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Estimating the survival and risk functions of a log-logistic distribution by using order statistics with practical application

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Abstract

The Log-Logistic distribution is one of the important statistical distributions as it can be applied in many fields (biological, chemical and physical experiments) and its importance comes from the importance of determining the survival and risk function for these experiments. The research will work to determine the characteristics of the distribution through the use of order statistics to estimate its parameters using the approved standard Bayes method On the squared loss function (Bayslf) and determining the optimal method by comparing it with the MLE method according to a simulation method by taking different models for default values for parameter and different sample sizes and with MSE, IMSE comparison criteria as well as applying it to real data for breast cancer patients and determine survival and risk function

Keywords: Survival function, risk function, log-logistic distribution, maximum likelihood estimation, Bays estimation ,Lindley approximation , order statistics .

1. Introduction

Order statistics is one of the most important branches of mathematical statistics as it plays an important and fundamental role in many branches of theoretical and applied sciences as well as in nonparametric problems from both theoretical and practical perspectives. In theoretical way, statistical inference theory depends on ordinal statistics and in practical side, the statistical inference which based on ordinal samples helps us to get a simple and applicable results that depend on the

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quality of fit and life tests for distributions based on ordinal samples, as these distributions are more suitable for studying the theory of outliers, probabilities, estimation theory and hypothesis tests .(A. Ragab and J. Green[10]).
In this research, two parameters of the log-logistic distribution used in income studies, survival and risk studies were estimated. It can be defined by the scale parameter α and the shape parameter β. Several previous studies of this distribution were conducted in 1992 (Howladera & Weissb[7]) studied approximations of Bayesian estimators of the survival function based on the censored data of a log-logistic distribution obtained under the quadratic error function Also, record the probabilities of this function. In 2005, the researchers (Singh & Guo[12]) made a study to estimate the parameters of the log-logistic distribution. They used the principle of maximum entropy (POME) to use a new method to estimate the parameters of the two-parameter log-logistic distribution. In the year 2020, the researcher (Dali Chen[4]) and others applied thermal time models to different distributions, the results showed that the model is more flexible with the log-logistic distribution.
This research is an estimation of the survival function and risk using the maximum likelihood estimation and the standard Bayes method based on the squared loss function and Lindley approximation.

2. Generalized order statistics

The random variables are called U(j,n,$\tilde{m}$,s), since j=1,2,. . . ,n are in generalized regular ordered statistics if the joint probability density function has the following formula:

$$f^{U(1,n,\tilde{m},s)..U(n,n,\tilde{m},s)}\left(t_1\ldots.t_n\right)=C_{n-1}[\prod_{i=1}^{n-1}\left(1-t_i\right)^{m_i}](1-t_n)^{s-1}$$

Where :

$$n\in N;n\geq 2;S\geq 1$$
$$\tilde{m}=\left(m_1,\ldots,6m_{n-1}\right)\in R^{n-1}$$
$$0<t_1\leq\cdots\leq t_n\leq 1$$
$$C_{n-1}=\prod_{i=1}^{n}Y_i=s\prod_{i=1}^{n}Y_i$$
$$Y_i=s+n-j+\sum_{i=j}^{n-1}m_i>0$$

for all values $j\in\{1,2,..,n-1\}$
The joint probability function can be obtained from generalized ordered statistics for n random variables T (j,n, $\tilde{m}$,s) as j∈{1,2,. . . ,n} which follows any probability distribution with a probability density function is f (x) and the cumulative distribution function is F(x) and takes the following form :(kamps [8])

$$f^{t(1,n,\tilde{m},s)..t(n,n,\tilde{m},s)}\left(t_1\ldots.t_n\right)=C_{n-1}\left[\prod_{i=1}^{n-1}\left(1-F(t_i)\right)^{m_i}f\left(t_i\right)\right]\left[\left(1-F\left(t_n\right)\right)^{s-1}f\left(t_n\right)\right] \tag{2.1}$$

Where : $F^{-1}\left(0\right)<t_1\leq\cdots\leq t_n<F^{-1}\left(1\right)$
The generalized order statistics model contains many ordered random variables models, and by choosing an appropriate parameter $(\widetilde{\mathrm{m}},\mathrm{s})$, we get the generalized order statistics models.

3. Ordinary order statistics

Ordinary ordered statistics can be obtained by putting $m_i = m = 0$ for the values of i=1,2,..,n-1, S=1, and $Y_i = n - i + 1$ in equation (2.1) we get the joint probability density function for n of the ordinary ordered statistics, as follows:

$$f\left(t_{(1)}, t_{(2)}, \ldots, t_{(n)}\right) = n! \prod_{i=1}^{n} f\left(t_i\right) \tag{3.1}$$

Where : $-\infty < t_1 \leq t_2 \leq \cdots \leq t_{(n)} < \infty$

4. The log-logistic distribution and its properties

The log-logistic distribution is one of the continuous probability distributions that has attracted the interest of researchers Raghab A. and Green.J (1984), it has been used extensively in models of population growth and biological problems and has some important applications in solving Many practical problems, especially in survival data as a model of events, the log-logistic distribution is also known as the Fisk distribution in economics. Suppose $(t_1 \leq t_2 \leq \cdots \leq t_n)$ an ordinal sample of size n drawn from a population that follows a log-logistic distribution whose probability function is as follows:

$$f\ (\mathrm{t_i},\alpha,\beta) = \frac{\frac{\beta}{\alpha}\ \left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta-1}}{\left[1+\left(\frac{\mathrm{t_i}}{\alpha}\right)^{\beta}\right]^2} \qquad t_i > 0 \quad \alpha, \beta > 0\ ,\ [0, \infty] \tag{4.1}$$

Where : α is Scale Parameter , β is Shape Parameter

The cumulative distribution function is given as follows:

$$F(\ t_i) = P_r\left(T_i < \ t_i\right) = \int_0^t f(x)\, dx = \frac{1}{1 + \left(\frac{t_i}{\alpha}\right)^{-\beta}} \tag{4.2}$$

Survival and risk function:

The survival function is the probability of the patient surviving during a certain period under special conditions and factors. The survival function is denoted by S (t), and the survival function is expressed by the following mathematical equation:

$$S(t) = \Pr(T > t) = \int_t^{\infty} f(y)\, dy = \frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}} \tag{4.3}$$

The risk function represents the probability that the patient under study will die during the period (t + Δt, t), noting that the patient was alive during the period t.

$$h(\ t_i) = \lim_{\Delta t \to 0} \frac{1}{\Delta \quad t_i} \frac{p_r(\ t_i \leq T_I < \ t_i + \Delta)}{p_r\left(T_i \geq \ t_i\right)} \tag{4.4}$$

$$h\left(t_i\right) = \frac{f\left(t_i\right)}{S\left(t_i\right)} \Rightarrow \frac{\frac{\beta}{\alpha}\left(\frac{t_i}{\alpha}\right)^{\beta-1}}{1 + \left(\frac{t_i}{\alpha}\right)^{\beta}} \tag{4.5}$$

properties of a log-logistic distribution :

1. **Central moment:**

$$\mathrm{ET^k}{=}\int_0^\infty \mathrm{T^k} f\left(t\right)\ dt{=}\alpha^{\mathrm{K}}\mathrm{B}\left(1{-}\frac{\mathrm{k}}{\beta},1{+}\frac{\mathrm{K}}{\mathrm{B}}\right);where\ K=1,2,\ldots;$$

$$;B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt$$

$$\alpha^k\left(k\pi/\beta\right)\csc\left(k\pi/\beta\right)\quad if\ \beta>1\qquad\qquad \textbf{AshaDixit}\ \ [\mathbf{7}]\ =$$

2. **Mean :**

$$M_t=\alpha\left(\pi/\beta\right)\csc\left(\pi/\beta\right)$$

3. **Variance :**

$$\sigma_t^2=\alpha^2\left[\left(2\pi/\beta\right)\csc\left(2\pi/\beta\right)\ -\left(\pi/\beta\right)^2\csc^2(\pi/\beta)\ \right]$$

4. **Median :**

$$\mathrm{Me}=\alpha$$

5. **Mode :**

$$Mo=\alpha\left(\frac{\beta_{-1}}{\beta+1}\right)^{\frac{1}{\beta}}\qquad \text{if } \beta>1$$

5. Estimation methods

5.1. Maximum likelihood estimation

This method is considered one of the most important estimation methods for its good properties, including stability, high efficiency and consistency in some cases. Suppose we have a number of observations from the log-logistic distribution, denoted by $t_1\leq t_2\leq ...\leq t_n$. The logarithmic probability function for α,β is as follows:

We get the Maximum likelihood estimation function when we substitute the function $f\left(t_i\right)$that was previously defined in equation (3.1) as follows:(Abdel Qader[1]).

$$L(\alpha,\beta\ |t)=n!\prod_{\ddot{I}=1}^{n} f\left(t_i\right)=n!\prod_{i=1}^{n}\frac{\beta}{\alpha}\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta-1}\left[1{+}\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}$$

$$=n!\beta^n\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1{+}\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}\tag{5.1}$$

$$\ln\ L\left(\alpha,\beta|t\right)=\ln\ n!+n\ \ln\ \beta-n\beta\ln\ \alpha+\left(\beta-1\right)\sum_{i=1}^{n}\ln\ \left(t_i\right)-2\sum_{i=1}^{n}\ln(1{+}\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta})\tag{5.2}$$

$$\frac{\partial\ln L(\alpha,\beta|t)}{\partial\alpha}=-\frac{n\beta}{\alpha}+\frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}}\tag{5.3}$$

$$\frac{\partial\ln\ L\left(\alpha,\beta|\ t\right)}{\partial\beta}=\frac{n}{\beta}-n\ln\left(\alpha\right)+\sum_{i=1}^{n}\ln\left(t_i\right)-2\sum_{i=1}^{n}\frac{\ln\left(\frac{t_i}{\alpha}\right)}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}}\tag{5.4}$$

Then we equate equations (5.3) and (5.4) with zero, and then solve them by one of the numerical methods to obtain the Maximum likelihood estimation for the parameters α and β.

6. Observed Fisher Information

In this section, we compute the observed Fisher information for the MLE. These will enable us to construct confidence intervals for the parameters based on the limiting s-normal distribution, and examine the probability coverage through simulation. We now derive the observed Fisher information for the likelihood using
equations (5.3) and (5.4). We have :

$$I_{(\widehat{\alpha},\widehat{\beta})} = \begin{bmatrix} \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial \boldsymbol{\alpha}^2} & \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha\partial\beta} \\ \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial \boldsymbol{\beta}^2} \end{bmatrix}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha^2} = \frac{n\beta}{\alpha^2} - \frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}} - \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} \tag{6.1}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\beta^2} = -\frac{n}{\beta^2} - 2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} \tag{6.2}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial\alpha\partial\beta} = -\frac{n}{\alpha} + \frac{2}{\alpha}\sum_{i=}^{n},\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} \tag{6.3}$$

The Maximum likelihood estimator of survival function S(t) and risk function h(t) where t>0 is based on the order statistics as follows:

$$\hat{s}(t_i)_{MLO} = \frac{1}{1+\left(\frac{t_i}{\widehat{\alpha}_{MLO}}\right)^{\widehat{\beta}_{MLO}}} \tag{6.4}$$

$$\hat{h}(t_i)_{MLO} = \frac{\frac{\widehat{\beta}_{MLO}}{\widehat{\alpha}_{MLO}}\left(\frac{t_i}{\widehat{\alpha}_{MLO}}\right)^{\widehat{\beta}_{MLO}-1}}{1+\left(\frac{t_i}{\widehat{\alpha}_{MLO}}\right)^{\widehat{\beta}_{MLO}}} \tag{6.5}$$

6.1. Standard Bayes estimation

The Bayesian method of estimation assumes that the parameters to be estimated are random variables, and these random variables have a prior distribution and prior probability density function $\pi(\alpha)$. It contains prior information about these parameters, as this method does not depend only on the sample data in the estimation process, because it is not sufficient according to this method (R. B. Hooge[6] pp. 364).

There are several types of prior probability density functions, and one of these functions is the non-informational probability density function, which is a function used in the absence of prior information about the parameter to be estimated and according to the Jeffery formula.

If the parameter period is a positive (0,∞) then the prior probability function is distributed logarithmically and in the case of a (log-logistic) distribution, we will assume that the distribution takes the following form (AL-Jassim[14] pp. 56) .

$$\pi_1(\alpha) \propto \frac{1}{\alpha}$$

$$\pi_2\left(\beta\right)\propto\frac{1}{\beta}$$

$$\pi\left(\alpha,\beta\right)=\frac{\mathrm{k}}{\alpha\beta}$$

After finding the prior distribution of the parameter, the joint distribution is found by multiplying this prior distribution of the parameter by the maximum likelihood function.

$$g\left(t_1,t_2,\ldots,t_n,\alpha\right)=f\left(t_1,\alpha\right)f\left(t_2,\alpha\right)\ldots f\left(t_n,\alpha\right)\pi\left(\alpha\right)$$

$$=L\left(t|\,\alpha\right)\pi\left(\alpha\right)$$

From the joint distribution, can be found the marginal density function of the sample observations

$$g\left(t_1,t_2,\ldots,t_n\right)=\int_{-\infty}^{\infty}g\left(t_1,t_2,\ldots,t_n,\alpha\right)d\alpha$$
$$=\int_{-\infty}^{\infty}L\left(t|\,\alpha\right)\pi\left(\alpha\right)$$

After dividing the joint distribution by the marginal density function, we get the posterior distribution that contains the prior and current information for the parameter to be estimated:

$$\pi\left(\alpha|\,t_1,t_2,\cdots,t_n\right)=\frac{L\left(t|\alpha\right)\pi\left(\alpha\right)}{\int_{-\infty}^{\infty}L\left(t|\,\alpha\right)\pi\left(\alpha\right)}\tag{6.6}$$

The Bayes estimator can be obtained by using the loss function which takes different forms

loss function:
The loss function $L(\widehat{\alpha},\alpha)$can be defined as the function through which you can measure the loss resulting from the process of estimating α by the estimator $\widehat{\alpha}$. That is, there is a difference between the estimator and the parameter, and in general the loss is measured by the difference function between the estimator and the parameter $\widehat{\alpha}-\alpha$ or For between the estimator and the parameter $\frac{\widehat{\alpha}}{\alpha}$ if $\widehat{\alpha}=\alpha$, then this means that there is no loss, and if $\widehat{\alpha}<\alpha$, then this means that there is a loss called (under estimation), and on the other hand if it is $\widehat{\alpha}>\alpha$ the loss is called (over estimation)

The loss function $L(\widehat{\alpha},\alpha)$can be defined as a real-valued function that achieves:

1. $L(\widehat{\alpha},\alpha)$>0 for all possible estimates $\widehat{\alpha}$ and for all values of the parameter α for the population of the study.

2. $L(\widehat{\alpha},\alpha)$=0 when $\widehat{\alpha}$=α

There are different loss functions, and each of these functions gives a Bayes estimator for α that differs from the estimators of other functions. In this research, we will use the squared loss function, which assumes the loss over estimation is equal to the loss under estimation, and the formula for this function is as follows:

$$L\left(\widehat{\alpha},\alpha\right)=a(\widehat{\alpha}-\alpha)^2$$

To find a Bayes estimator, we need to find the posterior expectation of the loss function $E\left[L\left(\widehat{\alpha},\alpha\right)\right]$by the formula $\int_{\forall\alpha}L\left(\widehat{\alpha},\alpha\right)p\left(\alpha|\,t\right)\;d\alpha$ which is called the risk function, then The Bayes estimator is the reduction of this function with respect to $\widehat{\alpha}$.

A bayes estimator using the non-informational density function and the squared loss function :

$$\pi\left(\alpha,\beta\right)=\frac{\mathrm{k}}{\alpha\beta}$$

$$n!\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right]^{-2} \qquad = L(\alpha,\beta\;|t_i)$$

$$L\left(\alpha,\beta|t\right).\pi\left(\alpha,\beta\right)=n!*k\left[\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}\right]$$

Posterior distribution

$$\pi\left(\alpha,\beta/\mathrm{t}_1...\mathrm{t_z}\right)=\frac{n!*k\left[\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}\right]}{n!*k\left[\int_0^{\infty}\int_0^{\infty}\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta\right]}$$

$$\pi\left(\alpha,\beta/\mathrm{t}_1...\mathrm{t_z}\right)=\frac{\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}}{\int_0^{\infty}\int_0^{\infty}\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta} \tag{6.7}$$

survival function:

$$S(t)=\frac{1}{1+\left(\frac{t}{\alpha}\right)^{\beta}}$$

Estimating the survival function using the squared loss function:

$$\widehat{\mathrm{S}}_{SLF}\left(\mathrm{t}\right)=\int_0^{\infty}\int_0^{\infty}S\left(t\right)\pi\left(\alpha,\beta|\,t_{1,}t_2..t_n\right)d\alpha d\beta$$

$$\widehat{\mathrm{S}}_{SLF}\left(\mathrm{t}\right)=\frac{\int_0^{\infty}\int_0^{\infty}\frac{1}{1+\left(\frac{t}{\alpha}\right)^{\beta}}\;\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta} \tag{6.8}$$

Risk function:

$$h\left(t\right)=\frac{\frac{\beta}{\alpha}\left(\frac{t}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t}{\alpha}\right)^{\beta}}$$

Estimating the risk function using the squared loss function:

$$\widehat{\mathrm{h}}_{SLF}\left(\mathrm{t}\right)=\int_0^{\infty}\int_0^{\infty}h\left(t\right)\pi\left(\alpha,\beta|\,t_{1,}t_2..t_n\right)d\alpha d\beta$$

$$\widehat{\mathrm{h}}_{SLF}\left(\mathrm{t}\right)=\frac{\int_0^{\infty}\int_0^{\infty}\frac{\frac{\beta}{\alpha}\left(\frac{t_i}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t_j}{\alpha}\right)^{\beta}}\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta}{\int_0^{\infty}\int_0^{\infty}\frac{1}{\alpha\beta}*\beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}{t_i}^{\beta-1}\prod_{i=1}^{n}\left[1+\left(\frac{\mathrm{t}_i}{\alpha}\right)^{\beta}\right]^{-2}d\alpha d\beta} \tag{6.9}$$

7. Lindley Approximation

Due to the difficulty of calculating the integrals of finding the estimator of the survival function and the risk function, so it is necessary to use approximate methods to solve these integrals and to find the estimator of the survival function and risk in the Bayesian method , In this research, we will use the Lindley approximation to calculate the proportion of the integrals, and this method was suggested by the researcher Lindley (D.V. Lindley, 1980). According to this method, the formulas (6.8) and (6.9) can be approximated to the following formula:

$$E\left(\phi\left(\alpha,\beta\right)\middle|\, t\right) \cong \phi\left(\alpha,\beta\right) + \frac{1}{2}\left[\, A + l_{30}B_{12} + l_{03}B_{21} + l_{21}C_{12} + l_{12}C_{21}\right] + P_1A_{12} + p_2A_{21} \tag{7.1}$$

Where:
$\phi(\alpha,\beta)$: is the function to be estimated, which in this research is the survival function S(t) and then the risk function h(t)

$$A = \sum_{i=1}^{2}\sum_{j=1}^{2} \omega_{ij}\ \tau_{ij} \tag{7.2}$$

$$l_{ij} = \frac{\partial^{i+j} L\left(t\middle|\, \boldsymbol{\alpha},\boldsymbol{\beta}\right)}{\partial\ \alpha^{i}\beta^{i}} \qquad i,\ j = 0\ ,\ 1\ ,\ 2\ ,\ 3 \quad ; \quad i+j = 3 \tag{7.3}$$

$$P_1 = \frac{\partial ln\ \pi(\alpha\ ,\ \beta)}{\partial\alpha} \tag{7.4}$$

$$P_2 = \frac{\partial \ln\ \pi(\alpha\ ,\ \beta)}{\partial\beta} \tag{7.5}$$

$$\omega_{ij} = \frac{\partial^2\phi(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha_i\partial\beta_j} \tag{7.6}$$

$$A_{ij} = \omega_i\ \tau_{ii} + \ \omega_j\tau_{ji} \tag{7.7}$$

$$B_{ij} = (\ \omega_i\ \tau_{ii} + \omega_j\ \tau_{ij})\,\tau_{ii} \tag{7.8}$$

$$C_{ij} = 3\omega_i\ \tau_{ii}\ \tau_{ij} + \omega_j\ \ \left(\tau_{ii}\ \tau_{jj} + \ 2(\ \tau_{ij}^2\right)\ ) \tag{7.9}$$

$$\omega_1 = \frac{\partial\phi(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha}$$

$$\omega_2 = \frac{\partial\phi(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\beta} \tag{7.10}$$

Observed Fisher Information:

$$I = \begin{bmatrix} \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\boldsymbol{\alpha}^2} & \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha\partial\beta} \\ \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\alpha\partial\beta} & \frac{\partial^2 \ln L(\boldsymbol{\alpha},\boldsymbol{\beta})}{\partial\boldsymbol{\beta}^2} \end{bmatrix}$$

τ_{ij}represents the negative inverse of the Fisher information matrix and its elements as follows:

$$\tau_{ii} = \frac{-I_{jj}}{(I_{ii}I_{jj} - I_{ij}I_{ji})}$$

$$\tau_{jj} = \frac{-I_{ii}}{(I_{ii}I_{jj} - I_{ij}I_{ji})}$$

$$\tau_{ij} = \frac{-I_{ji}}{(I_{ii}I_{jj} - I_{ij}I_{ji})} \qquad \text{i}\neq\text{j}$$

When using the Lindley approximation to estimate the survival function and risk function of the LL distribution, we get:

$$\boldsymbol{l_{30}}=\frac{\boldsymbol{\partial^3 \ln L\left(t|\,\alpha,\beta\right)}}{\boldsymbol{\partial\alpha^3 d\beta^0}}$$

$$=\frac{2\beta}{\alpha^3}\left[-\text{n}+2\sum_{\text{i}=1}^{\text{n}}\frac{1}{1+\left(\frac{\text{t}_\text{i}}{\alpha}\right)^{-\beta}}+\left(3\beta-\beta^2\right)\sum_{\text{i}=1}^{\text{n}}\frac{\left(\frac{\text{t}_\text{i}}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{\text{t}_\text{i}}{\alpha}\right)^{-\beta}\right)^2}+2\beta^2\sum_{\text{i}=1}^{\text{n}}\frac{\left(\frac{\text{t}_\text{i}}{\alpha}\right)^{-2\beta}}{\left(1+\left(\frac{\text{t}_\text{i}}{\alpha}\right)^{-\beta}\right)^3}\right]$$

$$\boldsymbol{l_{03}}=\frac{\boldsymbol{\partial^3 \ln L\left(t|\,\alpha,\beta\right)}}{\boldsymbol{\partial\alpha^0 d\beta^3}}$$

$$=\frac{2n}{\beta^3}+2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln^3\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{ti}{\alpha}\right)^{-\beta}\right)^2}-4\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-2\beta}\ln^3\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{ti}{\alpha}\right)^{-\beta}\right)^3}$$

$$\boldsymbol{l_{21}}=\frac{\boldsymbol{\partial^3 \ln L\left(t|\,\alpha,\beta\right)}}{\boldsymbol{\partial\alpha^2 d\beta^1}}$$

$$=\frac{1}{\alpha^2}\left[n-2\sum_{i=1}^{n}\frac{1}{1+\left(\frac{ti}{\alpha}\right)^{-\beta}}+2\beta(\beta-3)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\left[\ln\left(\frac{t_i}{\alpha}\right)\ -1\right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}-4\beta^2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^3}\right]$$

$$\boldsymbol{l_{12}}=\frac{\boldsymbol{\partial^3 \ln L\left(t|\,\alpha,\beta\right)}}{\boldsymbol{\partial\alpha^1 d\beta^2}}$$

$$=\frac{2\left(2-\beta\right)}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)\left[1+\ln\left(\frac{t_i}{\alpha}\right)\ \right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}+\frac{4\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-2\beta}ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^3}$$

$$\pi\left(\alpha\beta\right)=\frac{1}{\alpha\beta}=\left(\alpha B\right)^{-1}$$

$$log\left(\pi\left(\alpha\beta\right)\right)=-\ln\ \left(\alpha\beta\right)$$

$$\text{P1}=\frac{\partial\log\Pi\left(\alpha\beta\right)}{\partial\alpha}=\frac{-1}{\alpha}$$

$$\text{P2}=\frac{\partial\log\Pi\left(\alpha\beta\right)}{\partial\beta}=\frac{-1}{\beta}$$

$$I=\begin{bmatrix}\frac{n\beta}{\alpha^2}-\frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}}-\frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} & -\frac{n}{\alpha}+\frac{2}{\alpha}\sum_{i=}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}}+\frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}\\ -\frac{n}{\alpha}+\frac{2}{\alpha}\sum_{i=}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}}+\frac{2\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} & -\frac{n}{\beta^2}-2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}\end{bmatrix}$$

$$\tau_{ij} = -I^{-1}$$

$$= \frac{-\begin{bmatrix} -\frac{n}{\beta^2} - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} & \frac{n}{\alpha} - \frac{2}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{1+(\frac{t_i}{\alpha})^{-\beta}} - \frac{2\beta}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} \\ \frac{n}{\alpha} - \frac{2}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{1+(\frac{t_i}{\alpha})^{-\beta}} - \frac{2\beta}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} & \frac{n\beta}{\alpha^2} - \frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}} - \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2} \end{bmatrix}}{\left(\frac{n\beta}{\alpha^2} - \frac{2\beta}{\alpha^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{-\beta}} - \frac{2\beta^2}{\alpha^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}\right)\left(-\frac{n}{\beta^2} - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}\right) - \left(-\frac{n}{\alpha} + \frac{2}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{1+(\frac{t_i}{\alpha})^{-\beta}} + \frac{2\beta}{\alpha}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{-\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{-\beta}\right)^2}\right)^2}$$

survival function:

$$S(t_i) = \sum_{i=1}^{n}\frac{1}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}} \qquad t_i > 0,\ \alpha,\beta > 0$$

$$W1 = \frac{\partial s(t)}{\partial\alpha} = \frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}$$

$$W2 = \frac{\partial s(t)}{\partial\beta} = \sum_{i=1}^{n}\frac{-\left(\frac{t_i}{\alpha}\right)^{\beta}\ln(\frac{t_i}{\alpha})}{\left(1+(\frac{t_i}{\alpha})^{\beta}\right)^2}$$

$$W11 = \frac{\partial^2 s(t)}{\partial\alpha^2} = \frac{-\beta}{\alpha^2}\left[(\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2} - 2\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}\right]$$

$$W22 = \frac{\partial^2 s(t)}{\partial\beta^2} = 2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3} - \sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta}\ln^2\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}$$

$$W12 = \frac{\partial s(t)}{\partial\alpha\partial\beta} = \frac{1}{\alpha}\left[(\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta}\left[\ln\left(\frac{t_i}{\alpha}\right)+1\right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2} - 2\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}\right]$$

$$W21 = \frac{\partial s(t)}{\partial\beta\partial\alpha} = \frac{1}{\alpha}\left[(\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta}\left[\ln\left(\frac{t_i}{\alpha}\right)+1\right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2} - 2\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}\right]$$

$$A = w_{11}\tau_{11} + w_{12}\tau_{12} + w_{21}\tau_{21} + w_{22}\tau_{22}$$
$$A_{12} = \mathrm{w}_1\tau_{11} + \mathrm{w}_{12}\tau_{21}$$
$$A_{21} = 2\mathrm{w}_2\ \tau_{22} + \mathrm{w}_1\tau_{12}$$
$$B_{12} = 3(\mathrm{w}_1\tau_{11} + \mathrm{w}_2\tau_{12})\tau_{11}$$
$$B_{21} = (\mathrm{w}_2\tau_{22} + \mathrm{w}_1\tau_{21})\tau_{22}$$
$$\mathrm{C}_{12} = 3\mathrm{w}_1\tau_{11}\tau_{12} + \mathrm{w}_2(\tau_{11}\tau_{22} + 2{\tau_{12}}^{2p})$$
$$\mathrm{C}_{21} = 3\mathrm{w}_2\tau_{22}\tau_{21} + \mathrm{w}_1\left(\tau_{22}\tau_{11} + 2{\tau_{21}}^2\right)$$

By compensation the special derivations in equation (7.1) we get the survival function estimator of the Bayes method under the squared loss function risk function :

$$h\left(t_i\right)=\sum_{i=1}^{n}\frac{\frac{\beta}{\alpha}\left(\frac{t_i}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}} \qquad \mathrm{t}_i>0\ ,\alpha,\beta>0$$

$$Wh_1=\frac{\partial h\left(t\right)}{\partial\alpha}=\frac{\beta^2}{\alpha^2}\left[\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta-1}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}-2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]$$

$$Wh_2=\frac{\partial h\left(t\right)}{\partial\beta}=\frac{1}{\alpha}\left[(\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{a}\right)^{\beta-1}\left[\ln\left(\frac{t_i}{\alpha}\right)\ +1\right]}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}-\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{a}\right)^{2\beta-1}\ln\left(\frac{t_i}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}\right]$$

$$Wh_{11}=\frac{\partial h\left(t\right)}{\partial\alpha^2}=\frac{\beta^2}{\alpha^3}\left[2\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{3\beta-1}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}-(4\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta-1}}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}+2(\beta+1)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta-1}}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]$$

$$Wh_{12}=\frac{\partial h\left(t\right)}{\partial\alpha\partial\beta}=\frac{\beta}{\alpha^2}\left[-2\beta\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{3\beta-1}\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}+2\left(2\beta+1\right)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta-1}\left[\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)+1\right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}\right.$$
$$\left.-2\left(\beta+2\right)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta-1}\left[\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)+1\right]}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]$$

$$Wh_{21}=\frac{\partial h\left(t\right)}{\partial\beta\partial\alpha}=\frac{1}{\alpha^2}\left[-2\beta^2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{3\beta-1}\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}+\beta\left(3\beta+2\right)\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{2\beta-1}\left[3\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)+2\right]}{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}\right.$$
$$\left.-(\beta+1)^2\sum_{i=1}^{n}\frac{\left(\frac{t_i}{\alpha}\right)^{\beta-1}\left[3+2\ln\left(\frac{\mathrm{t_i}}{\alpha}\right)\right]}{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}\right]$$

$$Wh_{22}=\frac{\partial h\left(t\right)}{\partial\beta^2}=\boldsymbol{\frac{1}{\alpha}}\left[\boldsymbol{2\beta}\sum_{i=1}^{n}\frac{\boldsymbol{\left(\frac{t_i}{a}\right)^{3\beta-1}\ln^2\left(\frac{t_i}{\alpha}\right)}}{\boldsymbol{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^3}}-\boldsymbol{(3\beta+2)}\sum_{i=1}^{n}\frac{\boldsymbol{\left(\frac{t_i}{a}\right)^{2\beta-1}\ln\left(\frac{t_i}{\alpha}\right)\left[2\ln\left(\frac{t_i}{\alpha}\right)\ +2\right]}}{\boldsymbol{\left(1+\left(\frac{t_i}{\alpha}\right)^{\beta}\right)^2}}\right.$$
$$\left.+\ \boldsymbol{(\beta+2)}\sum_{i=1}^{n}\frac{\boldsymbol{\left(\frac{t_i}{a}\right)^{\beta-1}\left[\ln\left(\frac{t_i}{\alpha}\right)\ +1\right]^2}}{\boldsymbol{1+\left(\frac{t_i}{\alpha}\right)^{\beta}}}\right]$$

$$A=wh_{11}\tau_{11}+wh_{12}\tau_{12}+wh_{21}\tau_{21}+wh_{22}\tau_{22}$$
$$A_{12}=\mathrm{wh}_1\tau_{11}+\mathrm{wh}_2\tau_{21}$$
$$A_{21}=\mathrm{wh}_2\ \tau_{22}+\mathrm{wh}_1\tau_{12}$$
$$B_{12}=(wh_1\tau_{11}+wh_2\tau_{12})\tau_{11}$$
$$B_{21}=(wh_2\tau_{22}+wh_1\tau_{21})\tau_{22}$$
$$\mathrm{C}_{12}=3wh_1\tau_{11}\tau_{12}+wh_2(\tau_{11}\tau_{22}+2{\tau_{12}}^{2p})$$
$$\mathrm{C}_{21}=3wh_2\tau_{22}\tau_{21}+wh_1\left(\tau_{22}\tau_{11}+2{\tau_{21}}^2\right)$$

By compensation the special derivations in equation (7.1) we get the risk function estimator of the Bayes method under the squared loss function.

8. The Simulation Approach

The simulation Approach was used in the Monte Carlo method to compare between the different estimation methods, as this Approach is characterized by flexibility and saves a lot of costs by considering the different sample sizes and the different values of the distribution parameters and the repetition of the experiment each time. In this Approach, data is generated without resorting to real data with Without prejudice to the required accuracy, this Approach is summarized in the following steps:

- **The first step**: **Define the default values**

  Default values were used for the two parameters based on the estimated values of the practical side and the selection of three sample sizes, which are (50,75,100)

- **The second step: Data generation**

  If the random variable is generated by the inverse transformation method as follows:

$$\mathrm{t_i}{=}\alpha\left(\frac{1{-}u}{\mathrm{u}}\right)^{-\frac{1}{\beta}}, \qquad i=1,2,3,\ldots,n \tag{8.1}$$

- **The third step: Sort the generated data in ascending order $t_1 < t_2 < \cdots < t_i$**

- **The fourth step: Solve the equations obtained by numerical methods**

- **The Fifth step: The best method was determined by the comparison scale (IMSE) in the case of estimating the survival function, risk function and probability distribution function.**

$$IMSE(\widehat{\mathrm{S}}\,(\mathrm{t})) = \frac{1}{r}\sum_{i=1}^{r}\left[\frac{1}{n_t}\sum_{\mathrm{j}=1}^{n_t}(\widehat{\mathrm{s}}_{\mathrm{i}}\,(\mathrm{t_j})\,-\mathrm{s}\,(\mathrm{t_j}))^2\right] \tag{8.2}$$

$$IMSE(\widehat{\mathrm{S}}\,(\mathrm{t})) = \frac{1}{r}\sum_{i=1}^{r}\left[\frac{1}{n_t}\sum_{\mathrm{j}=1}^{n_t}(\widehat{\mathrm{s}}_{\mathrm{i}}\,(\mathrm{t_j})\,-\mathrm{s}\,(\mathrm{t_j}))^2\right] \tag{8.3}$$

$$IMSE(\hat{f}\,(\mathrm{t})) = \frac{1}{r}\sum_{i=1}^{r}\left[\frac{1}{n_t}\sum_{\mathrm{j}=1}^{n_t}\left(\hat{f}_i\,(\mathrm{t_j})\,-\mathrm{f}\,(\mathrm{t_j})\right)^2\right] \tag{8.4}$$

where:
r: The number of repetitions of the experiment (1000)times

n_t: The number of data generated per sample

$\widehat{\mathrm{f}}_{\mathrm{i}}\,(\mathrm{t_j})\,,\widehat{\mathrm{s}}_{\mathrm{i}}\,(\mathrm{t_j})\,,\widehat{\mathrm{h}}_{\mathrm{i}}\,(\mathrm{t_j})$: The probability distribution function, survival function, and risk function estimator respectively.

$f\,(\mathrm{t_j})$, $S\,(\mathrm{t_j})\,,h\,(\mathrm{t_j})$: Probability distribution function, survival function and risk function by initial values, respectively.

- **Sixth step: Calculate the mean square error (MSE) for each value of the variable (t_i) for the α and β distribution parameters.**

$$\mathrm{MSE}(\widehat{\alpha}_i)=\frac{\sum_{i=1}^{r}\left(\hat{a}_i-\alpha_i\right)^2}{r} \tag{8.5}$$

$$\mathrm{MSE}(\widehat{\beta}_i)=\frac{\sum_{i=1}^{r}\left(\widehat{\beta}_i-\beta_i\right)^2}{r} \tag{8.6}$$

where:
r: The number of repetitions of the experiment (1000) times

$\widehat{\alpha}_i, \widehat{\beta}_i$: estimators of distribution parameters, respectively.

α, β: The parameters of the distribution according to the initial values, respectively.

The results are represented by the following tables:

Table 1: shows the default and estimated parameter values for $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, sample sizes, and for all experiments

		$\widehat{\alpha}$		$\widehat{\beta}$	
model	n	MLE	Bayslf	MLE	Bayslf
α=2.0,β=2.0	50	2.05412	1.054029	2.021349	1.023302
	75	1.041727	.048618 1	1.028554	1.021891
	100	1.038246	1.039385	1.02942	1.019945
α=2.5,β=2.5	50	2.490443	2.48064	2.49015	2.480195
	75	2.399779	1.601469	2.423485	2.434578
	100	2.384991	1.496439	1.411661	2.439427
α=2.7,β=4.6	50	2.709721	2.69999	4.691514	4.681057
	75	2.682198	2.692053	4.608248	4.688603
	100	2.501577	2.523006	4.590082	4.60075
α=3.0,β=5.3	50	2.093172	1.090602	5.241676	5.231614
	75	1.040121	1.042101	4.714326	5.114657
	100	1.022057	1.020615	4.507596	4.506071

Table 2: shows the MSE values for parameter estimation ($\boldsymbol{\alpha}$) and ($\boldsymbol{\beta}$) in all methods, sample sizes and for all experiments.

		$\widehat{\alpha}$			$\widehat{\beta}$		
model	n	MLE	Bayslf	Best	MLE	Bayslf	Best
α=2.0,β=2.0	50	2.72E-02	2.70E-02	Bayslf	1.51E-02	1.50E-02	Bayslf
	75	2.06E-02	2.13E-02	MLE	1.20E-02	1.19E-02	Bayslf
	100	2.00E-02	2.03E-02	MLE	1.15E-02	1.18E-02	MLE
α=2.5,β=2.5	50	4.08E-02	4.07E-02	Bayslf	2.97E-02	2.90E-02	Bayslf
	75	4.99E-03	4.06E-02	MLE	1.16E-02	1.58E-02	MLE
	100	3.20E-03	3.22E-03	MLE	1.05E-02	1.73E-02	MLE
α=2.7,β=4.6	50	1.52E-02	1.49E-02	Bayslf	4.60E-02	4.50E-02	Bayslf
	75	6.29E-03	6.40E-03	MLE	9.72E-03	1.79E-02	MLE
	100	1.30E-03	2.26E-03	MLE	0.009538	0.01063	MLE
α=3.0,β=5.3	50	2.06E-02	2.01E-02	Bayslf	0.022297	0.022174	Bayslf
	75	6.73E-03	6.97E-03	MLE	0.014484	0.014569	MLE
	100	2.48E-03	3.39E-03	MLE	0.011319	0.013953	MLE

We notice from Table No. 2 and in all experiments, when sample size n = 50, the Bayes method best, but in the case of sample size n = 75,100, the method of Maximum likelihood best in estimating the parameters of the distribution (α and β).

Table 3: shows the IMSE values for estimating the survival function by all methods, sample sizes and for all experiments

model	n	MLE	Bayslf	best
α=2.0,β=2.0	50	8.91E-02	8.80E-02	Bayslf
	75	1.10E-02	1.39E-02	MLE
	100	8.41E-03	8.50E-03	MLE
α=2.5,β=2.5	50	1.04E-02	1.02E-02	Bayslf
	75	1.13E-03	1.14E-03	MLE
	100	1.04E-03	1.12E-03	MLE
α=2.7,β=4.6	50	7.07E-02	7.06E-02	Bayslf
	75	9.21E-03	9.22E-03	MLE
	100	6.80E-03	6.81E-03	MLE
α=3.0,β=5.3	50	8.00E-03	7.96E-03	Bayslf
	75	4.90E-03	4.92E-03	MLE
	100	1.26E-03	1.30E-03	MLE

Table 4: shows the IMSE values for estimating the risk function for all methods, sample sizes and for all experiments.

model	n	MLE	Bayslf	best
α=2.0,β=2.0	50	4.22E-02	1.10E-02	Bayslf
	75	4.00E-03	4.01E-03	MLE
	100	4.94E-04	4.95E-04	MLE
α=2.5,β=2.5	50	6.42E-02	6.42E-03	Bayslf
	75	6.30E-03	6.41E-03	MLE
	100	6.05E-03	6.16E-03	MLE
α=2.7,β=4.6	50	0.279779	0.249545	Bayslf
	75	0.223695	0.233841	MLE
	100	0.20355	0.223485	MLE
α=3.0,β=5.3	50	0.449907	0.410512	Bayslf
	75	0.239492	0.328949	MLE
	100	0.221082	0.271311	MLE

Table 5: shows the IMSE values for estimating the probability density function for all methods, sample sizes and for all experiments.

model	n	MLE	Bayslf	best
α=2.0,β=2.0	50	3.07E-02	3.05E-02	Bayslf
	75	3.03E-02	3.05E-02	MLE
	100	3.00E-02	3.03E-02	MLE
α=2.5,β=2.5	50	0.048972	4.85E-02	Bayslf
	75	4.79E-02	4.83E-02	MLE
	100	4.78E-02	4.81E-02	MLE
α=2.7,β=4.6	50	4.89E-02	4.80E-02	Bayslf
	75	4.63E-02	4.67E-02	MLE
	100	0.032048	4.23E-02	MLE
α=3.0,β=5.3	50	2.41E-02	2.40E-02	Bayslf
	75	2.22E-02	2.34E-02	MLE
	100	0.022249	2.32E-02	MLE

We notice from Tables No. 3, 4 and 5, in all experiments, and when sample size n = 50, the Bayes method best, but in the case of sample size n = 75,100, the method of Maximum likelihood best in estimating the survival function, risk function and the probability density function.

9. the practical Approach

Real data were collected for (50) breast cancer patients at Medical City Hospital - Baghdad for the year 2020. As the time of the patient's entry to the center until discharge was recorded, and that all of them were in a state of death upon discharge, and this data is considered complete data .

Table 6: shows the real data

t_i	2.1 7.2 1.2 3.4 8.6 2.1 6.4 3 8.12 5.4 5.2 3.4 2.1 2.1 3.4 6.4 6.4
t_i	3.2 3.2 3.2 6.2 12.4 1.9 5.6 4.1 2.1 3.4 9.8 3.1 2.1 2.1 12.4 11.4
t_i	3.4 3.4 6.5 3.2 12.1 5.4 6.1 6.1 6.1 2.4 8.4 10.5 10.5 8.5 9.1 9.1 5.3

Table No. 6 represents the failure times for breast cancer patients in months, and the fractional number represents the failure times in days

9.1. Data fitting test

To show the suitability of the real data to the log-logistic distribution, a test was conducted for the above data using the Standard Easy Fit 5.5 program to test the following statistical hypothesis at the 0.05 level of significance.
: The data follow a log-logistic distribution Ho
: The data does not follow a log-logistic distribution H1
The results of the test were as in the following table 7:

Table 7: shows the fitting tests for the application data

Test	Sig
Kolmogorov-smirnov	0.13623
Anderson Darling	0.9763
Chi-squared	8.977

We notice from Table 7 that the sig value for all tests is greater than 0.05, so we accept the null hypothesis Ho, meaning that the selected real data follow a log-logistic distribution.

Table 8: shows the values of estimator parameters log-logistic distribution of estimation methods

$\widehat{\alpha}_{MLE}$ $\widehat{\beta}_{MLE}$	$\widehat{\alpha}_{OLS}$ $\widehat{\beta}_{OLS}$	$\widehat{\alpha}_{WLS}$ $\widehat{\beta}_{WLS}$	$\widehat{\alpha}_{Q.E}$ $\widehat{\beta}_{Q.E}$	$\widehat{\alpha}_{Bay}$ $\widehat{\beta}_{Bay}$
3.279519	3.405548	3.001948	2.714018	3.51449
4.889562	5.133424	5.37474	5.360724	5.309038

Table 9: shows the estimation of survival and risk function for application data

t_i	$\widehat{S}(t)_{MLE}$	$\widehat{S}(t)_{Bay}$	$\widehat{h}(t)_{MLE}$	$\widehat{h}(t)_{Bay}$	t_i	$\widehat{S}(t)_{MLE}$	$\widehat{S}(t)_{Bay}$	$\widehat{h}(t)_{MLE}$	$\widehat{h}(t)_{Bay}$
1.2	0.217695	0.210239	2.96E-02	0.014681	5.4	8.58E-03	7.14E-03	0.912776	0.921946
1.9	0.107034	0.100814	0.166841	0.102777	5.4	8.58E-03	7.14E-03	0.912776	0.921946
2.1	8.89E-02	8.32E-02	0.236563	0.154217	5.6	7.67E-03	6.35E-03	0.933676	0.934333
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.1	5.87E-03	4.78E-03	0.93478	0.956106
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.1	5.87E-03	4.78E-03	0.93478	0.956106
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.1	5.87E-03	4.78E-03	0.93478	0.956106
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.2	5.57E-03	4.53E-03	0.955093	0.966213
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.4	5.03E-03	4.06E-03	0.965998	0.976489
2.1	8.89E-02	8.32E-02	0.236563	0.154217	6.4	5.03E-03	4.06E-03	0.965998	0.976489
2.4	6.82E-02	6.31E-02	0.363627	0.257937	6.4	5.03E-03	4.06E-03	0.965998	0.976489
3	4.17E-02	3.78E-02	0.640197	0.533498	6.5	4.78E-03	3.85E-03	0.996619	0.996712
3.1	3.86E-02	3.48E-02	0.680781	0.581149	7.2	3.41E-03	2.69E-03	1.664887	0.991351
3.2	3.58E-02	3.22E-02	0.718202	0.62727	8.12	2.26E-03	1.74E-03	1.895094	0.996245
3.2	3.58E-02	3.22E-02	0.718202	0.62727	8.4	2.01E-03	1.53E-03	1.896291	0.995898
3.2	3.58E-02	3.22E-02	0.718202	0.62727	8.5	1.93E-03	1.46E-03	1.96983	0.996789
3.2	3.58E-02	3.22E-02	0.718202	0.62727	8.6	1.85E-03	1.40E-03	1.963499	1.80204
3.4	3.09E-02	2.75E-02	0.782313	0.712278	9.1	1.51E-03	1.13E-03	1.963683	1.839699
3.4	3.09E-02	2.75E-02	0.782313	0.712278	9.1	1.51E-03	1.13E-03	1.963683	1.839699
3.4	3.09E-02	2.75E-02	0.782313	0.712278	9.8	1.15E-03	8.49E-04	1.976583	1.899408
3.4	3.09E-02	2.75E-02	0.782313	0.712278	10.5	8.95E-04	6.47E-04	1.984104	1.904113
3.4	3.09E-02	2.75E-02	0.782313	0.712278	10.5	8.95E-04	6.47E-04	1.986104	1.904113
3.4	3.09E-02	2.75E-02	0.782313	0.712278	11.4	6.57E-04	4.65E-04	1.987941	1.934805
4.1	1.90E-02	1.65E-02	0.892904	0.898431	12.1	5.23E-04	3.64E-04	1.993415	1.938145
5.2	9.61E-03	8.06E-03	0.890959	0.907573	12.1	5.23E-04	3.64E-04	1.993415	1.938145
5.3	9.08E-03	7.58E-03	0.892018	0. 91957	12.4	4.76E-04	3.29E-04	1.993729	1.977619

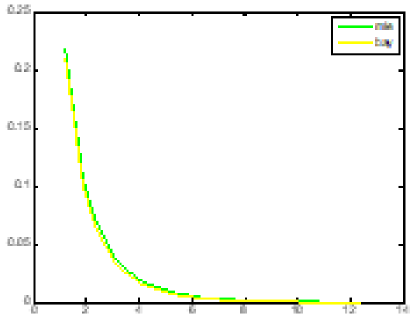

Figure 1: represents the behavior of estimating the survival function

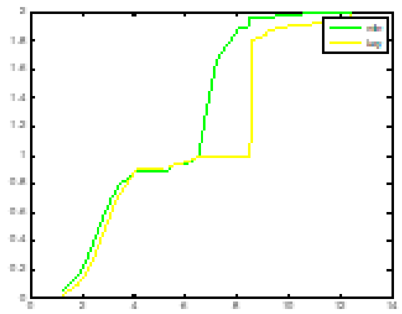

Figure 2: represents the behavior of estimating the Risk function

We notice from Table No. 9 and Figures 1 and 2, that the survival function appeared decreasing, and the risk function appeared increasingly, and this corresponds to the theory of the two functions.

Table 10: shows the estimation of probability density function for application data

t_i	$\hat{f}(t)_{MLE}$	$\hat{f}(t)_{Bay}$	t_i	$\hat{f}(t)_{MLE}$	$\hat{f}(t)_{Bay}$	t_i	$\hat{f}(t)_{MLE}$	$\hat{f}(t)_{MLE}$
1.2	2.94E-02	1.46E-02	3.4	0.356744	0.387369	6.4	0.02697	0.02697
1.9	0.156025	9.90E-02	3.4	0.356744	0.387369	6.5	2.47E-02	2.47E-02
2.1	0.212528	0.14481	3.4	0.356744	0.387369	7.2	1.39E-02	1.39E-02
2.1	0.212528	0.14481	3.4	0.356744	0.387369	8.12	6.99E-03	6.99E-03
2.1	0.212528	0.14481	3.4	0.356744	0.387369	8.4	5.74E-03	5.74E-03
2.1	0.212528	0.14481	4.1	0.22437	0.275073	8.5	5.36E-03	5.36E-03
2.1	0.212528	0.14481	5.2	8.09E-02	0.100801	8.6	5.01E-03	5.01E-03
2.1	0.212528	0.14481	5.3	7.35E-02	0.091328	9.1	3.61E-03	3.61E-03
2.1	0.212528	0.14481	5.4	6.69E-02	8.27E-02	9.1	3.61E-03	3.61E-03
2.4	0.298726	0.22786	5.4	6.69E-02	8.27E-02	9.8	2.34E-03	2.34E-03
3	0.388731	0.372667	5.6	5.54E-02	0.067978	10.5	1.56E-03	1.56E-03
3.1	0.386944	0.383943	6.1	3.51E-02	4.20E-02	10.5	1.56E-03	1.56E-03
3.2	0.380625	0.390109	6.1	3.51E-02	4.20E-02	11.4	9.65E-04	9.65E-04
3.2	0.380625	0.390109	6.1	3.51E-02	4.20E-02	12.1	6.80E-04	6.80E-04
3.2	0.380625	0.390109	6.2	3.21E-02	3.82E-02	12.1	6.80E-04	6.80E-04
3.2	0.380625	0.390109	6.4	0.02697	3.17E-02	12.4	5.89E-04	5.89E-04
3.4	0.356744	0.387369	6.4	0.02697	3.17E-02			

We notice from Table No. 10 and Figure 3 that the probability density function appeared increasingly and then gradually decreased

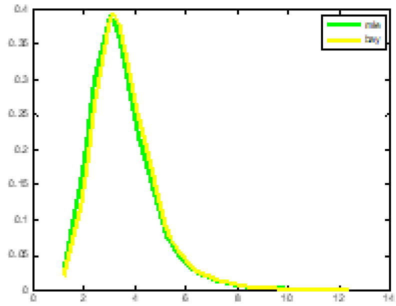

Figure 3: represents the behavior of estimating the probability density function

10. Conclusion

1. Through the results of the simulation experiments, the Bayes estimator for survival, risk, and probability density function at sample size (n = 50) best was shown. under Squared loss function and Lindley approximation.

2. In the case of samples (n = 75,100), it was found that the estimator of survival function, risk and probability distribution function using the method of Maximum likelihood were the most efficient.

3. It has been found the values of the estimators converge more to the true values as the time increases in the presence of order statistics .

4. From the conclusions of the practical side, it became clear that the values of the risk function are increasing with the increase in the time of infection for a group of breast cancer patients under study, and this corresponds to the theoretical properties of this function as it is a monotone increasing function.

5. The values of the survival function appeared decreasing as the values of (t) increased, and this corresponds to the theory of the survival function

6. The values of the probability density function appeared increasingly and then gradually decreased

11. Recommendations

1. Using other loss functions (general entropy loss function, linear exponential loss function, Deckert loss function) to estimate the two parameters or the survival function or the risk function of the (LL) distribution using the standard Bayes method.

2. the using of log-logistic regression in estimating the survival and risk function of living organisms by using different estimation methods.

3. using other models of order statistics to estimate the survival and risk function of the log-logistic distribution

4. the necessity of accrediting the Ministry of Health the estimators of the survival function, the risk function, and the probability density function in the simulation aspect of breast cancer, and to benefit in the development of future treatment plans and follow of patients.

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