Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 2293-2301
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.5929

Nearly locally hollow modules

Nada Khalid Abdullah[a,*]

[a] *Department of Mathematics, College of Education for Pure Science, University of Tikrit, Tikrit, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

Let R be a commutative ring with identity and let M be an R-module. This study presents the nearly locally hollow module that's a strong form of a hollow module. We present that an R-module M is nearly locally hollow if M has a unique semi-maximal sub-module that contains all small sub-modules of M. The current study deals with this class of modules and gives several fundamental properties related to this concept.

Keywords: Nearly locally hollow module, R-module, sub-module, hollow module

1. Introduction

In this paper, R represents a commutative ring with identity, and each R-module M is a left unity. A proper sub-module L of an R-module M is called small if $L + K \neq M$ for each proper sub-module K of M [1]. A non-zero module M is called a hollow module if each proper sub-module of M is small [5]. A proper submodule N of an R-module M is called a maximal in M if and only if M/N is a simple R-module [8]. As an extension of this concept, a fact of semi-maximal sub-module in M is explained, so a submodule N of R-module M is represented as semi-maximal if and only if M/N is a semi-simple R-module [8]. The R-module M is said to be a local if M has a unique maximal sub-module that contains all proper submodules of M [3]. This study gives a strong formula of a hollow module that calls a nearly locally hollow module which has a unique semi-maximal submodule that contains all small sub-modules of M.

This paper can be classified into four sections. Section one is the introduction of the study, while section two contains the definition of the nearly locally hollow module as a strong form of a hollow module. The main properties of this class of modules have been investigated. In section three, some conditions under which hollow modules and nearly locally hollow modules are equivalent are introduced. The 3rd section investigates the relation among the nearly locally hollow modules and some other modules such as amply indecomposable, supplemented modules and lifting modules.

*Corresponding author
Email address: nada.khalid@tu.edu.iq (Nada Khalid Abdullah)

Received: December 2021 *Accepted:* December 2021

2. Nearly Locally Hollow Modules (NLH)

In this section, the concept of NLH modules and the basic properties of this type of module has been presented.

Definition 2.1. *An R-module M is nearly locally hollow NLH module if M has a unique semi-maximal submodules that contains all small sub-modules of M.*

Example 2.2.

1. *Z-module Z_4 is NLH module, but the Z-module Z_6 is not NLH module .*
2. *Each NLH module is a hollow module.*
   Proof . *Suppose that M is NLH module , then there exists a unique semi-maximal sub-module contains all small sub-module say N in M. And since N is a sub-module of M, each small is contains in M. By definition hollow module then N is a small sub-module of M, implies that M is a hollow module, while the converse is not true in general. For example Z_p^{∞} is hollow module but not NLH module.* □
3. *Every local module is NLH module, while the converse is not true in general, for example $Z_2 \oplus Q$ is NLH module. But not local module since $\{0\} \oplus Q$ is a unique semi-maximal sub-module of $Z_2 \oplus Q$ and $\{0\} \oplus \{0\}$ is a small sub-module of $Z_2 \oplus Q$ and contained in $\{0\} \oplus Q$, but $Z_2 \oplus \{0\}$ is a proper sub-module of $Z_2 \oplus Q$, but $Z_2 \oplus \{0\}$ is not contained in $\{0\} \oplus Q$.*
4. *It is not necessary that each simple module is NLH module , for example the Z-module Z_5 is a simple module , but it is not semi local hollow module. And is not necessary each NLH module is a simple module , for example the Z-module Z_8 is NLH module , but it is not a simple module.*

Via the following proposition, the study presents some of the basic properties of NLH modules.

Proposition 2.3. *Epimorphic image of NLH module is NLH module.*

Proof . Let M_1 be a NLH module and let $f : M_1 \to M_2$ be an epimorphism with M_2 is an R-module. Suppose that A be a unique semi-maximal sub-module of M_2 with $A + B = M_2$ where B is a proper submodule of M_2. Now, f^{-1}(N) is a unique semi-maximal sub-module of M_1 since otherwise f^{-1}(A)= M_1 and hence $f(f^{-1})(N) = f(M_1) = M_2$ implies that $A = M_2$ which is contradiction with A is a unique semi-maximal sub-module of M_2, thus f^{-1}(A) is a unique semi-maximal sub-module of M_1 and since M_1 is nearly locally hollow module, therefore $f^{-1}(A)$ contains each small sub-module of $M_1$and hence $f(f^{-1}(A))$ is a small sub-module of f (M_1). That means A is small sub-module of M_2. So M_2 is nearly locally hollow module. □

Proposition 2.4. *Let M be an R-module, if M is NLH module then M/N is NLH module for each proper sub-module N of M.*

Proof . Let M be a nearly locally hollow module then there exists a unique semi-maximal contains all small sub-module. Let N be a proper sub-module of NLH module M and let $\pi : M \to M/N$ be the natural epimorphism then M/N is NLH module by(prop. (2.3)). □

Corollary 2.5. *If M is a factor module and M/K is an NLH module, then M is NLH , where K is a small sub-module of M.*

Proof . Suppose that M/K is an NLH module with a small sub-module K of M then there exists a unique semi-maximal sub-module N/K of M/K with $A+L=M$ where L is a sub-module of M then $(A+L)/K = M/K$, implies that $((A+K)/K)+((L+K)/K) = M/K$ since $(A+K)/K$ is a proper sub-module of N/K and M/K is NLH module, then $(A+K)/K$ is a small sub-module of M/K. Thus $(L+K)/K = M/K$, so $L+K = M$, since K is a small sub-module of M, then $L = M$. So that M is an NLH module. □

Definition 2.6. *[[4]] A pair (P, f) is a projective cover of the module M in case P is a projective module and $f : P \to M$ where f is an epimorphism and kerf is a small sub-module of P (we call P itself is a projective cover of M).*

Proposition 2.7. *Let $f : M_1 \to M_2$ be a projective cover of M_2, if M_2 is an NLH module. Then M_1 is an NLH module.*

Proof . Let M_2 be NLH module and since $f : M_1 \to M_2$ is an epimorphism then$M_1/kerf$ is isomorphism to M_2 and hence it is a NLH module and kerf is a small submodule of M_1. Thus by (prop.(1-4)) we get M_1 is a NLH module. □

Proposition 2.8. *Let M be an R-module, then M is an NLH module and finitely generated module if and only if M is a cyclic and has a unique semi-maximal sub-module.*

Proof . : Let M be a finitely generated NLH module then $M = R_{x_1} + R_{x_2} + \cdots + R_{x_n}$. If $M \neq R_{x_1}$ then R_{x_1} is a proper sub-module of M which implies that R_{x_1} is a small sub-module of M. Hence $M = R_{x_2} + R_{x_3} + \cdots + R_{x_n}$. So, we delete the summand one by one until we have $M = R_{x_i}$ for some i. Thus M is a cyclic module and since M is a NLH module. So, M has a unique semi-maximal sub-module by (Def. 2.1).
Conversely, let M be a cyclic module have a unique semi-maximal sub-module say N, then M is finitely generated. Let L be a proper submodule of M with $L+K = M$ where K is a submodule of M. Now, if L is not small submodule of M implies that $K \neq M$. Then K is a proper submodule of M and K is submodule of N and since M is finitely generated, then K is contained in a semi-maximal sub-module. But by assumption M has a unique semi-maximal sub-module N. Thus L is a sub-module of N (L is contained in N). So that $L+N = N = M$ which is a contradiction. Hence $K = M$, L is a sub-module of N and L is a small sub-module of M. Then M is a NLH module. □

Proposition 2.9. *Let N be a semi-maximal sub-module of a module M. If M is an NLH module and M/N is finitely generated then M is finitely generated.*

Proof . Let N be a semi-maximal sub-module of a NLH module M with M/N is a finitely generated. Then $M/N =$R(x_1+N)+$R(x_2+N\ )+\cdots+R(x_n+N)$ where $x_i \in M$ for all i =1,2,$\cdots$,n we claim that $M = Rx_1 + Rx_2 + \cdots + Rx_n$. Let $m \in M$ then m+$N \in M/N$, implies that, $m + N = r_1(x_1 + N) + r_2(x_2+N)+\cdots+r_nx_n+N = r_1x_1+r_2x_2+\cdots+r_nx_n$+$N$. That gives $m = r_1x_1+r_2x_2+\cdots+r_nx_n+n$ for some $n \in N$. So $M = r_1x_1+r_2x_2+\cdots+r_nx_n+N$ and since M is NLH module, then N is a small sub-module of M which implies that $M = r_1x_1 + r_2x_2 + \cdots + r_nx_n$. Thus M is a finitely generated. □

3. NLH modules and hollow modules.

In section one, the fact that every NLH module is a hollow module has been introduced, and an example shows that the converse is not true has been shown. In this section we investigate the conditions under which hollow modules can NLH modules .

Proposition 3.1. *Let M be an R-module such that each prime sub-module is semi-maximal, then M is a NLH module if and only if M is a hollow and cyclic module.*

Proof . Suppose that M is a NLH module then it has a unique semi-maximal sub-module N that have all small sub-module of M. Since N is a prime sub-module, N is a maximal sub-module of M. Let $x \in M$ with $x \notin N$ then R_x is a sub-module of M. We claim that $R_x = M$. If $R_x \neq M$ then R_x is a proper small sub-module of M and hence R_x is a sub-module of N which implies that $x \in N$ which is a contradiction. Thus $R_x = M$ then M is a cyclic module. Now, since M is a NLH module then M is a hollow module by (Remark(2-2))(2).

Conversely, suppose that M is a hollow and cyclic module then it is a finitely generated module and it has a maximal sub-module of M say N, so it is a semi-maximal sub-module of M by (Remark(1-2)) (2). has a Let L be a proper small sub-module of M ,if L is not contained in N then $\mathrm{L}+N = M$, but L is a small sub-module of M, thus $N = M$ which is a contradiction. This implies that every proper small sub-module of M is contained in N. So that M is a NLH module . □

Corollary 3.2. *Let M be an R-module such that $[N : K] = \{N : M\}$ for each semi-maximal sub-modules N and K of M. Then M is a NLH module if and only if M is a hollow and cyclic module.*

Corollary 3.3. *Let M be an R-module such that $[N : K] = \{N : Rx\} \forall x \notin N$ and N is a semi-maximal sub-module of M. Then M is a NLH module if and only if M is a hollow and cyclic module.*

Corollary 3.4. *Let M be an R-module such that $N = \{N : Rx\} \forall x \notin [N : M]$ and N is a semi-maximal sub-module of M. Then M is a NLH module if and only if M is a hollow and cyclic module.*

Proposition 3.5. *Let M be an R-module, M is a NLH module if and only if M is a hollow module and has a unique semi-maximal sub-module.*

Proof . suppose that M is a NLH module, then M is a hollow module by (Remark1-2) (2). And by (Def. 2.1), then M has a unique semi-maximal sub-module.
Conversely, let M be a hollow module which has a unique semi-maximal sub-module say N, it is enough to prove that M is a cyclic module. Let $x \in M$ and $x \notin N$, then $R_x + N = M$ and since M is a hollow module then N is a small sub-module of M and since $M = R_x$ so M is a cyclic module, and then by (prop. 2-1), M is NLH module. □

Proposition 3.6. *Let M ba an R-module. Then a submodule N of M is an NLH module if and only if it is a cyclic module and every non-zero factor module of M is indecomposable.*

Proof . Let M be a NLH module, then by (Prop. 2.3) M is a hollow and cyclic module. Now by (Prop. **??**) every non-zero factor module of M is indecomposable.

Conversely, let M be a cyclic module and every non-zero factor module of M is indecomposable, then by [6,prop.(4-41)]. M is a hollow module and by (prop. 2.3). Thus M is a NLH module. □

Proposition 3.7. *M is a NLH module if and only if M is a hollow module and $RadM \neq M$.*

Proof . Let M be a NLH module, then M is a hollow and cyclic module by (prop. (2-1)). And since M is a cyclic module, then M is a finitely generated and then Rad$M \neq M$.
Conversely; let M be a hollow module and Rad$M$ ≠M, then Rad$M$ is a small sub-module of M. Also by [4,prop.1,3,13.P.36]. Rad$M$ is the a unique semi-maximal sub-module of M and thus $\frac{M}{\mathrm{RadM}}$ is a simple module and hence its cyclic. Implies that $M/Rad\ M =< m + Rad >$ for some $m \in M$. We claim that $M = Rm$, let $w \in M$ then $w + Rad\ M \in M/Rad\ M$, and therefore there is $r \in R$ such that $w + Rad\ M = r\,(m + Rad\ M) = rm + Rad\ M$. Implies that $w - rm \in RadM$ which implies that $w - rm = y$ for some $y \in Rad\ M$. Thus $w = rm + y \in Rm + \mathrm{Rad}M$, hence $M = Rm + \mathrm{Rad\ M}$. But Rad$M$ is a small submodule of M implies $M = Rm$. Thus M is a cyclic module and by (prop. 3-1) we get M is a NLH module. □

Proposition 3.8. *M is said to be NLH module if and only if $RadM$ is a small and semi-maximal in M.*

Proof . Suppose that Rad$M$ be a small and semi-maximal sub-module. To prove that M is NLH module, first we want to show that Rad$M$ is a unique semi-maximal sub-module of M. Suppose that L is another semi-maximal sub-module of M, then $M = L + \mathrm{RadM}$, but Rad$M$ is a small sub-module that means $L = M$, which is a contradiction. Thus Rad$M$ is aunique semi-maximal sub-module in M. We conclude that every small sub-module of M is contained in Rad$M$. Let N be a small sub-module of M, if N is not contained in Rad$M$, then $N+$ Rad$M = M$. but Rad$M$ is a small sub-module of M that gives $N = M$ which is a contradiction. Therefore M is a NLH module.
Conversely; suppose that M is a NLH module then by (Remark(2-2)) (2), we get M is hollow module and by [9]. Then Rad$M$ is a maximal sub-module so Rad M is a semi-maximal sub-module [7] and since M is nearly locally hollow module. Thus Rad$M$ is a unique semi-maximal sub-module of M, hence Rad$M$ +$N = M$ for some proper submodule N of M. If Rad$M$ is not small sub-module of M then N is a small sub-module of M. thus Rad$M = M$ which is contradiction by [2].Hence Rad$M$ is small sub-module of M. □

4. NLH modules and some other modules.

This section studies the relation between NLH module and other modules such as amply supplemented modules, indecomposable modules and lifting module.

Definition 4.1. *[[7]] A module M is called amply supplemented, if for every two sub-modules A and B of M with $M = A+B$ then there exists a supplement L of B in M, such that $L \leq B$.*

Proposition 4.2. *Every NLH module is an amply supplemented.*

Proof . Let M be a NLH module and let N be a unique semi-maximal sub-module of M. Since M is NLH module, then N+M=M and $N \cap M + N$ is a small sub-module of M. Therefore M is amply supplemented. □

Remark 4.3. *The converse of proposition (3-2) is not true in general, for example the Z-module Z_6 is amply supplemented, but not NLH module*

Definition 4.4 ([6]). *An R-module M is indecomposable if $M \neq 0$ and the only direct summands of M are $\overline{0}$ and M. Implies that M has no a direct sum of two non-zero sub-module.*

Note: It is clear that each simple module is indecomposable, but the Z-module Z_6 is indecomposable , but it is not simple .

Proposition 4.5. *Every NLH module is indecomposable.*

Proof . Let M be a NLH module then there exists a unique semi-maximal sub-module N which contains all small submodule of M, suppose that M is decomposable, then there are proper sub-modules K and L such that K and L are sub-modules of N and $M = K \oplus L$. But M is a NLH module then either L is a small sub-module of M with L is sub-module of N implies that $K = M$ or K is small sub-module of M with K is sub-module of N implies that $L = M$ which is a contradiction. Then M is indecomposable. $\square$

Proposition 4.6. *Let M be a cyclic module, then M is a NLH module if and only if every non-zero factor module of M is indecomposable.*

Proof . Let $\frac{M}{A}$ be a non-zero factor module of M. Since M is a NLH module. Then $\frac{M}{A}$ is NLH module by (corollary 1-5). And by (prop.3-5), $\frac{M}{A}$ is indecomposable. Conversely; let N be a semi-maximal sub-module of M and let L be a sub-module of N. Suppose that $M = L + K$, where K is a sub-module of M by [7], so $M\ L \cap K \cong \frac{M}{L} \oplus M\ K$. But $\frac{M}{L\cap K}$ is indecomposable then either $M\ L = 0$ or $M\ K$. Since L is a sub-module of N, and N is a submodule of M. Hence L is a proper submodule of M. Then $M/L \neq 0$ implies that $M/K = 0$ and since $M = K$ then L is a small sub-module of M. Thus M is a hollow module and since M is a cyclic module then by (prop.2-1) M is a NLH module. $\square$

Definition 4.7 ([6]). *Let M be a module, M is called a lifting module (or satisfying Def. 2.1) if for every sub-module N of M there are sub-module K and L of M such that $M = K \oplus L$, K is a sub-module of N and $N \cap K$ is a small sub-module of K.*

Proposition 4.8. *Every NLH module is a lifting module.*

Proof . Let M be NLH module, then there exists a unique semi-maximal sub-module N of M contains all small sub-module, then $M = M \oplus \{0\}$ where $\{0\}$ is a sub-module of $N, N \cap M = N$ and since M is a NLH module. Then $N \cap M = N$ is a small sub-module of M. Thus M is lifting module. $\square$

Remark 4.9. *The converse of proposition(4-8) is not true in general, for example. The Z-module Z_{10} is a lifting module. But it is not a NLH module .*

Proposition 4.10. *Let M be a cyclic indecomposable module. If M is lifting module, then M is a NLH module.*

Proof . Let N be a proper sub-module of M, since M is lifting module. Then A+B=M , where A and B are sub-modules of M and $N \cap A$ is small sub-module of A. But M is an indecomposable module, thus $B = 0$ and hence $A = M$, which implies that $N \cap M = N$, hence N is a small submodule of M. Hence M is hollow module and since M is cyclic module. Then M is a NLH module by (Prop. 3.1). $\square$

Proposition 4.11. *Let K be a semi-maximal sub-module of a module M. If L is a supplement sub-module of K in M, then L is a NLH module.*

Proof . Let L be a supplement of K and let L_1 be a proper sub-module of L with $L_1 + L_2 = L$ for some sub-module L_2 of L. Now,$k + L = M = K + L_1 + L_2 = M$ and $L_1$is a sub-module of K, since otherwise $K+ L_1 = M$ and since K is semi-maximal sub-module of M we get $L_1 = L$, which is a contradiction. Thus $K + L_2 = M$ and since K is semi-maximal sub-module of M we get $L_2 = L$. Implies that L is a hollow module. Now to show that L is a cyclic module, let $x \in M$ and $x \notin K$ then $R_x+ K = M$ and hence $R_x = L$ by minimality of L. And by(prop. 3-1), then L is a NLH module . □

Definition 4.12 ([9]). *A sub-module N of an R- module M is called co-closed in M if $\frac{N}{K}$is a small sub-module of M/K implies that $N = K$ for all sub-module K of M contained in N.*

Proposition 4.13. *If M is a NLH module then each non-zero co-closed sub-module of prime sub-module of M is an NLH module.*

Proof . Let M be a NLH module and let N be a unique semi-maximal sub-module of M. Let A be a non-zero coclosed sub-module of N, suppose that L is a proper sub-module of A. Since M is a NLH module so L is a small sub-module of M contained in N. And hence A is co-closed sub-module of M. Thus L is a small submodule of A. Hence A is nearly locally hollow module. □

Proposition 4.14. *Let A be a sub-module of an R-module M. If A is a NLH module, then either A is a small sub-module of M or co-closed sub-module of M, but not both.*

Proof . Suppose that A is not co-closed sub-module of M. To prove that A is a small sub-module of M, then there exists a proper sub-module of $\frac{M}{B}$. But A is a NLH module so, A is hollow module by (Remark 2-2) (2). Then by [2] , B is a small sub-module of A s a small submodule of M. Now, since A is a NLH module, A is a zero sub-module, so A is a co-closed and small sub-module. □

Proposition 4.15. *Let M be a cyclic module, and let f:P→M be a projective cover of M and then the following statements are equivalent.*

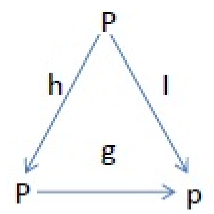
(1) M is a NLH module.
(2) M is hollow module.
(3) P is hollow module.
(4) P is indecomposable and supplemented.
(5) The End(P) is local ring.

Proof . (1) $\Longrightarrow$ (2) is clear by Remark 1-2(2). (2) $\Longrightarrow$ (3) Let M be a hollow module and since $f : P \to M$ is an epimorphism, $\frac{P}{\mathrm{ker}f}$ is isomorphism to M and hence it is a hollow module and since kerf is small sub-module of P, P is a hollow module by [7,prop.1-3-3.P.31].

(3) $\Longrightarrow$ (4) is clear by [7, prop.1-3-5.P.32] and [7, prop.1-3-9.P.34].

(4) $\Longrightarrow$ (5) Let $g : P \longrightarrow P$ be a homomorphism then two cases are hold.

Case1: g is onto. Since P is a projective module consider the following diagram :



Where $I : P \to P$ be the identity homomorphism and there exists a homomorphism $h : P \to p$ such that $f \circ h = I$, implies that g has a right inverse, this implies that $P = kerg \oplus h(P)$, but P is indecomposable by [3]. Then $kerg = 0$, thus $P = h(P)$. Then g is one to one. Hence g is an isomorphism.

Case2: g is not onto. We know that $P = g(P) + (I - g)(P)$, P is amply supplemented by [7], then there exists a supplement K of $g(P)$ in $(I–g)(P)$. implies that $P = g(P) + K$ and $g(p) \cap K$ is a small sub-module of K, and there exists a supplement L of K in $g(P)$. Implies that $P = L + K$ and $L \cap K$ is a small sub-module of K. Now, L and K are mutual supplements and hence $L \cap K = 0$. Then $P = L \oplus K$, but P is indecomposable and $K \neq 0$ then $K = P$. Now, K is a sub-module of $(I - g)(P)$ this implies that $(I - g)(P) = P$. Implies that $(I - g)(P)$ is onto and by the previous argument $I - g$ is an isomorphism.

(5) $\Longrightarrow$ (1) To show that M is a hollow module we need only to show that P is hollow by [7], define $g : P \to p/L \cap K$ as follows, for $x \in P, x = s + t$ for some $s \in L$ and $t \in K$. Let $g(x) = s + L \cap K, g$ is a well-defined and homomorphism and since P is a projective module, there exists a homomorphism $\Psi : P \to P$ such that the following diagram is commutative

$$\begin{tikzcd} & P \arrow[dl, "\Psi"'] \arrow[dr, "g"] & \\ P \arrow[rr, "\pi"] & & P/L\cap K \end{tikzcd}$$

where $\pi : P \to P/L \cap K$ is the natural epimorphism. We claim that $\Psi(P)$ is a sub-module of L. To see this, let y $\in \Psi(P)$ then there exists w $\in$ P such that $y = \Psi(P)$ Now, $(\pi \circ \Psi)(w) = g(w)$ where $w = s + t$ for some $s \in L$ and $t \in K$. Implies that $\Psi(w) + L \cap K = s + L \cap K$ implies that $\Psi(w) - s \in L \cap K$ is a sub-module of L. Then $\Psi(w) \in L$, and so $\Psi(P)$ is a sub-module of L. As the same, one can show that $(I - \Psi)(P)$ is a sub-module of K. Now, $\Psi \in End(P)$ and by (5), $End(P)$ is a local ring then Ψ or $(I - \Psi)$ is onto, but Ψ is not onto. Since otherwise $\Psi(P)$ is a sub-module of L, we have $L = P$ which is a contradiction. So $(I - \Psi)$ is onto. Implies that $K = P$ thus P is a hollow module. Since P is a hollow module, M is a hollow module and since M is cyclic module, M is a NLH module by Proposition 3.1. □

5. Conclusion

The aim of this manuscript is to introduce a new strongly nearly locally hollow module which is the hollow module and local module held, but the converse is not true. Furthermore, proofed for some concepts via properties as factor module, maximal, semi-maximal and finitely generated etc. additively, show that every nonzero nearly locally hollow module co-closed and if hold their then prime and other results.

6. Future Work

We study and development of concepts generalization nearly locally hollow modules and extended prime and some prime sub-module on nearly locally hollow modules also primary.

References

[1] M.S. Abbas and F.M. Manhal, *d-small submodules and d-small projective modules*, Int. J. Algebra 12 (2018) 25–30.

[2] K.M. Ali, *Hollow Modules and Semihollow modules*, Thesis in College of Science, University of Baghdad, 2005.

[3] Z.T. Khlaif and N.K. Abdullah, *L-hollow modules*, Tikrit J. Pure Sci. 24 (2019) 104–109.

[4] M.F. Manhal, *Essential-Small M-Projective Modules*, Thesis College of science, Mustansiriyah University, 2018.

[5] M. Mohammed and A.G. Ahmed, *On Some Properties of hollow and hollow dimension modules*, Pure Appl. Math. J. 2 (2013) 156–161.

[6] K.I. Nagam, *Local lifting module with some of there generalizations*, Thesis in Collge of Science, University of Tikrit, 2019

[7] T. Ünsal, *On multiplication modules*, Int. Math. Forum 2 (2007) 1415–1420.

[8] Y.K. Hatem, *Semimaximal Submodules*, M. Sc. Thesis, University of Baghdad, 2007

[9] Y. Talebi, A.R. Moniri Hamzekolaee, M. Hosseinpour and S. Asgari, *A new generalization of t-lifting modules*, J. Algebra Related Topics 8 (2020) 1–13.