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Extended Hermite-Hadamard $(H-H)$ and Fejer's inequalities based on (h_1, h_2, s)-convex functions

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Abstract

In this paper, (h_1, h_2)-convex and s-convex functions are merged to form (h_1, h_2, s)-convex function. Inequalities of the Hermite-Hadamard (H-H) and Fejer's types will then be extended by using the (h_1, h_2, s)-convex function and its derivatives. Some special cases for these extended H-H and Fejer's inequalities are also explored in order to get the previously specified results. The relationship between newly constructed Hermite-Hadamard $(H-H)$ and Fejer's types of inequalities with the average (mean) values are also discussed.

Keywords: Inequality, Hermite-Hadamard (H-H), Fejer, Convex function
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1. Introduction

The theory of convexity has a significant influence in the development of pure and applied mathematics [2]. A convex function is defined as follows.

Definition 1.1. *[16] A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is convex for $x, y \in \mathbb{I}$ and $a_1 \in [0, 1]$, if*

$$f(a_1 x + (1 - a_1)y) \leq a_1 f(x) + (1 - a_1)f(y). \tag{1.1}$$

Since the introduction of Definition 1.1, many researchers have shown interest to investigate its usability, resulting to various additional useful and interactive extensions of this definition have been proposed. Some of the definitions, which are related our study are stated below.

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Definition 1.2. *[16] Let $h : \mathbb{J} = [0,1] \to \mathbb{R}$ be a non-negative function. We say that $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is h-convex function for $x, y \in \mathbb{I}$, $a_1 \in [0,1]$ if*

$$f(a_1x + (1-a_1)y) \leq h(a_1)f(x) + h(1-a_1)f(y). \tag{1.2}$$

In the same fashion as Definition 1.2, Varošanec [1] introduced h-convex function and is defined as follows.

Definition 1.3. *[1] A real valued function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ is called s-convex function or K_s^2 for $x, y \in \mathbb{I}$ and $a_1 \in [0,1]$, $s \in (0,1]$ if*

$$f(a_1x + (1-a_1)y) \leq a_1^s f(x) + (1-a_1)^s f(y). \tag{1.3}$$

Remark 1.4. *If we choose $s = 1$ in Definition 1.3, then s-convex function will be reduced to the original Definition 1.1 of convex function.*

Recently, some efforts have been made to merge various families of convex functions in order to improve their roles in inequality theory [17, 8, 5, 10, 12, 13, 14, 6, 9, 4]. Noor and Awan [11] further merged h-convex and s-convex functions and defined as follows.

Definition 1.5. *[11] Let $h : \mathbb{J} = [0,1] \to \mathbb{R}$ be a positive function. A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{I}$, $a_1 \in [0,1]$ is called $(h-s)$-convex if*

$$f(a_1x + (1-a_1)y) \leq h^s(a_1)f(x) + h^s(1-a_1)f(y). \tag{1.4}$$

In relation to the above definition, Awan, Noor, Khalida, and Khan [9] introduced the definition of (h_1, h_2)-convex function as

Definition 1.6. *[9] Let $h_1, h_2 : (0,1) \subseteq \mathbb{I} \to \mathbb{R}$ be two positive function such that $h_1, h_2 \not\equiv 0$. A function $f : \mathbb{I} \subset \mathbb{R} \to \mathbb{R}$ and $x, y \in \mathbb{I}$, $a_1 \in [0,1]$ is called (h_1, h_2)-convex function if*

$$f(a_1x + (1-a_1)y) \leq h_1(a_1)h_2(1-a_1)f(x) + h_2(a_1)h_1(1-a_1)f(y). \tag{1.5}$$

In 1998, Dragomir and Agarwal [3] proposed Lemma 1.7 for differentiable convex function and then expanded the H-H inequality based on this lemma.

Lemma 1.7. *[3] Let $f : \mathbb{I} = [p,q] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping with $p < q$. If $f' \in L[p,q]$, then*
$\frac{f(p)+f(q)}{2} - \frac{1}{(q-p)} \int_p^q f(x)d(x) = \left(\frac{q-p}{2}\right) \int_0^1 [1-2a_1] f'(pa_1 + (1-a_1)q)da_1,$
where $L[p,q]$ represents the space of integrable function on $[p,q]$.

The relationship between the H-H inequality and means (averages) was also established for real numbers in applications. Furthermore, some error estimations for trapezoidal formula were also computed. Meanwhile, Sarikaya, Set, Yaldiz, and Basak [15] constructed Lemma 1.8 for convex function with fractional integrals.

Lemma 1.8. *[15] Let $f : \mathbb{I} = [p,q] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping with $p < q$. If $f' \in L[p,q]$, then*
$\frac{f(p)+f(q)}{2} - \frac{\Gamma(1+\alpha)}{2(q-p)^\alpha} \left[J_{p+}^\alpha f(q) + J_{q-}^\alpha f(p)\right] = \left(\frac{q-p}{2}\right) \int_0^1 \left[(1-a_1)^\alpha - (a_1)^\alpha\right] \times f'(pa_1 + (1-a_1)q)\, da_1,$
where $J_{p+}^\alpha f(q)$, $J_{q-}^\alpha f(p)$ represents the Riemann-Liouville integrals and $\Gamma(\alpha)$ is the Gamma function. Based on Lemma 1.8, H-H inequality was further extended for fractional integral. Lemma 1.8 was also employed to verify the results and applications for differentiable convex function presented in [3] were verified by using Lemma 1.8. Further, Kirmaci extended the H-H inequality by introducing Lemma 1.9 for differentiable convex functions.

Lemma 1.9. *[7] Let $f:\mathbb{I}=[p,q]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping with $p<q$. If $f'\in L[p,q]$, then*

$$\frac{1}{(q-p)}\int_p^q f(x)dx-f\left(\frac{q+p}{2}\right)=(q-p)\int_0^{\frac{1}{2}}a_1f'\left(pa_1+(1-a_1)q\right)da_1+(q-p)\times\int_{\frac{1}{2}}^1(1-a_1)f'\left(pa_1+(1-a_1)q\right)da_1.$$

Some error estimations for mid-point formula were obtained by using Lemma 1.9. Motivated by these works of merging different convex functions, this article attempts to merge (h_1,h_2)-convex and s-convex functions.

2. Preliminaries

In this section, some new definitions are obtained as a result of merging (h_1,h_2)-convex and s-convex functions.

Definition 2.1. *Let $h_1,h_2:(0,1)\subseteq\mathbb{J}\to\mathbb{R}$ be two positive functions such that $h_1,h_2\not\equiv 0$. A function $f:\mathbb{I}\subset\mathbb{R}\to\mathbb{R}$ and $x,y\in\mathbb{I}$, $a_1\in[0,1]$, and $s\in[0,1]$ is called (h_1,h_2,s)-convex function if*

$$f(a_1x+(1-a_1)y)\leq h_1^s(a_1)h_2^s(1-a_1)f(x)+h_2^s(a_1)h_1^s(1-a_1)f(y). \tag{2.1}$$

Remark 2.2.
1. *If $h_2^s(a_1)=h_2^s(1-a_1)=1$, then Definition 2.1 will be reduced to $(h-s)$-convex function.*
2. *If $s=1$ in Definition 2.1, then definition for (h_1,h_2)-convex function will be produced.*

In order to extend Hermite-Hadamard (H-H) inequality for differentiable functions, related lemmas for (h_1,h_2,s)-convex function will be constructed.

Lemma 2.3. *Let $f,g:\mathbb{I}=[p,q]\subset\mathbb{R}\to\mathbb{R}$ be two differentiable mappings with $p<q$. If $f',g'\in L[p,q]$ and (h_1,h_2,s)-convex functions, then*

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2}-\frac{1}{2(q-p)}\int_p^q\left[g'\left(\frac{x-p}{q-p}\right)+g'\left(\frac{q-x}{q-p}\right)\right]f(x)dx\quad\left(\frac{q-p}{2}\right)\int_0^1[g(1-a_1)-g(a_1)]f'\left(pa_1+(1-a_1)q\right)da_1 \tag{2.2}$$

Proof . By applying integration by parts on $\int_0^1[g(1-a_1)-g(a_1)]f'(pa_1+(1-a_1)q)da_1$, we get

$$\int_0^1 [g(1-a_1)-g(a_1)]f'(pa_1+(1-a_1)q)da_1 = \frac{[g\,(1-a_1)-g(a_1)]\,f\,(pa_1+(1-a_1)\,q)}{p-q}\,|_0^1$$
$$-\int_0^1 \frac{[g'(1-a_1)-g'(a_1)]f(pa_1+(1-a_1)q)}{p-q}da_1, \tag{2.3}$$
$$=\left[\frac{[g(1-1)-g(1)]f(p(1)+(1{-}1)q)}{p-q}-\frac{[g(1-0)-g(0)]f(p(0)+(1-0)q)}{p-q}\right]$$
$$-\int_0^1 \frac{[g'(1-a_1)-g'(a_1)]f(pa_1+(1-a_1)q)}{p-q}da_1,$$
$$=\left[\frac{[g(0)-g(1)]f(p)}{p-q}-\frac{[g(1)-g(0)]f(q)}{p-q}\right]-\int_0^1 \frac{[g'(a_1)-g'(1-a_1)]f(pa_1+(1-a_1)q)}{p-q}da_1.$$
$$=-\left[\frac{[g(1)-g(0)]f(p)}{p-q}+\frac{[g(1)-g(0)]f(q)}{p-q}\right]-\int_0^1 \frac{[g'(a_1)-g'(1-a_1)]f(pa_1+(1-a_1)q)}{p-q}da_1$$
$$=\left[\frac{[g(1)-g(0)]f(p)}{q-p}+\frac{[g(1)-g(0)]f(q)}{q-p}\right]$$
$$-\int_0^1 \frac{[g'(1-a_1)-g'(a_1)]f(pa_1+(1-a_1)q)}{q-p}da_1. \tag{2.4}$$

□

Let $x=pa_1+(1-a_1)\,q$ in Equation (2.4), then $a_1=\frac{q-x}{(q-p)}$ and $1-a_1=\frac{x-p}{(q-p)}$.
For $a_1=0$, $x{=}p$ and for $a_1=1$, then $x=q$. By applying limit on Equation (2.4), gives

$$\int_0^1 [g(1-a_1)-g(a_1)]f'(pa_1+(1-a_1)q)da_1 = [g(1)-g(0)]\frac{f(p)+f(q)}{p-q}$$
$$-\frac{1}{(q-p)^2}\int_p^q \left[g'(\frac{x-p}{q-p})+g'(\frac{q-x}{q-p})\right]f(x)dx. \tag{2.5}$$

Multiplying both sides by $\left(\frac{q-p}{2}\right)$ produces

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2}-\frac{1}{2(q-p)}\int_p^q [g'(\frac{x-p}{q-p})+g'(\frac{q-x}{q-p})]f(x)dx$$
$$=(\frac{q-p}{2})\int_0^1 [g(1-a_1)-g(a_1)]f'(pa_1+(1-a_1)q)da_1. \tag{2.6}$$

Equality (2.6) completes Lemma 2.3.

Remark 2.4. 1. *If $g(a_1)=a_1$ in Lemma 2.3, then it will be reduced to Lemma 1.7.*
2. *If $s=\frac{a_1^{\alpha}}{\Gamma(1+\alpha)}$ in Lemma 2.3, then we get Lemma 1.8.*

Lemma 2.5. *Let $f,g:\mathbb{I}=[p,q]\subset\mathbb{R}\to\mathbb{R}$ be two differentiable mappings with $p<q$. If $f',g'\in L[p,q]$ and (h_1,h_2,s)-convex functions, then*

$$\left|g(0)(f(p)+f(q))-2g(\frac{1}{2})f(\frac{q+p}{2})+(\frac{1}{q-p})\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)dx+\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right]\right|$$
$$=(q-p)\left[\int_0^{\frac{1}{2}} g(a_1)f'(pa_1+(1-a_1)q)da_1-\int_{\frac{1}{2}}^{1} g(1-a_1)f'(pa_1+(1-a_1)q)da_1\right]. \tag{2.7}$$

Proof . By applying integration by parts on $\int_0^{\frac{1}{2}} g(a_1)f'(pa_1+(1-a_1)q)da_1$ we obtain

$$\int_0^{\frac{1}{2}} g(a_1)f'(pa_1+(1-a_1)q)da_1$$
$$=\frac{g(a_1)f\left(pa_1+(1-a_1)q\right)}{p-q}|_0^{\frac{1}{2}}-\int_0^{\frac{1}{2}}\frac{g'(a_1)f(pa_1+(1-a_1)q)}{p-q}da_1 \tag{2.8}$$
$$=\left[\frac{g\left(\frac{1}{2}\right)f\left(p\left(\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)q\right)}{p-q}-\frac{g(0)f(p(0)+(1-0)q)}{p-q}\right]-\int_0^{\frac{1}{2}}\frac{g'(a_1)f(pa_1+(1-a_1)q)}{p-q}da_1$$
$$=\left[\frac{g\left(\frac{1}{2}\right)f\left(\frac{p+q}{2}\right)}{(p-q)}-\frac{g(0)f(q)}{p-q}\right]-\int_0^{\frac{1}{2}}\frac{g'(a_1)f(pa_1+(1-a_1)q)}{p-q}da_1 \tag{2.9}$$

□

Put $x=pa_1+(1-a_1)q$ in Equation (2.9), then $a_1=\frac{q-x}{(q-p)}$ and $1-a_1=\frac{x-p}{(q-p)}$.

For $a_1=0$, then x=p and for $a_1=1$, then $x=\frac{p+q}{2}$. After substitute all these values in Equation (2.9), gives

$$\int_0^{\frac{1}{2}} g(a_1)f'(pa_1+(1-a_1)q)da_1=\frac{g\left(\frac{1}{2}\right)\left(\frac{p+q}{2}\right)}{q-p}-\frac{g(0)f(q)}{q-p}+\left(\frac{1}{2(q-p)^2}\right)$$
$$\times\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)dx. \tag{2.10}$$

In similar manner, integrating $\int_{\frac{1}{2}}^{1} g(1-a_1)f'(pa_1+(1-a_1)q)da_1$ gives

$$\int_{\frac{1}{2}}^{1} g(1-a_1)f'(pa_1+(1-a_1)q)da_1$$
$$=\left.\frac{g\left(1-a_1\right)f\left(pa_1+\left(1-a_1\right)q\right)}{p-q}\right|_{\frac{1}{2}}^{1}-\int_{\frac{1}{2}}^{1}\frac{g'\left(1-a_1\right)f\left(pa_1+\left(1-a_1\right)q\right)}{p-q}da_1 \tag{2.11}$$
$$=\left[\frac{g\left(\frac{1}{2}\right)f\left(\frac{p+q}{2}\right)}{q-p}-\frac{g(0)f(q)}{q-p}\right]-\frac{1}{2(q-p)^2}\int_p^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx \tag{2.12}$$

Substituting Equation (2.10) and Equation (2.11) in Equation (2.7) yields

$$\left|[g(0)](f(p)+f(q))-2g(\frac{1}{2})f(\frac{q+p}{2})+(\frac{1}{q-p})\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)\,dx+\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right]\right|$$
$$=(q-p)\left[\frac{g\left(\frac{1}{2}\right)f(p+q)}{q-p}+\frac{g(0)\,f(q)}{q-p}+\frac{1}{2(q-p)^2}\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)dx-\frac{g\left(\frac{1}{2}\right)f(p+q)}{q-p}-\frac{g(0)\,f(q)}{q-p}\right.$$
$$\left.-\frac{1}{2(q-p)^2}\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)\,dx\right], \tag{2.13}$$

After multiplying $(q-p)$ on both sides of Equation (2.13) results

$$\left|[g(0)](f(p)+f(q))-2g(\frac{1}{2})f(\frac{q+p}{2})+(\frac{1}{q-p})\left[\int_{\frac{p+q}{2}}^{q} g'\left(\frac{q-x}{q-p}\right)f(x)dx+\int_{p}^{\frac{p+q}{2}} g'\left(\frac{x-p}{q-p}\right)f(x)dx\right]\right|$$
$$=(q-p)\left[\int_{0}^{\frac{1}{2}} g(a_1)\,f'(pa_1+(1-a_1)\,q)\,da_1-\int_{\frac{1}{2}}^{1} g(1-a_1)\,f'(pa_1+(1-a_1)q)da_1\right]. \tag{2.14}$$

Equality (2.14) completes the proof for Lemma 2.5.

Remark 2.6. 1. *If $g(a_1)=a_1$ in Lemma 2.5, then Lemma 1.9 will be produced.*
2. *If $s=\frac{a_1^{\alpha}}{\Gamma(1+\alpha)}$ in Lemma 1.8, then it will be converted to Lemma 1.8.*

3. Main Results

In this section, we extend Hermite-Hadamard inequality based on the newly constructed (h_1,h_2,s)-convex function.

Theorem 3.1. *Let $f:\mathbb{I}=[p,q]\subset\mathbb{R}\to\mathbb{R}$ be a (h_1,h_2,s)-convex function and $s\in[0,1]$ with ${h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)\not\equiv 0$, then*

$$\frac{1}{2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)}f\left(\frac{p+q}{2}\right)\leq\frac{1}{q-p}\int_p^q f(x)dx\leq[f(p)+f(q)]\int_0^1 {h_1}^s(t){h_2}^s(1-t)dt \tag{3.1}$$

Proof . Let f be a (h_1,h_2,s)-convex function and $x=a_1p+(1-a_1)q$, $y=a_1q+(1-a_1)p$ with $a_1=\frac{1}{2}$ in Definition 2.1 gives

$$f\left(\frac{p+q}{2}\right)\leq h_1^s\left(\frac{1}{2}\right)h_2^s\left(\frac{1}{2}\right)[f(a_1p+(1-a_1)q)+f(a_1q+(1-a_1)p)] \tag{3.2}$$

Integrating Inequality (3.2) on both sides with respect to $a_1\in[0,1]$ produces

$$\frac{1}{2h_1^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)}f\left(\frac{p+q}{2}\right)\leq\frac{1}{q-p}\int_p^q f(x)dx. \tag{3.3}$$

Using Definition 2.1 yields

$$f(a_1p+(1-a_1)q)\leq {h_1}^s(a_1){h_2}^s(1-a_1)f(x)+{h_2}^s(a_1){h_1}^s(1-a_1)f(y) \tag{3.4}$$

Integrating Inequality (3.4) on both sides with respect to $a_1 \in [0,1]$ leads to

$$\frac{1}{q-p} f(x)dx \leq [f(p)+f(q)] \int_0^1 {h_1}^s(t){h_2}^s(1-t)dt. \tag{3.5}$$

Now, by combining Inequality (3.3) and Inequality (3.5), the following extension of H-H inequality using (h_1, h_2, s)-convex function is obtained.

$$\frac{1}{2{h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) \leq \frac{1}{q-p}\int_p^q f(x)dx \leq [f(p)+f(q)]\int_0^1 {h_1}^s(t){h_2}^s(1-t)dt.$$

This completes the theorem. □

Remark 3.2. 1. *If ${h_2}^s(a_1) = {h_2}^s(1-a_1) = 1$ in Inequality* (3.1)*, then original H-H inequality will be obtained.*

2. *If $s=1$ in Inequality* (3.1)*, then result will be the H-H inequality for (h_1,h_2)-convex function.*

Theorem 3.3. *Let $f, g : \mathbb{I} = [p,q] \subset \mathbb{R} \to \mathbb{R}$ be (h_1,h_2,s)-convex functions for $s \in [0,1]$, and $a_1 \in [0,1]$ with ${h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right) \not\equiv 0$, then*

$$\begin{aligned}&\frac{1}{2{h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)} f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) - [M(p,q)\int_0^1 [{h_1}^s(a_1){h_2}^s(a_1){h_1}^s(1-a_1){h_2}^s(1-a_1)]da_1 \\ &+ N(p,q)\int_0^1 {h_1}^{2s}(a_1){h_2}^{2s}(1-a_1)] \leq \frac{1}{q-p}\int_0^1 f(x)g(x)dx, .\end{aligned} \tag{3.6}$$

where $M(p,q) = f(p)g(p) + f(q)g(q)$ and $N(p,q) = f(p)g(q) + f(q)g(p)$.

Proof . Since f and g are (h_1,h_2,s)-convex functions, then

$$\begin{aligned}f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) &\leq {h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)[f(a_1p+(1-a_1)q) + f(a_1q+(1-a_1)p)] \\ &\times {h_1}^s\left(\frac{1}{2}\right){h_2}^s\left(\frac{1}{2}\right)[g(a_1p+(1-a_1)q) + g(a_1q+(1-a_1)p)],\end{aligned} \tag{3.7}$$

$$\begin{aligned}&f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) \leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)[f(a_1p+(1-a_1)q)g(a_1p+(1-a_1)q) \\ &+ f((1-a_1)p+a_1q)g((1-a_1)p+a_1q) + f(a_1p+(1-a_1)q)g((1-a_1)p+a_1q) \\ &f((1-a_1)p+a_1q)g(a_1p+(1-a_1)q)],\end{aligned} \tag{3.8}$$

$$\begin{aligned}&f\left(\frac{p+q}{2}\right) g\left(\frac{p+q}{2}\right) \leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)[f(a_1p+(1-a_1)q)g(a_1p+(1-a_1)q) \\ &+ f((1-a_1)p+a_1q)g((1-a_1)p+a_1q) + 2{h_1}^s(a_1){h_2}^s(a_1){h_1}^s(1-a_1){h_2}^s(1-a_1)] \\ &\times [f(p)g(p)+f(q)g(q)] + {h_1}^{2s}(a_1){h_2}^{2s}(1-a_1) + {h_2}^{2s}(a_1){h_1}^{2s}(1-a_1) \times [f(p)g(q)+f(q)g(p)].\end{aligned} \tag{3.9}$$

Apply integration by parts on Inequality (3.9) with respect to a_1 gets

$$f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) \leq {h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)\left[\frac{1}{q-p}\int_0^1 f(x)g(x)dx + M(p,q)\int_0^1 [{h_1}^s(a_1){h_2}^s(a_1)\ ,\right.$$
$$\left.\times {h_1}^s(1-a_1){h_2}^s(1-a_1)]\,da_1 + N(p,q)\int_0^1 {h_1}^{2s}(a_1){h_2}^{2s}(1-a_1)da_1\right] \tag{3.10}$$

which can be written as

$$\frac{1}{2{h_1}^{2s}\left(\frac{1}{2}\right){h_2}^{2s}\left(\frac{1}{2}\right)}f\left(\frac{p+q}{2}\right)g\left(\frac{p+q}{2}\right) - [M(p,q)\int_0^1 [{h_1}^s(a_1){h_2}^s(a_1)\ \times {h_1}^s(1-a_1){h_2}^s(1-a_1)]\,da_1$$
$$+ N(p,q)\int_0^1 {h_1}^{2s}(a_1){h_2}^{2s}(1-a_1)da_1] \leq \frac{1}{q-p}\int_0^1 f(x)g(x)dx \tag{3.11}$$

This completes the theorem. □

Theorem 3.4. *Let $f, g : \mathbb{I} = [p, q] \subset \mathbb{R} \to \mathbb{R}$ be two differentiable functions. If f be (h_1, h_2, s)-convex function, g be increasing function with $p < q$, $s \in [0, 1]$, and $a_1 \in [0, 1]$ then*

$$\left|[g(1)-g(0)](\frac{f(p)+f(q)}{2}) - \frac{1}{2(q-p)}\int_p^q [g'\left(\frac{x-p}{q-p}\right) + g'\left(\frac{q-x}{q-p}\right)]f(x)dx\right|$$
$$\leq \left(\frac{q-p}{2}\right)(|f'(p)| + |f'(q)|)\int_0^{\frac{1}{2}}[g(1-a_1)-g(a_1)][{h_1}^s(a_1) + {h_2}^s(1-a_1)]da_1. \tag{3.12}$$

Proof . By using Lemma 2.3 and $|f'|$ is (h_1, h_2, s)-convex function, then

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2} - \frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right) + g'\left(\frac{q-x}{q-p}\right)\right]$$
$$\leq \left(\frac{q-p}{2}\right)\int_0^1 [g(1-a_1)-g(a_1)]\left[{h_1}^s(a_1)|f'(p)| + {h_2}^s(1-a_1)|f'(q)|\right]da_1. \tag{3.13}$$

Since g is increasing, then we have

$$[g(1)-g(0)]\frac{f(p)+f(q)}{2} - \frac{1}{2(q-p)}\int_p^q \left[g'\left(\frac{x-p}{q-p}\right) + g'\left(\frac{q-x}{q-p}\right)\right]$$
$$\leq \left(\frac{q-p}{2}\right)\left\{\int_0^{\frac{1}{2}}[g(1-a_1)-g(a_1)]\left[{h_1}^s(a_1)|f'(p)| + {h_2}^s(1-a_1)|f'(q)|\right]da_1\right.$$
$$\left.\times \int_0^{\frac{1}{2}}[g(a_1)-g(1-a_1)]\left[{h_1}^s(a_1)|f'(p)| + {h_2}^s(1-a_1)|f'(q)|\right]da_1\right\}, \tag{3.14}$$
$$= \left(\frac{q-p}{2}\right)(|f'(p)| + |f'(q)|)\left\{\int_0^{\frac{1}{2}}[g(1-a_1)-g(a_1)]\,{h_1}^s(a_1)\,da_1\right.$$
$$\left.+ \int_0^{\frac{1}{2}}[g(a_1)-g(1-a_1)]\,{h_2}^s(1-a_1)\,da_1\right\}. \tag{3.15}$$

Inequality (3.15) completes Theorem 3.4. □

Corollary 3.5. *Suppose Theorem 3.4 with* $g(a_1) = a_1$, *then Inequality* (3.12) *becomes*

$$\left|\left(\frac{f(p)+f(q)}{2}\right) - \frac{1}{(q-p)}\int_p^q f(x)dx\right| \leq \left(\frac{q-p}{2}\right)\left(|f'(p)|+|f'(q)|\right)\int_0^{\frac{1}{2}}[1-2a_1]\left[{h_1}^s(a_1)+{h_2}^s(1-a_1)\right]da_1. \tag{3.16}$$

Remark 3.6. *If* ${h_i}^s(a_1) = a_1$ *for all* $i = 1, 2$ *in Corollary 3.5 then it will be reduced to Lemma 1.7.*

Corollary 3.7. *In Theorem 3.4, put* $g(a_1) = \frac{{a_1}^\alpha}{\Gamma(1+\alpha)}$, *becomes*

$$\left|\left(\frac{f(p)+f(q)}{2}\right) - \frac{\Gamma(1+\alpha)}{2(q-p)^\alpha}\left[J^\alpha_{p+}f(q) + J^\alpha_{q-}f(p)\right]\right| \leq \left(\frac{q-p}{2\Gamma(1+\alpha)}\right)\left(|f'(p)|+|f'(q)|\right)\int_0^{\frac{1}{2}}\left[(1-a_1)^\alpha - {a_1}^\alpha\right]\left[{h_1}^s(a_1)+{h_2}^s(1-a_1)\right]da_1. \tag{3.17}$$

Remark 3.8. *If* ${h_i}^s(a_1) = a_1$ *for all* $i = 1, 2$ *in Corollary 3.7 then it will be reduced to Inequality for fractional integral* (h_1, h_2, s)*-convex function.*

Theorem 3.9. *Let* $f, g : \mathbb{I} = [p, q] \subset \mathbb{R} \to \mathbb{R}$ *be two differentiable functions. Let* f *be the* (h_1, h_2, s)*-convex function and* g *be increasing function with* $p < q$, $s \in [0, 1]$, *and* $a_1 \in [0, 1]$, *then*

$$\left|g(0)(f(p)+f(q)) - 2g\left(\frac{1}{2}\right)f'\left(\frac{p+q}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^q g'\left(\frac{x-p}{q-p}\right)f(x)dx + \int_p^{\frac{p+q}{2}} g'\left(\frac{q-x}{q-p}\right)f(x)dx\right]\right| \leq (q-p)\left(|f'(p)|+|f'(q)|\right) \times \int_0^{\frac{1}{2}}[g(a_1)]\left[{h_1}^s(a_1)+{h_2}^s(1-a_1)\right]da_1. \tag{3.18}$$

Proof . By utilizing Lemma 2.5 and suppose $|f'|$ is (h_1, h_2, s)-convex function, then

$$\left|[g(0)](f(p)+f(q)) - 2g\left(\frac{1}{2}\right)f\left(\frac{q+p}{2}\right) + \left(\frac{1}{q-p}\right)\left[\int_{\frac{p+q}{2}}^q g'\left(\frac{q-x}{q-p}\right)f(x)dx + \int_p^{\frac{p+q}{2}} g'\left(\frac{q-x}{q-p}\right)f(x)dx\right]\right| \leq (q-p) \left[\int_0^{\frac{1}{2}} g(a_1)f'(pa_1+(1-a_1)q)da_1 - \int_{\frac{1}{2}}^1 g(1-a_1)f'(pa_1+(1-a_1)q)da_1\right], \tag{3.19}$$

$$\leq (q-p)\left[\int_0^{\frac{1}{2}} g(a_1)\left[{h_1}^s(a_1)|f'(p)| + (1-{h_2}^s(a_1))|f'(q)|\right]da_1 + \int_{\frac{1}{2}}^1 g(1-a_1)\left[{h_1}^s(a_1)|f'(p)| + (1-{h_2}^s(a_1))|f'(q)|\right]da_1\right], \tag{3.20}$$

$$= (q-p)\left[|f'(p)|+|f'(q)|\right]\left[\int_0^{\frac{1}{2}} g(a_1)\left[{h_1}^s(a_1) + (1-{h_2}^s(a_1))\right]da_1\right]. \tag{3.21}$$

This Inequality (3.21) completes the proof of Theorem 3.9. □

Corollary 3.10. *Suppose Theorem 3.9 with $g(a_1) = a_1$, then Inequality* (3.17) *becomes*

$$\left|\left(\frac{1}{q-p}\right)\left[\int_p^q f(x)\,dx - f\left(\frac{p+q}{2}\right)\right]\right|$$
$$\leq (q-p)\left(|f'(p)| + |f'(q)|\right)\int_0^{\frac{1}{2}} a_1\left[{h_1}^s(a_1) + {h_2}^s(1-a_1)\right]da_1. \tag{3.22}$$

Remark 3.11. *If ${h_i}^s(a_1) = a_1$ for all $i = 1, 2$ in Corollary 3.10 then Lemma 1.7 will be obtained [7].*

4. Application to Special Means (Averages)

In this section, the relationship between the previously defined results by different scholars and the means (averages) are defined. Let us consider the following means (averages) for real numbers $p, q \in \mathbb{R}$ such as

Arithmetic Mean

$$A(p,q) = \frac{p+q}{2}, \quad p, q \in \mathbb{R}$$

Harmonic Mean

$$H(p,q) = \frac{2}{\frac{1}{p} + \frac{1}{q}}, \quad p, q \in \mathbb{R}\backslash\{0\}$$

Logarithmic Mean

$$\overline{L}(p,q) = \frac{q-p}{\ln|q| \; - \ln|p|}, \quad p, q \in \mathbb{R}\backslash\{0\}$$

Generalized log-Mean

$$L_n(p,q) = \left[\frac{q^{n+1} - p^{n+1}}{(n+1)(q-p)}\right]^{\frac{1}{n}}, \quad p, q \in \mathbb{R}\backslash\{0\} \;\; and \;\; n \in \mathbb{R},$$

Consider the following proposition to highlight the relationship between previously defined inequalities and mean (average) for real numbers.

Proposition 4.1. *Let $p, q \in \mathbb{R}$, $f(x) = \frac{1}{x}, x > 0$ and $g(x) = x$ with ${h_i}^s(a_1) = {a_1}^k$, where $k \in (-\infty, -2) \bigcup (-1, 1]$ in Theorem 3.4, then*

$$\left|H^{-1}(p,q) - \overline{L}(p,q)\right| \leq \frac{(q-p)\,A\left(|p|^{-2}, |q|^{-2}\right)}{(k+1)(k+2)}\left(k + \frac{1}{2^k}\right). \tag{4.1}$$

Corollary 4.2. *Suppose $k = 1$ in Proposition 4.1, we get Proposition 4.1 of [3].*

5. Conclusion

This paper extends the family of convex function by merging s-convex and (h_1, h_2)-convex functions. The H-H and Fejer's inequalities are extended by using newly constructed (h_1, h_2, s)-convex function. H-H and Fejer's inequalities are also used to produce the results. As application, the relationship between newly constructed inequalities and mathematical means (averages) are also discussed.

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