frac{1}{2} f\left[\left(\left(\frac{\lambda}{h(\lambda)}\right)^s+1\right)b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right]$$

$$\leq \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right)b-\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\diamond_{\frac{1}{2}} t$$

$$\leq \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\Delta t+\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\nabla t\right]. \quad (2.5)$$

Adding up (2.3) and (2.5) side by side, we obtain

$$2^s\left(h(\frac{1}{2})\right)^s\frac{1}{2}\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s+1\right)a\right)\right]$$

$$+\,2^s\left(h(\frac{1}{2})\right)^s\frac{1}{2}\left[\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\left(\left(\frac{\lambda}{h(\lambda)}\right)^s+1\right)b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)\right]$$

$$\leq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s-1\right)a}\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\diamond_{\frac{1}{2}} t$$

$$+\frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right)b-\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\diamond_{\frac{1}{2}} t$$

$$\leq \left(\frac{\lambda}{h(\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t+\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right]$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\Delta t+\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\nabla t\right].$$

That is,

$$l_f(\lambda) \leq \frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s-1\right)a}\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\diamond_{\frac{1}{2}} t$$

$$+\frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right)b-\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(t)\diamond_{\frac{1}{2}} t \leq L_F(\lambda). \quad (2.6)$$

Since f is F_h-convex on $I_{\mathbb{T}}$, we have that

$$2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right)$$

$$= 2^s \left(h(\frac{1}{2})\right)^s \times$$

$$f\left[\left(\frac{\left(\frac{\lambda}{h(\lambda)}\right)^s\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s+1\right)a\right)}{2}\right)+\left(\frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\left(\left(\frac{\lambda}{h(\lambda)}\right)^s+1\right)b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)\right]$$

$$\leq 2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\frac{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s+1\right)a}{2}\right)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(\frac{\left(\left(\frac{\lambda}{h(\lambda)}\right)^s+1\right)b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}{2}\right)\right]$$

$$\leq 2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\frac{a+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(\frac{b+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)\right]$$

By definition 1.3, F is a Δ antiderivative of f if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$; G is a ∇ antiderivative of g if $G^{\nabla}(t) = g(t)$ for all $t \in \mathbb{T}_k$. Thus

$$2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\frac{a+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(\frac{b+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)\right]$$

$$= 2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s F^{\Delta}\left(\frac{a+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s G^{\nabla}\left(\frac{b+\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)\right], \tag{2.7}$$

wish by equations (1.3) and (1.4) is

$$2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(k)\Delta k+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^{b} f(k)\nabla k\right]$$

$$= 2^s \left(h(\frac{1}{2})\right)^s \times$$

$$\left[\left(\frac{\lambda}{h(\lambda)}\right)^s \left(\frac{F\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)-F(a)}{2}\right)\right.$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{G(b)-G\left(\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a\right)}{2}\right)\Bigg]$$

$$\leq\left(\frac{\lambda}{h(\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t+\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t\right]$$

$$+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\Delta t+\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\nabla t\right]$$

$$=\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\left(\frac{\lambda}{h(\lambda)}\right)^s\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\Delta t+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\Delta t\right]$$

$$+\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\left(\frac{\lambda}{h(\lambda)}\right)^s\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\nabla t+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\nabla t\right]$$

$$\leq\left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s\left[\int_a^b f(t)\Delta t+\int_a^b f(t)\nabla t\right]. \tag{2.8}$$

Thus we obtain (2.1) by (2.6) and (2.8). □

Remark 2.2. *Theorem 2.1. shows that there exists estimations better than (1.12) in the case where f is F_h-convex on time scales. It equally refines and generalizes previous results in literature as follows:*

(i) The first three inequalities in (1.7) are obtained by applying (1.14) for $\lambda=\frac{1}{2}$, $s=1$, $h(\lambda)=1$ and $h(\frac{1}{2})=\frac{1}{2}$ in (2.1).

(ii) If $\mathbb{T}=\mathbb{R}$, then (2.1) is the same as inequality (1.6) [11].

The following is an immediate consequence of the above Theorem.

Corollary 2.3. *Let $h:\mathbb{J}_{\mathbb{T}}\subset\mathbb{T}\to\mathbb{R}$ be a non zero, non negative function with the property that $h(\frac{1}{2})\neq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$ and $f:I_{\mathbb{T}}\to\mathbb{R}$ be a continuous F_h-convex function, $a,b,t\in I_{\mathbb{T}}$, with $a<b$ and $s\in[0,1]$. Then we have the following inequalities for lower and upper bounds $l_f(\lambda)$ and $L_f(\lambda)$ of real numbers respectively.*

$$2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right)\leq\sup_{\lambda\in[0,1]} l_f(\lambda)\leq\frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s-1\right)a}\int_a^{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a} f(t)\diamond_{\frac{1}{2}}t$$

$$+\frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{\left(1-\left(\frac{\lambda}{h(\lambda)}\right)^s\right)b-\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}\int_{\left(\frac{\lambda}{h(\lambda)}\right)^s b+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s a}^b f(t)\diamond_{\frac{1}{2}}t$$

$$\leq\inf_{\lambda\in[0,1]} L_f(\lambda)\leq\left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^s\left[\int_a^b f(t)\Delta t+\int_a^b f(t)\nabla t\right], \tag{2.9}$$

where $l_f(\lambda)$ and $L_f(\lambda)$ are as stated in Theorem 2.1.

We then state the following corollaries, which are refinements of (1.6).

Corollary 2.4. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous convex function on $I_{\mathbb{T}}$. Then for all $\lambda \in [0,1]$, we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right) \\ &\leq \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}}t \leq \frac{1}{2}[f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)] \\ &\leq \frac{f\left(a\right)+f\left(b\right)}{2}. \end{aligned} \tag{2.10}$$

Corollary 2.5. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a,b] \to \mathbb{R}$ be a continuous convex function on $I_{\mathbb{T}}$. Then we have the following inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \sup_{\lambda\in[0,1]}\left[\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}}t \leq \inf_{\lambda\in[0,1]}\left[\frac{1}{2}[f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)]\right] \\ &\leq \frac{f\left(a\right)+f\left(b\right)}{2}. \end{aligned} \tag{2.11}$$

Remark 2.6. *Corollaries 2.4 and 2.5 are improvements of some results for a convex function on the interval I of $\mathbb{R}$. See [12, Theorem 1.1 and Corollary 1.1].*

Thus, we can have the following corollaries as refinements and improvements to Theorem 1.8 and Corollary 1.9.

Corollary 2.7. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a,b] \to \mathbb{R}$ be convex and rd-continuous on $[a,b]$. Then we have the following*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \left[\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\Delta t \leq \left[\frac{1}{2}[f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)]\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.12}$$

Corollary 2.8. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$. Let $f\colon [a,b] \to \mathbb{R}$ be convex and rd-continuous on $[a,b]$. Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \sup_{\lambda\in[0,1]}\left[\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(t)\Delta t \leq \inf_{\lambda\in[0,1]}\left[\frac{1}{2}[f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)]\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{2.13}$$

Example 2.9. *Let $\mathbb{T}$ be a time scale and $a, b \in \mathbb{T}$ such that $0 \le a < b$. Suppose* $l(\lambda) = \left[\lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right) + (1-\lambda) f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)\right]$ *and* $L(\lambda) = \left[\frac{1}{2}[f(\lambda b + (1-\lambda)a) + \lambda f(a) + (1-\lambda) f(b)]\right]$ *are the lower and upper bounds in the inequalities (2.10)-(2.13), then*

(i) For $\lambda{=}\frac{a}{a+b}$, *we obtain*

$$l\left(\frac{a}{a+b}\right) = \frac{a}{a+b} f\left(\frac{a^2+3ab}{2(a+b)}\right) + \frac{a}{a+b} f\left(\frac{b^2+3ab}{2(a+b)}\right)$$

and

$$L\left(\frac{a}{a+b}\right) = \frac{1}{2}\left(f\left(\frac{2ab}{a+b}\right) + \frac{af(a)+bf(b)}{a+b}\right).$$

(ii) For $\lambda{=}\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$, *we get*

$$l\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} f\left(\frac{\sqrt{a}(\sqrt{a}+\sqrt{b})}{2}\right) + \frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}} f\left(\frac{\sqrt{b}\ (\sqrt{a}+\sqrt{b})}{2}\right)$$

and

$$L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) = \frac{1}{2}\left(f(\sqrt{ab}) + \frac{\sqrt{a}f(a)+\sqrt{b}f(b)}{\sqrt{a}+\sqrt{b}}\right).$$

Remark 2.10. *Example 2.9 shows that there exist estimations better than (1.1) for functions f convex on classical and time scales intervals. Also, there are various refinements and generalizations of the following results in literature;*

(i) Using inequality (1.6) for $\lambda{=}\frac{1}{2}$ *gives (1.7) (see [4]).*

(ii) A result of [2] is obtained from the right hand side of (2.11) for $\mathbb{T}{=}\mathbb{R}$ *and* $L\left(\frac{1}{2}\right)$.

(iii) For $\mathbb{T}{=}\mathbb{R}$ *and* $L\left(\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right)$, *the right hand side of (2.11) reduces to a previously obtained result [17].*

(iv) For $\mathbb{T}{=}\mathbb{R}$ *and* $L\left(\frac{a}{a+b}\right)$, *the right hand side of (2.11) is the same as that of [5] as previously obtained.*

(v) If $\mathbb{T}{=}\mathbb{R}$, *then (2.11) is the same as inequality (1.6) [11].*

The following corollary is a middle-point variant of Theorem 2.1. for $\lambda \in [0, 1]$ and is a refinement of Theorem 1.4 of [10].

Corollary 2.11. *Let* $h{:}\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ *be a non zero, non negative function with the property that* $h(\frac{1}{2}) \ne 0$, *where* $\mathbb{J}_{\mathbb{T}}$ *is a* F_h*-convex subset of the real* $\mathbb{T}$ *and* $f{:}I_{\mathbb{T}} \to \mathbb{R}$ *be a continuous* F_h*-convex function,* $a, b, t \in I_{\mathbb{T}}$, *with* $a{<}b$ *and* $s \in [0, 1]$. *Then the following holds*

$$2^s\left(h(\frac{1}{2})\right)^s f\left(\frac{a+b}{2}\right) \le 2^s\left(h(\frac{1}{2})\right)^s \frac{1}{2}\left[f\left(\frac{a+3b}{4}\right) + f\left(\frac{3a+b}{4}\right)\right] \le \frac{1}{b-a}\int_a^b f(t)\diamond_{\frac{1}{2}} t$$
$$\le \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \frac{1}{2}\left[\int_a^b f(t)\Delta t + \int_a^b f(t)\nabla t\right] \le \left(\frac{\frac{1}{2}}{h(\frac{1}{2})}\right)^s \left[\int_a^b f(t)\Delta t + \int_a^b f(t)\nabla t\right]. \tag{2.14}$$

Proof . The proof follows easily from Theorem 2.1. □

Remark 2.12. *Applying*

$$2^{s}\left(h(\frac{1}{2})\right)^{s} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f\left(t\right) \diamond_{\frac{1}{2}} t \leq \left(\frac{\frac{1}{2}}{h\left(\frac{1}{2}\right)}\right)^{s}\left[\int_{a}^{b} f\left(t\right)\Delta t+\int_{a}^{b} f\left(t\right)\nabla t\right]$$

for $\lambda{=}\frac{1}{2}$, $s{=}\,1$ *and* $h(\lambda)=1$ *in (2.1) gives (2.14).*

Thus, we can have the following corollary as a refinement of Theorem 1.8. above, see [19].

Corollary 2.13. *Let* $\mathbb{T}$ *be a time scale and* $a, b \in \mathbb{T}$. *Let* f: $[a, b] \to \mathbb{R}$ *be rd-continuous and convex on* $[a, b]$. *Then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{a+3b}{4}\right)+f\left(\frac{3a+b}{4}\right)\right] \leq \frac{1}{b-a}\int_{a}^{b} f(t)\Delta t \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f\left(a\right)+f\left(b\right)}{2}. \end{aligned} \tag{2.15}$$

3. Applications to Economics and Optimization

A fundamental condition for the application of time scales calculus theory is description of dynamic processes using discrete and continuous models. Hence, the field of Economics, with its many dynamic models, finds applications for time scales calculus. Most dynamic optimization problems in Economics are set up in the following form: a representative consumer seeks to maximize his/her lifetime utility U subject to certain budgetary constraints A. There is the (constant) discount factor δ, which satisfies $0 \leq \delta \leq 1$, C_s is consumption during period s, $u(C_s)$ is the utility the consumer derives from consuming C_s units of consumption in periods $s{=}\,0, 1, 2, ...,T$. Utility is assumed to be concave: $u(C_s)$ has $u(C_s)'{>}0$ and $u(C_s)''{<}0$. The consumer receives some income Y in a time period s and decides how much to consume and save during that same period. If the consumer consumes more today, the utility or satisfaction he derives from consumption, is forgone tomorrow as the deterrence. Normally, the consumer is insatiable. However, each additional unit consumed during the same period generates less utility than the previous unit consumed within the same period (Law of diminishing marginal utility, LDMU).

The individual is constrained by the fact that the value function of his consumption, $u(C)$ must be equal to the value function of his income Y_s, plus the assets/debts, A_s that he might accrue in a period s. Hence, A_{s+1} is the amount of assets held at the beginning of period $s+1$. Also, A could be positive or negative; the consumer might save for the future or borrow against the future at interest rate r in any given period s but the value of A_T, which is the debt accrued with limit or the last period asset holding, has to be nonnegative (the optimal level is naturally zero).

In order to state the necessary and sufficient condition for optimization in the formulation of a dynamic optimization problem, it is important to present the simplest form of optimal control problem in terms of the diamond-F_h integral as;

$$\max\ J_{\diamond_{F_h}}[x,u]=\int_{a}^{b} L(t,x,u)\diamond_{F_h} t=\left(\frac{\lambda}{h\left(\lambda\right)}\right)^{s}\int_{a}^{b} L\left(t,x^{\sigma},u^{\sigma}\right)\Delta t+\left(\frac{1-\lambda}{h\left(1-\lambda\right)}\right)^{s}\int_{a}^{b} L\left(t,x^{\rho},u^{\sigma}\right)\nabla t, \tag{3.1}$$

for all $s \in [0,1]$ and $0 \leq \lambda \leq 1$, among all pairs (x,u) such that $x^{\Delta}=f(t,x^{\sigma},u^{\sigma})$ and $x^{\Delta}=f(t,x^{\rho},u^{\sigma})$, together with appropriate endpoint conditions $u^{\diamond_{F_h}\prime}(t)=L(t,u,p)$, $x(0)=u_0$, $u(T)$ free for all $t \in [0,T]$.

Thus, a simple utility maximization model of household consumption in Economics for a function of single variable can be refined and solved in time scales settings in order to obtain better estimates of the maximized utility function, using the same intuition as that of the dynamic optimization problem presented earlier, by employing our developed concepts in section 2 as below. Thus, the model assumes a perfect foresight.

Theorem 3.1. *Let $\mathbb{T}$ be a time scale and h:$\mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \rightarrow \mathbb{R}$ be a nonzero non negative function with the property that $h(\frac{1}{2}) \neq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. Then, the value function of the lifetime utility $U_{\diamond_{\phi_h}}$ to be maximized is;*

$$Maximize\ U_{\diamond_{F_h}} = \sup_{\lambda\in[0,1]} l_F(\lambda) \leq \int_0^T u(C(t))\, e_{-\delta}(t,\ 0) \diamond_{F_h} t \leq \inf_{\lambda\in[0,1]} L_F(\lambda), \tag{3.2}$$

subject to the budget constraints

$$a^{\nabla}(t) = (rA + Y - C)(\rho(t)), \quad \frac{r}{1\ +\ r\mu(t)} a^{\sigma}(t) + \frac{1}{1+r\mu(t)} y^{\sigma}(t) - \frac{1}{1+r\mu(t)} c^{\sigma}(t), \tag{3.3}$$

$$a^{\Delta}(t) = \qquad a(0) = a_0, \qquad\qquad a(T) = a_T,$$

where u is F_h-concave ($u'(C) > 0$ and $u''(C) < 0$), $0 \leq \lambda \leq 1$, $s \in [0,1]$, $l_F(\lambda)$, $L_F(\lambda)$, $A^{\Delta}, A^{\nabla}, r, \delta, A, Y$ and e are as defined above.

Proof . Let $f(t)$ be a function satisfied by the consumption function path that would maximize lifetime utility $U(C(t))e_{-\delta}(t,\ 0)$ in (3.2), then the condition for a functional of the form

$$\int_a^b L(t,x,u) \diamond_{F_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b L(t,x^{\sigma},u^{\sigma})\,\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b L(t,x^{\rho},u^{\sigma})\,\nabla t,$$

for all $s \in [0,1]$ and $0 \leq \lambda \leq 1$, to have a local extremum for a function $u(t)$ and the sufficient condition for an absolute maximum(minimum) of the functional hold.

Since both local and absolute extreme hold, the functional satisfies the sufficient conditions for optimization, which in turn satisfies Theorem 2.1.

Therefore, the model (3.2)-(3.3) can be analysed by writing (3.2) in terms of diamond-F_h integral (1.11), stating the maximum principle and giving the Hamiltonian function for the model. □

4. Conclusion

In this article, new improvements and generalizations of Hermite-Hadamard integral inequalities for the generalized class of F_h-convex functions on time scales are hereby established. Examples and applications to Optimization and Economics are given to support our results. The new diamond-F_h utility maximization model of household consumption in Economics constructed, provides better estimates to the diamond-F_h time scale model.

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