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Comparsion between weighted quadratic loss function and quadratic loss function to estimate asymmetric Laplace distribution parameters

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Abstract

The skewness and scale parameters of the asymmetric Laplace distribution are estimated with Bayesian methods using quadratic loss function and the weighted quadratic loss function, respectively, based on the functions of the prior of the gamma distribution and the exponential distribution for each of the skewness and scale parameters. These estimates were compared using integral mean square error, which was based on the real data technique of the stock prices the Iraqi market. The results revealed that the bayes estimator outperformed the quadratic loss function under the weighted quadratic loss function.

Keywords: Asymmetric Laplace distribution, Bays estimator underweighted quadratic loss function, Quadratic loss function, Lindley approximation.

1. Introduction

The skewed Laplace distribution (asymmetric) is one of the skewed distributions, and the significance of the distribution is evident in the financial statements, which are characterized by asymmetry with the presence of a sharp peak around the origin and the density of the tail when compared to the normal symmetric Laplace distribution.

Skewness is defined as a distribution's degree of symmetry or distance from symmetry. The frequency curve of a distribution with a larger tail to the right is said to be right-skewed (positive

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skew), while the frequency curve of a distribution with a long tail to the left is said to be left-skewed (negative skewness).

The presence of outliers or extreme observations is one of the reasons for the expansion of one end of the distribution. In addition to the importance of this distribution, it stands out in that it is a monomodal distribution, and thus provides a suitable alternative to the AL distribution is one of the stable geometric distributions and this is what appears in the financial statements models [4]. The results showed that these distributions have an accurate mathematical form, and the practical aspect of these distributions is easily implemented. In addition to that, the researcher [2] estimated the parameter of the exponential distribution in the presence of symmetric and asymmetric loss functions and the comparison among them as well as with the estimator of the possible function and they found that the unconventional estimator is better than the conventional [2].

2. Asymmetric Laplace Distribution

The AL distribution is one of the marginal distributions for independent and symmetrically distributed random variables with a specific variance. For other symmetric distributions, The p.d.f can be defined as in the following [4].

$$f\,(x,\sigma,k) = \frac{\sqrt{2}}{\sigma}\;\frac{k}{1+k^2}\begin{cases} e^{\left(-\frac{\sqrt{2}k}{\sigma}x\right)} & , \quad x\geq 0 \\ e^{\left(\frac{\sqrt{2}}{\sigma k}x\right)} & , \quad x<0 \end{cases} \tag{2.1}$$

where k skewness parameter and σ scale parameter. Figure (1).shows p.d.f for the AL distribution by choosing more than one value for k when the value of k = 1 makes the p.d.f for the ALdistribution reduced to the Symmetric Laplace distribution, while The value of k<1 makes the p.d.f of the ALdistribution skewed to the right as in the black colored curve compared with the blue colored curve which represents the SLdistribution and in this case the skewness is positive.but if the value of k>1.this makes the p.d.f function of the ALdistribution skewed to the left as shown The curve has a dot black color compared to the SL distribution, and the skewness is negative. It is worth noting that the ALdistribution suffers from skewness if the tail of the curve is longer than the other side of the distribution curve [3] .

For the cumulative distribution function (cdf) with respect to the AL distribution (σ, k) through the following equation [4] :-

$$F(x,\sigma,k) = \begin{cases} 1-\frac{1}{1+k^2}\; e^{\left(-\frac{\sqrt{2}k}{\sigma}x\right)}, & x>0 \\ \frac{k^2}{1+k^2}\quad e^{\left(\frac{\sqrt{2}}{\sigma k}x\right)}\;, & x\leq 0 \end{cases} \tag{2.2}$$

Figure (2).shows the cumulative distribution function of the function curve with a fixed value for the scale parameter σ=1 and a variable value for the skewness parameter k. The curve in blue shows the cdf function with respect to the SLdistribution, because the value of k = 1. While the black curve shows the cdf function for the AL distribution is right skewed, because the value of k<1. But if the value of k>1, this makes the cdf function for the AL distribution skewed to the left and this is shown by the a dot black curve [3] .

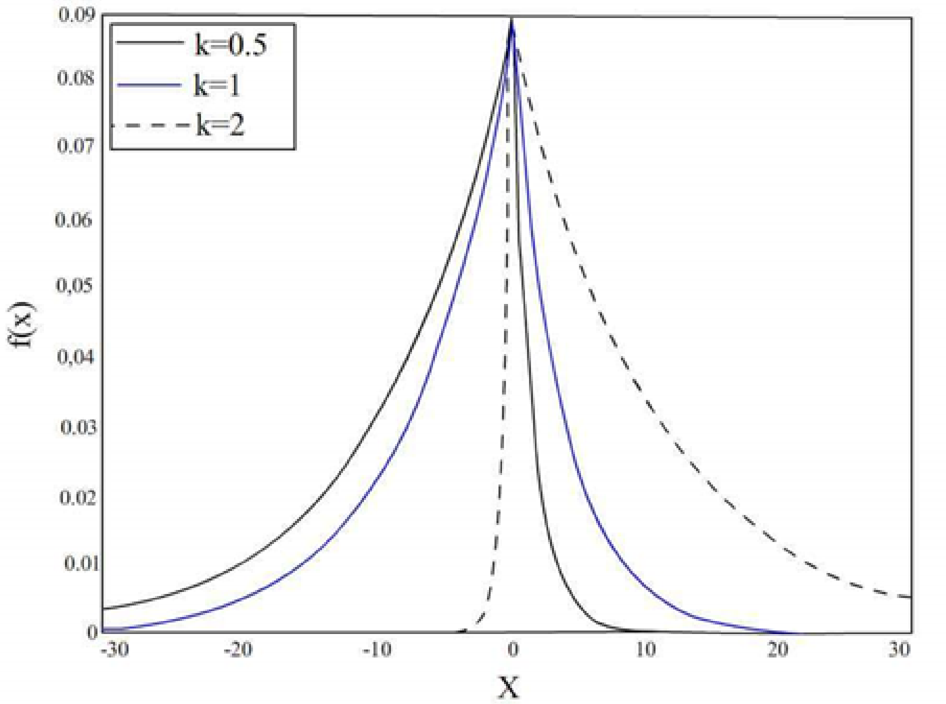

Figure 1: shows the probability density function of the asymmetric Laplace distribution (AL).

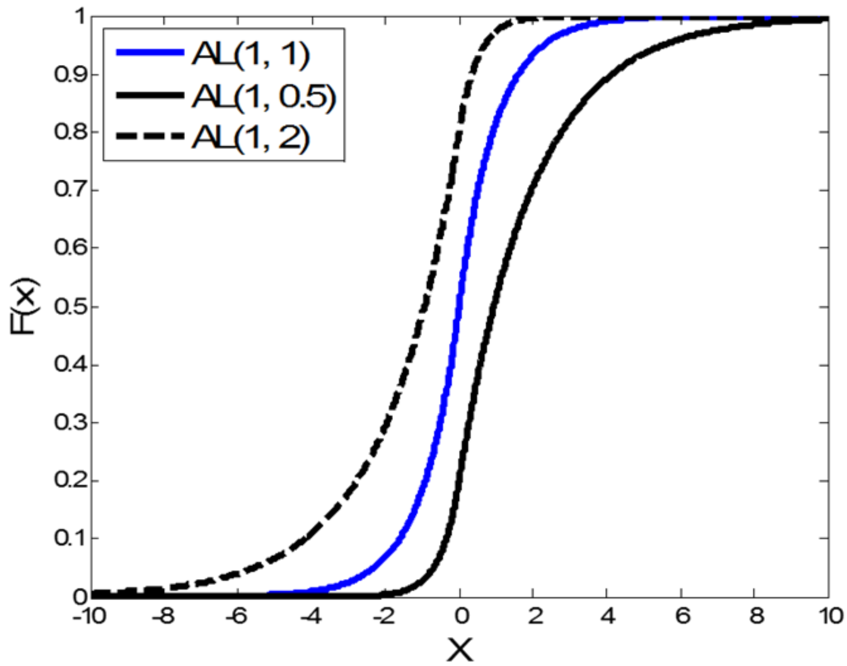

Figure 2: shows the cumulative function of the asymmetric Laplace distribution AL(σ,k)

3. The Bayesian Method For Estimating The Parameters of The Asymmetric Laplace Distribution

The bayes theory dates back to the middle of the eighteenth century, and the bayes method is considered one of the important methods for obtaining the best estimator for the parameters of a particular distribution, This method of estimation assumes that the parameter to be estimated is a random variable, and in the event of its estimation, preliminary information must be available about it with a probability distribution called Prior distribution[1]:

Joint Density Function Using The Prior Functions of The Gamma Distribution and The Exponential Distribution

To obtain an estimate of the skewness and scale parameters of the ALdistribution, we assume the skewness parameter k has an initial distribution $\pi_1(.)$ follows the distribution of $k \sim \Gamma(a, b)$

Also, we assume that the scale parameter σ has an initial distribution $\pi_2(.)$follows the distribution of $\sigma \sim \exp(c)$ where they are independent of each other:

$$\pi_1(k) = \frac{(b)^{a}\,(k)^{a-1}\,e^{-kb}}{\Gamma(a)}, \qquad a > 0 \quad , b > 0 \ \ , k > 0 \tag{3.1}$$

$$\pi_2(\sigma) = C\, e^{-c\sigma} \qquad C{>}0, \quad \sigma \geq 0 \tag{3.2}$$

Because of the difficulty of finding the best estimators for the parameters of the skewness k and the scale σ of the AL distribution by integration, the method of lindley approximation was used, as follows [7] .

$$E(k|\underline{x}) \approx \widehat{k} + P_1 u_1 \sigma_{11} + \frac{1}{2}\left(L_{30} u_1 \sigma_{11}^2\right) + \frac{1}{2}\left(L_{12} u_1 \sigma_{11} \sigma_{22}\right) \tag{3.3}$$

And the j.p.d.f for both the skewness and scale parameters σ, k is as follows:

$$\pi(\sigma, k) = \frac{(b)^{a}\,(k)^{a-1}\,e^{-bk}}{\Gamma(a)}\ .Ce^{-c\sigma}$$

$$\text{Ln}\,\pi(\sigma, k) = a\text{Ln}(b) + (a-1)\,\text{Ln}k - bk + \text{Ln}\,(c) - c\sigma$$

$$P_1 = \frac{\partial P}{\partial k} = \frac{a-1}{k} - b$$

$$P_2 = \frac{\partial P}{\partial \sigma} = -C$$

When :

$$L_{ij} = \ \text{Ln}\ L\ f(x, \sigma, k) \qquad\qquad i, j = 0, 1, 2, 3$$

$$\text{Ln}Lf(x, \sigma, k) = \frac{n}{2}\text{Ln}2 - n\text{Ln}\sigma + n\text{Ln}k - n\text{Ln}\left(1 + k^2\right) - \frac{\sqrt{2}}{\sigma}(k\sum_{i=1}^{n} xi + \frac{1}{k}\sum_{i=1}^{n} xi)$$

That is, (L_{12}) represents the first derivative with respect to the parameter k with the second derivative with respect to the parameter σ

$$L_{12}= -\frac{2\sqrt{2}\ nx}{\sigma^3} - \frac{2\sqrt{2}\ nx}{k^2\sigma^3}$$

$$L_{21} = -\frac{2\sqrt{2}nx}{k^3\ \sigma^2}$$

$$L_{03}= -\ \frac{2n}{\sigma^3} + \frac{6\sqrt{2}knx}{\sigma^4} -\ \frac{\sigma\sqrt{2}knx}{k\sigma^4}$$

$$L_{30}= \frac{2n}{k^3} + \frac{12nk}{(k^2+1)^{\ 2}} - \frac{16nk^3}{(k^2+1)^{\ 3}} - \frac{6\sqrt{2}\ nx}{\sigma k^4}$$

$$L_{20}= -\frac{n}{k^2} - \frac{2n}{k^2+1} + \frac{4nk^2}{(k^2+1)^{\ 2}} + \frac{2\sqrt{2}nx}{\sigma k^3}$$

$$L_{02}= \frac{n}{\sigma^2} - \frac{2\sqrt{2}knx}{\sigma^3} + \frac{2\sqrt{2}nx}{\sigma^3 k}$$

$$\sigma_{11}= -\ \frac{1}{L_{20}} \frac{k^3\left(k^2+1\right)^{\ 2}\sigma}{n\left(2\sqrt{2}\ k^4x + k^5\sigma + 4\sqrt{2}\ k^2x - 4\sigma k^3 + 2\sqrt{2}x - \sigma k\right)}$$

$$\sigma_{22}= -\ \frac{1}{L_{02}} \Longrightarrow = \frac{\sigma^3 k}{n(2\sqrt{2}k^2x - 2\sqrt{2}x - \sigma k)}$$

Quadratic Loss Function

The quadratic loss function is one of the most common and widely used loss functions. It is an asymmetrical function and the mathematical formula is as follows[5] :-

Thus, the bayes estimator for the skewness and scale parameters, respectively, under quadratic loss function $\hat{k}_s$, $\widehat{\sigma}_s$ is as follows:

$$\hat{k}_s = E\left(k \mid \underline{x}\right) \quad (3.4)$$

$$\widehat{\sigma}_s = E\left(\sigma \mid \underline{x}\right) \quad (3.5)$$

a) ***Bays Estimator For The Skewness Parameter k Using Quadratic Loss Function***

To get the bayes estimator for the skewness parameter k under the quadratic loss function, we assume that:

$$u(k,\sigma) = k, \qquad u_1 = \frac{\partial u\left(\sigma, k\right)}{\partial k} = 1, \qquad u_2 = \frac{\partial u(\sigma, k)}{\partial \sigma} = 0$$

Substitute the equations into the law of lindley approximation into eq (3.3):-

$$\text{E}(\text{k}|\underline{x}) \approx \hat{k} + \left(\frac{a-1}{\hat{k}} - b\right)\left(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\hat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x + \hat{k}^5\hat{\sigma} + 4\sqrt{2}\hat{k}^2x - 4\hat{\sigma}\hat{k}^3 + 2\sqrt{2}x - \hat{\sigma}\hat{k}\right)}\right)$$
$$+\frac{1}{2}\left(\frac{2n}{\hat{k}^3} - \frac{12n\hat{k}}{\left(\hat{k}^2+1\right)^2} + \frac{16n\hat{k}^3}{\left(\hat{k}^2+1\right)^3} - \frac{6\sqrt{2}nx}{\hat{\sigma}\hat{k}^3}\right)$$
$$\times\left(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\hat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x + \hat{k}^5\hat{\sigma}4\sqrt{2}\hat{k}^2x - 4\hat{\sigma}\hat{k}^3 + 2\sqrt{2}x - \hat{\sigma}\hat{k}\right)}\right)^2$$
$$+\frac{1}{2}\left(-\frac{2\sqrt{2}nx}{\hat{\sigma}^3} - \frac{2\sqrt{2}nx}{\hat{k}^2\hat{\sigma}^3}\right)\left(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\hat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x + \hat{k}^5\hat{\sigma}4\sqrt{2}\hat{k}^2x - 4\hat{\sigma}\hat{k}^3 + 2\sqrt{2}x - \hat{\sigma}\hat{k}\right)}\right)$$
$$\times\left(\frac{\hat{\sigma}^3\hat{k}}{n\left(2\sqrt{2}\hat{k}^2x - 2\sqrt{2}x - \hat{\sigma}\hat{k}\right)}\right) \tag{3.6}$$

Substitute eq (3.6) into eq (3.4) as follows:

$$\hat{k}_s \approx E(k|\underline{x})$$

b) ***Bayes Eestimator For The Scale Parameter σ Using The Quadratic Loss Function***

To get the bayes estimator for the scale parameter under the quadratic loss function, we assume the following:

$$u(k,\sigma) = \sigma, \qquad u_1 = \frac{\partial u\,(k,\sigma)}{\partial\sigma} = 0, \qquad u_2 = \frac{\partial u\,(k,\sigma)}{\partial\sigma} = 1$$

$$\text{E}(\sigma|\underline{x}) \approx \hat{\sigma} + p_2u_2\sigma_{22} + \frac{1}{2}\left(L_{03}u_2\sigma_{22}^2\right)$$
$$\text{E}(\sigma\,|\underline{x}) \approx \hat{\sigma} + (-c)\left(\frac{\sigma^3k}{n\left(2\sqrt{2}k^2x - 2\sqrt{2}x - \sigma k\right)}\right)$$
$$+\frac{1}{2}\left(-\frac{2n}{\sigma^3} + \frac{6\sqrt{2}Knx}{\sigma^4} - \frac{6\sqrt{2}nx}{\sigma^4K}\right)\left(\frac{\sigma^3k}{n\left(2\sqrt{2}k^2x - 2\sqrt{2}x - \sigma k\right)}\right)^2 \tag{3.7}$$

Substitute Equation (3.7) into eq (3.5) as follows:

$\text{E}(\sigma|\underline{x}) \approx \hat{\sigma}_s$

Weighted Quadratic Loss Function

The weighted quadratic loss function is considered an asymmetric function, so the formula for the skewness parameter k is as follows [6]:-

$$L(\hat{k},k)=\frac{(\sum_{j=0}^{t} aj\ k^{j})(\hat{k}-k)^{\ 2}}{k^{c}}\qquad k>0\ \ ,a_j,j=0.1\ ,2,\ \dots t$$

k: is a positive integer number and c is a constant

Thus, the bayess estimator for the skewness parameter k under the weighted quadratic error loss function is as follows:

$$\hat{k}=\frac{a_0E\left(\frac{1}{k^{c-1}}\mid \underline{x}\right)+a_1E\left(\frac{1}{k^{c-2}}\mid \underline{x}\right)+\cdots+a_tE\left(\frac{1}{k^{c-(t+1)}}\mid \underline{x}\right)}{a_0E\left(\frac{1}{k^{c}}\mid \underline{x}\right)+a_1E\left(\frac{1}{k^{c-1}}\mid \underline{x}\right)+\cdots+a_tE\left(\frac{1}{k^{c-t}}\mid \underline{x}\right)} \tag{3.8}$$

Therefore, the bayes estimator for the scale parameter σ under the weighted quadratic loss function is as follows:

$$\widehat{\sigma}=\frac{a_0E\left(\frac{1}{\sigma^{c-1}}\mid \underline{x}\right)+a_1E\left(\frac{1}{\sigma^{c-2}}\mid \underline{x}\right)+\cdots+a_tE\left(\frac{1}{\sigma^{c-(\ t+1)}}\mid \underline{x}\right)}{a_0E\left(\frac{1}{\sigma^{c}}\mid \underline{x}\right)+a_1E\left(\frac{1}{\sigma^{c-1}}\mid \underline{x}\right)+\cdots+a_{\ t}\ E\left(\frac{1}{\sigma^{c-t}}\mid \underline{x}\right)} \tag{3.9}$$

a) Bays Estimator For The Skewness Parameter k Using The Weighted Quadratic Loss Function:-

To get a bayes estimator, let us suppose each of (t = 1, c = 0) and substitute it in equation (3.8) as follows:

$$\widehat{k}_{10}=\frac{a_0E\left(k\mid \underline{x}\right)+a_1E\left(k^2\mid \underline{x}\right)}{a_0+a_1E\left(k\mid \underline{x}\right)} \tag{3.10}$$

$$u(\sigma,k)=k^2,\qquad u_1=\frac{\partial u(\sigma,k)}{\partial k}=2k,\qquad u_{11}=\frac{\partial^2 u(\sigma,\ k)}{\partial k^2}=2,$$

$$E(k^2|\underline{x})\approx\hat{k}^2+\frac{1}{2}\left(u_{11}\sigma_{11}\right)+p_1u_1\sigma_{11}+\frac{1}{2}\left(L_{30}u_1\sigma_{11}^2\right)+\frac{1}{2}\left(L_{12}u_1\sigma_{11}\sigma_{22}\right)$$

$$E(k^2\,|\underline{x})\approx\hat{k}^2+\frac{1}{2}\left(2\ \frac{\hat{k}^3\left(\hat{k}^2+1\right)\ ^2\widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x+\ \hat{k}^5\ \widehat{\sigma}+4\sqrt{2}\hat{k}^2x-4\ \widehat{\sigma}\hat{k}^3+2\sqrt{2}x-\widehat{\sigma}\hat{k}\right)}\right)$$
$$+\left(\frac{a-1}{\hat{k}}-b\right)(2k)\left(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x+\hat{k}^5\widehat{\sigma}+4\sqrt{2}\hat{k}^2x-4\widehat{\sigma}\hat{k}^3+2\sqrt{2}x-\widehat{\sigma}\hat{k}\right)}\right)+etc \tag{3.11}$$

And by substituting equations (3.6), (3.11) with eq (3.10) to find a bayes estimator for the skewness parameter k under the weighted quadratic loss function as follows:

$$\hat{k}_{10}=\frac{a_0(\hat{k}_s)+a_1(\hat{k}_s^2)}{a_0+a_1(\hat{k}_s)}$$

when (t = 1, C = 1):

$$\widehat{k}_{11}=\frac{a_0+a_1E(k|\underline{x}\ )}{a_0E(\frac{1}{k}|\underline{x}\ )+a_1} \tag{3.12}$$

We find the value of $E\ (\frac{1}{k}\,|\underline{x}\ )$using lindley approximation

$$u(\sigma,k)=\frac{1}{k},\qquad u_1=\frac{\partial u(\sigma,k)}{\partial k}=-k^{-2},\qquad u_{11}=\frac{\partial^2 u(\sigma,k)}{\partial k^2}=2k^{-3},$$

$$E(\frac{1}{k}\,|\underline{x})\approx\frac{1}{\hat{k}}+\frac{1}{2}\,(u_{11}\sigma_{11})+p_1u_1\sigma_{11}+\frac{1}{2}\left(L_{30}u_1\sigma_{11}^2\right)+\frac{1}{2}\,(L_{12}u_1\sigma_{11}\sigma_{22})$$

$$E(\frac{1}{k}\,|\underline{x})\approx\frac{1}{\hat{k}}+\frac{1}{2}(2k^{-3})(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x+\hat{k}^5\widehat{\sigma}+4\sqrt{2}\hat{k}^2x-4\widehat{\sigma}\hat{k}^3+2\sqrt{2}x-\widehat{\sigma}\hat{k}\right)})\left(\frac{a-1}{\hat{k}}-b\right)(-k^{-2})$$
$$+\,etc \tag{3.13}$$

Substituting eq (3.6), (3.13) in eq (3.12) as follows:

$$\hat{k}_{\ 11}=\frac{a_0+a_1\ (\ \hat{k}_s)}{a_0\ \ (\frac{1}{\hat{k}_s}\ )+a_1} \tag{3.14}$$

when (t = 1, C = 2):

$$\widehat{k}_{12}=\frac{a_0E\left(\frac{1}{k}\ \middle|\ \underline{x}\ \right)+a_1}{a_0E\left(\frac{1}{k^2}\ \middle|\ \underline{x}\ \right)+a_1E\left(\frac{1}{k}\ \middle|\ \underline{x}\ \right)} \tag{3.15}$$

$$u(\sigma,k)=\frac{1}{k^2},\qquad u_1=\frac{\partial u(\sigma,k)}{\partial k}=-2k^{-3},\qquad u_{11}=\frac{\partial^2 u(\sigma,k)}{\partial k^2}=6k^{-4},$$

$$E(\frac{1}{k^2}|\underline{x})\approx\frac{1}{\hat{k}^2}+\frac{1}{2}\,(u_{11}\sigma_{11})+p_1u_1\sigma_{11}+\frac{1}{2}\left(L_{30}u_1\sigma_{11}^2\right)+\frac{1}{2}\,(L_{12}u_1\sigma_{11}\sigma_{22})$$

$$E(\frac{1}{k^2}|\underline{x})\approx\frac{1}{\hat{k}^2}+\frac{1}{2}(-2k^{-3})(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2\widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x+\hat{k}^5\widehat{\sigma}+4\sqrt{2}\hat{k}^2x-4\widehat{\sigma}\hat{k}^3+2\sqrt{2}x-\widehat{\sigma}\hat{k}\right)})+etc \tag{3.16}$$

By substituting equations (3.16) and (3.13) with eq (3.15), it becomes as follows:

$$\widehat{k}_{12}=\frac{a_0\ \left(\frac{1}{\hat{k}_s}\right)+a_1}{a_0\left(\frac{1}{\hat{k}_s^2}\right)+a_1\left(\frac{1}{\hat{k}_s}\right)}$$

when (t = 2, C = 0) :

$$\widehat{k}_{20}=\frac{a_0E(k|\underline{x}\ )+a_1E(k^2|\underline{x}\ )+a_2E(k^3|\underline{x}\ )}{a_0+a_1E\ (\ k|\underline{x})+a_2E(k^2|\underline{x}\ )} \tag{3.17}$$

$$u(\sigma,k)=k^3,\qquad u_1=\frac{\partial u(\sigma,k)}{\partial k}=-3k^2,\qquad u_{11}=\frac{\partial^2 u(\sigma,k)}{\partial k^2}=6k,$$

$$E(k^3|\underline{x}\ )\ \approx\hat{k}^3+\ \frac{1}{2}(u_{11}\sigma_{11})(p_1u_1\sigma_{11}+\frac{1}{2}\left(L_{30}u_1\sigma_{11}^2\right)+\frac{1}{2}(L_{12}u_1\sigma_{11}\sigma_{22})$$

$$E(k^3|\underline{x}\ )\approx\hat{k}^3+\ \frac{1}{2}(6k)(\ \frac{\hat{k}^3\left(\hat{k}^2+1\right)\ {}^2\widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4x+\ \hat{k}^5\ \widehat{\sigma}+4\sqrt{2}\hat{k}^2x-4\ \widehat{\sigma}\hat{k}^3+2\sqrt{2}x-\ \widehat{\sigma}\hat{k}\right)})+etc \tag{3.18}$$

By substituting equations (3.18), (3.11) and (3.6) with eq (3.17), it becomes as follows:

$$\widehat{k}_{20}=\frac{a_0\ (\hat{k}_s)+a_1\left(\hat{k}_s^2\right)+a_2(\hat{k}_s^3)}{a_0+a_1(\hat{k}_s)+a_2\ \left(\hat{k}_s^2\right)}$$

when (t = 2, C = 1):

$$\widehat{k}_{21}=\frac{a_0+a_1E(k|\underline{x}\ )+a_2E(k^2|\underline{x}\ )}{a_0E\left(\ \frac{1}{k}|\underline{x}\right)+a_1+a_2E(k|\underline{x}\ )} \tag{3.19}$$

By substituting equations (3.6), (3.11) and (3.13) with eq (3.19), it becomes as follows:

$$\widehat{k}_{21}=\frac{a_0+a_1\ (\hat{k}_s\ )+a_2\ (\hat{k}_s^2\ )}{a_0E\left(\ \frac{1}{\hat{k}_s}|\underline{x}\right)+a_1+a_2\ (\hat{k}_s\ )}$$

when (t = 2, C = 2):

$$\widehat{k}_{22}=\frac{a_0\ \ E\left(\ \frac{1}{k}|\underline{x}\right)+\ a_1\ +a_2\ E(k|\underline{x}\ )}{a_0\ E(\frac{1}{k^2}/\underline{x}\ )+\ a_1E(\frac{1}{k}|\underline{x}\ )+a_2} \tag{3.20}$$

By substituting equations (3.6), (3.16) and (3.13) with eq (3.20), it becomes as follows:

$$\widehat{k}_{22}=\frac{a_0\ \ \left(\ \frac{1}{\hat{k}_s}\right)+\ a_1\ +a_2\ (\hat{k}_s)}{a_0\ (\frac{1}{\hat{k}_s^2}\ )+\ a_1(\frac{1}{\hat{k}_s})+a_2}$$

***b) Bayes Estimator For The Scale Parameter σ Using The Weighted Quadratic Loss Function*:**

To find Bayes estimator when (t = 1, C = 0) by substituting in eq (3.9), we get the following:

$$\widehat{\sigma}_{\ 10}=\frac{a_0E(\sigma|\underline{x}\ )+a_1E(\sigma^2|\underline{x}\ )}{a_0+a_1E(\sigma|\underline{x}\ )} \tag{3.21}$$

$$u\ (\sigma,k)=\sigma^2,\qquad u_2=\frac{\partial u\,(\sigma,k)}{\partial\sigma}=2\sigma,\qquad u_{22}=\frac{\partial u\,(\sigma,k)}{\partial\sigma^2}=2$$

$$E(\sigma^2|\underline{x}\ )\approx\widehat{\sigma}^2+\frac{1}{2}\,(u_{22}\sigma_{22})+p_2u_2\sigma_{22}+\frac{1}{2}\left(L_{03}u_2\sigma_{22}^2\right)$$

$$E(\sigma^2|\underline{x}\ )\ \approx\widehat{\sigma}^2+\ \frac{1}{2}\,(2)\left(\frac{\widehat{\sigma}^3\ \hat{k}}{n\left(2\sqrt{2}\,\hat{k}^{\ 2}\ x-\ 2\sqrt{2}\ x-\ \widehat{\sigma}\ \widehat{k}\right)}\right)+etc \tag{3.22}$$

Where $\hat{k}$& $\widehat{\sigma}$ estimators by (mle)

By substituting equations (3.7), (3.22) with eq (3.21), it becomes as follows:

$$\widehat{\sigma}_{\ 10}=\frac{a_0\ (\widehat{\sigma}_s)+a_1\ (\widehat{\sigma}_s^2\ )}{a_0+a_1\ \ (\widehat{\sigma}_s)}$$

when (t = 1, C = 1):

$$\widehat{\sigma}_{11} = \frac{a_0 + \ a_1 \ E(\sigma|\underline{x} \ )}{a_0 \ E\left(\ \frac{1}{\sigma}|\underline{x}\right) + \ a_1} \tag{3.23}$$

$$u(\sigma, k) = \frac{1}{\sigma}, \qquad u_2{=}\frac{\partial u\,(\sigma,k)}{\partial \sigma}{=}-\sigma^{-2}, \qquad u_{22}{=}\frac{\partial u\,(\sigma,k)}{\partial \sigma^2}{=}2\sigma^{-3},$$

$$E(\frac{1}{\sigma}|\underline{x} \ ) \ ) \approx \frac{1}{\widehat{\sigma}} + \frac{1}{2} \ (u_{22}\sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2}\left(L_{03} u_2 \sigma_{22}^2\right) \ + \frac{1}{2}(L_{21} u_2 \sigma_{11} \sigma_{22})$$

$$E(\frac{1}{\sigma}|\underline{x} \ ) \ ) \approx \frac{1}{\widehat{\sigma}} + \frac{1}{2}\left(-\sigma^{-2}\right) \left(\frac{\widehat{\sigma}^3 \ \hat{k}}{n\left(2\sqrt{2}\,\hat{k}^{\ 2} \ x - \ 2\sqrt{2} \ x - \ \widehat{\sigma}\,\widehat{k}\right)} \right) + etc \tag{3.24}$$

And by substituting equations (3.7), (3.24) with eq (3.23), it becomes as follows:

$$\widehat{\sigma}_{\ 11} = \frac{a_0 + \ a_1 \ (\widehat{\sigma}_s)}{a_0 \ (\frac{1}{\widehat{\sigma}_s}) + \ a_1}$$

when (t = 1, C = 2):

$$\widehat{\sigma}_{12}{=}\frac{a_0 E(\frac{1}{\sigma}|\underline{x} \ ){+}a_1}{a_0 E(\frac{1}{\sigma^2}|\underline{x} \ ){+}a_1 E(\frac{1}{\sigma}|\underline{x} \ )} \tag{3.25}$$

$$u(\sigma, k) = \frac{1}{\sigma^2}, \qquad u_2 = \frac{\partial u(\sigma,k)}{\partial \sigma} = -2\sigma^{-3}, \qquad u_{22}{=}\frac{\partial^2 u(\sigma,k)}{\partial \sigma^2} = 6\sigma^{-4}$$

$$E(\frac{1}{\sigma^2}|\underline{x}) \approx \frac{1}{\widehat{\sigma}^2} + \frac{1}{2}\,(u_{22}\sigma_{22}) + p_2 u_2 \sigma_{22} + \frac{1}{2}\left(L_{03} u_2 \sigma_{22}^2\right) + \frac{1}{2}(L_{21} u_2 \sigma_{11} \sigma_{22})$$

$$E(\frac{1}{\sigma^2}|\underline{x}) \approx \frac{1}{\widehat{\sigma}^2} + \frac{1}{2}\left(6\sigma^{-4}\right) \left(\frac{\widehat{\sigma}^3 \hat{k}}{n\left(2\sqrt{2}\hat{k}^2 x - 2\sqrt{2}x - \widehat{\sigma}\widehat{k}\right)} \right)$$
$$+ (-c)(-2\sigma^{-3}) \left(\frac{\hat{k}^3\left(\hat{k}^2+1\right)^2 \widehat{\sigma}}{n\left(2\sqrt{2}\hat{k}^4 x + \hat{k}^5 \widehat{\sigma} + 4\sqrt{2}\hat{k}^2 x - 4\widehat{\sigma}\hat{k}^3 + 2\sqrt{2}x - \widehat{\sigma}\hat{k}\right)} \right) + etc \tag{3.26}$$

And by substituting equations (3.26), (3.24) with eq (3.25), get the following:

$$\widehat{\sigma}_{12}{=}\frac{a_0(\frac{1}{\widehat{\sigma}_s}) \ {+}a_1}{a_0(\frac{1}{\widehat{\sigma}^2}) \ {+}a_1(\frac{1}{\widehat{\sigma}_s})}$$

when (t = 2, C = 0):

$$\widehat{\sigma}_{20}{=}\frac{a_0 \ E\left(\ \sigma|\underline{x}\right) {+}a_1 E\left(\sigma^2 \mid \underline{x}\right) {+}a_2 E\left(\sigma^3 \mid \underline{x}\right)}{a_0{+}a_1 E\left(\sigma \mid \underline{x}\right) {+}a_2 E\left(\ \sigma^2 \mid \underline{x}\right)} \tag{3.27}$$

$$u(\sigma, k) = \sigma^3, \qquad u_2{=}\frac{\partial u\,(\sigma,k)}{\partial \sigma}{=}\ 3\sigma^2, \qquad u_{22}{=}\frac{\partial u\,(\sigma,k)}{\partial \sigma^2}{=}6\sigma$$

$$E(\sigma^3|\underline{x}\ )\approx\widehat{\sigma}^3+\frac{1}{2}\left(u_{22}\sigma_{22}\right)+p_2u_2\sigma_{22}+\frac{1}{2}\left(L_{03}u_2\sigma_{22}^2\right)$$

$$E(\sigma^3|\underline{x}\ )\approx\widehat{\sigma}^3+\frac{1}{2}\left(2\sigma\right)\left(\frac{\widehat{\sigma}^3\ \widehat{k}}{n\left(2\sqrt{2\ \widehat{k}}^{\ 2}\ x-\ 2\sqrt{2}\ \ x-\ \widehat{\sigma}\ \widehat{k}\right)}\right)+etc \quad (3.28)$$

And by substituting equations (3.7), (3.22) and (3.28) with eq (3.27), get the following:

$$\widehat{\sigma}_{20}=\frac{a_0(\ \widehat{\sigma}_s)+a_1(\ \widehat{\sigma}_s)+a_2\ (\widehat{\sigma}_s^3)}{a_0+a_1\ (\ \widehat{\sigma}_s)\ +a_2(\widehat{\sigma}_s^2)}$$

when (t = 2, C = 1):

$$\widehat{\sigma}_{21}=\frac{a_0+a_1E(\sigma|\underline{x})+a_2E(\sigma^2|\underline{x})}{a_0E(\frac{1}{\sigma}|\underline{x})+a_1+a_2E(\sigma|\underline{x})} \quad (3.29)$$

$$\widehat{\sigma}_{21}=\frac{a_0+a_1\ (\ \widehat{\sigma}_s)\ +a_2(\ \widehat{\sigma}_s^2)}{a_0\left(\frac{1}{\widehat{\sigma}_s}\right)\ +a_1+a_2\ (\ \widehat{\sigma}_s)}$$

when (t = 2, C = 2):

$$\widehat{\sigma}_{22}=\frac{a_0E\left(\frac{1}{\sigma}\ \middle|\ \underline{x}\right)\ +a_1+a_2E\left(\sigma\ |\ \underline{x}\right)}{a_0E\left(\frac{1}{\sigma^2}\ \middle|\ \underline{x}\right)\ +a_1E\left(\frac{1}{\sigma}\ \middle|\ \underline{x}\right)\ +a_2} \quad (3.30)$$

$$\widehat{\sigma}_{22}=\frac{a_0\left(\frac{1}{\widehat{\sigma}_s}\right)\ +a_1+a_2\ (\ \widehat{\sigma}_s)}{a_0(\frac{1}{\widehat{\sigma}^2})+a_1+a_2\left(\frac{1}{\widehat{\sigma}_s}\right)}$$

4. Application

This part includes estimating the parameters of the asymmetric Laplace distribution using the current stock prices of the Bank of Baghdad, which were obtained from the Iraqi Stock Exchange, and then the comparison between these methods is done according to the statistical criterion, the mean of the integral error squares. The data was collected from the financial statements that were obtained from the Iraqi market for securities.

The data obtained from all companies and sectors is classified. The Iraqi Listed Company, on the Iraq Stock Exchange, during the period (2019) which consists of (90) regular companies. During the study years .was distributed over (8) different sectors. for the final sample size (30).

Table 1: It Shows The Classification of The Sample Size according To The Special Sectors in The Iraqi Market

Number	Sector Name	Number of Companies	Number of Companies	Study Sample
1	Banks	38	38	13
2	Insurance	5	5	1
3	Invest	6	6	1
4	Services	9	9	4
5	Industry	14	14	3
6	Hotels and tourism	10	10	1
7	Agriculture	6	6	6
8	contacts	2	2	2
Total	—	90	90	30

Comparison of Estimating Parameters of The Asymmetric Laplace Distribution

The bayes method, under the loss function, weighted quadratic, is better. estimating method for estimating Parameters of the AL distribution, so that this method can be applied to real data, where the results of the estimators (IMSE) of the bayesian estimator are under (the quadratic loss function, and the weighted quadratic loss function) as shown in Table 2.

Table 2: TheValues of The Parameter Estimators and The Results Comparsion between (WSELF &SELF)

Parameter	Estimator distribution	Parameter Estimators
IMSE Bayes(k)	9.5601e-04	0.0017
IMSE Bayes (σ)	2.3490	
IMSE Bayes $\hat{k}_{W10}$	8.6969e-04	3.1209e-05
IMSE Bayes $\widehat{\sigma}_{W10}$	0.5304	
IMSE Bayes $\hat{k}_{W11}$	8.3079e-04	0.0750
IMSE Bayes $\widehat{\sigma}_{W11}$	41.7434	
IMSE Bayes $\hat{k}_{W20}$	7.9344e-05	2.2652e-06
IMSE Bayes $\widehat{\sigma}_{W20}$	0.0278	
IMSE Bayes $\hat{k}_{W21}$	7.2835e-05	0.0869
IMSE Bayes $\widehat{\sigma}_{W21}$	42.7654	
IMSE Bayes $\hat{k}_{W22}$	4.6379e-04	2.6777e-04
IMSE Bayes $\widehat{\sigma}_{W22}$	39.9520	

Through the table 2. note that the results of the Bays estimator under the weighted quadratic loss function (IMSE $_{W11}$) resulting from the difference between the real values and the estimated values of each of the parameters of the skewness k and the scale σ is the best method after it has been estimated and compared with the methods other, followed by a Bays estimator under the quadratic loss function

5. Conclusions

The results of the values of the positions of the two parameters of the skewness and scale parameters of the Bays estimator under the weighted quadratic loss function (IMSE $_{W11}$) resulting from the difference between the real values and the estimated values showed the best method after it was estimated and compared with other methods, followed by the Bays estimator under quadratic loss function.

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