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A new subclass of analytic functions involving Pascal distribution series

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Abstract

In this work, we introduce and investigate a new class $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ of analytic functions in the open unit disc U with negative coefficients. The aim of this study is to determine coefficient estimates, neighborhoods and partial sums for functions u belonging to this class.

Keywords: analytic, starlike, coefficient estimate, neighborhood, partial sums.
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1. Introduction

Let A denote the class of analytic functions u defined on the unit disk $U = \{z : |z| < 1\}$ with normalization $u(0) = 0$ and $u'(0) = 1$. Such a function has the Taylor series expansion about the origin in the form

$$u(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

denoted by S, the subclass of A consisting of functions that are univalent in U.

A function $u \in A$ is said to be in $k-UST(\gamma)$, the class of $k-$uniformly starlike functions of order $\gamma, 0 \le \gamma < 1$, if satisfies the condition

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > k\left|\frac{zu'(z)}{u(z)} - 1\right| + \gamma, \quad (k \ge 0), \tag{1.2}$$

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and a function $u \in A$ is said to be in $k - UCV(\gamma)$, the class of $k-$uniformly convex functions of order $\gamma, 0 \leq \gamma < 1$, if satisfies the condition

$$\Re\left\{1 + \frac{zu''(z)}{u'(z)}\right\} > k\left|\frac{zu''(z)}{u'(z)}\right| + \gamma, \quad (k \geq 0). \tag{1.3}$$

Uniformly starlike and uniformly convex functions were first introduced by Goodman [6] and studied by Ronning [12].

In [14], Sakaguchi defined the class ST_s of starlike functions with respect to symmetric points as follows:

Let $u \in A$. Then u is said to be starlike with respect to symmetric points in U if and only if

$$\Re\left\{\frac{2zu'(z)}{u(z) - u(-z)}\right\} > 0, \; (z \in U).$$

Recently, Owa et al. [11] defined the class $ST_s(\alpha, \varsigma)$ as follows:

$$\Re\left\{\frac{(1-\varsigma)zu'(z)}{u(z) - u(\varsigma z)}\right\} > \alpha, \; (z \in U),$$

where $0 \leq \alpha < 1, |\varsigma| \leq 1, \varsigma \neq 1$. Note that $ST_s(0, -1) = ST_s$ and $ST_s(\alpha, -1) = ST_s(\alpha)$ is called Sakaguchi function of order α.

For $u \in A$ given by (1.1) and $\varrho(z)$ given by

$$\varrho(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

their convolution (or Hadamard product), denoted by $(u * \varrho)$, is defined as

$$(u * \varrho)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (\varrho * u)(z) \quad (z \in U). \tag{1.5}$$

Note that $u * \varrho \in A$.

It is well known that the special functions play an important role in Geometric Function Theory, especially in the proof given by de Branges [3] for the famous Bieberbach conjecture. The surprising use of special functions (hypergeometric functions) has prompted renewed interest in function theory in the last few decades. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized Gaussian hypergeometric functions [2, 7, 8, 15, 18, 19].

A variable x is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \cdots$ with probabilities $(1-q)^m, \frac{q^m(1-q)^m}{1!}, \frac{q^2 m(m+1)(1-q)^m}{2!}, \frac{q^3 m(m+1)(m+2)(1-q)^m}{3!}, \cdots$ respectively, where q and m are called the parameters, and thus

$$P(x = n) = \binom{n+m-1}{m-1} q^n (1-q)^m, \quad n = 0, 1, 2, 3, \cdots.$$

Very recently, El-Deeb [4] introduced a power series whose coefficients are probabilities of Pascal distribution

$$\Phi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-1}{m-1} q^{n-1} (1-q)^m z^n, z \in U,$$

where $m \geq 1, 0 \leq q \leq 1$, and we note that, by ratio test, the radius of convergence of above series is infinity.

We considered the linear operator $\mathscr{I}_q^m : A \to A$ defined in terms of convolution (or Hadamard) product by

$$\mathscr{I}_q^m u(z) = \Phi_q^m(z) * u(z),$$

we have

$$\mathscr{I}_q^m u(z) = z + \sum_{n=2}^{\infty} C(n,m) q^{n-1}(1-q)^m a_n z^n \tag{1.6}$$

$$\text{where,} \quad C(n,m) = \binom{n+m-1}{m-1}, m \geq 1 \text{ and } \ 0 \leq q \leq 1.$$

Motivated by the results in connections between various subclasses of analytic univalent functions, by using hypergeometric functions (see [2, 7]) and Poisson distributions [9, 10], we obtain necessary and sufficient condition for the function I_q^m to be in the class $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$.

Now, we define a new subclass of functions belonging to the class A.

Definition 1.1. *A function $u \in A$ is said to be in the class $k - UST_s(q, m, \gamma, \varsigma)$ if for all $z \in U$*

$$\Re\left\{\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)}\right\} \geq k\left|\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)} - 1\right| + \gamma,$$

for $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \gamma < 1$.

Furthermore, we say that a function $u \in k - UST_s(q, m, \gamma, \varsigma)$ is in the subclass $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$ if $u(z)$ is of the following form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0, n \in \mathbb{N}, \ z \in U). \tag{1.7}$$

The aim of the present paper is to study the coefficient bounds, partial sums and certain neighborhood results of the class $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$.

In order to prove our results we shall need the following lemmas [1].

Lemma 1.2. *Let $w = u + iv$. Then*

$$\Re\ (w) \geq \alpha \quad \textit{if and only if} \quad |w - (1+\alpha)| \leq |w + (1-\alpha)|.$$

Lemma 1.3. *Let $w = u + iv$ and α, γ be real numbers. Then*

$$\Re\ (w) > \alpha|w-1| + \gamma \quad \textit{if and only if} \quad \Re\{w(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\} > \gamma.$$

2. Coefficient bounds of the function class $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$

Theorem 2.1. *The function u defined by (1.7) is in the class $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$ if and only if*

$$\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m|n(k+1) - u_n(k+\gamma)|a_n \leq 1-\gamma, \tag{2.1}$$

where $k \geq 0, |\varsigma| \leq 1, \varsigma \neq 1, 0 \leq \gamma < 1$ and $u_n = 1 + \varsigma + \cdots + \varsigma^{n-1}$.
The result is sharp for the function $u(z)$ given by

$$u(z) = z - \frac{1-\gamma}{C(n,m)q^{n-1}(1-q)^m|n(k+1) - u_n(k+\gamma)|} z^n.$$

Proof . By Definition (1.1), we get

$$\Re\left\{\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)}\right\} \geq k\left|\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)} - 1\right| + \gamma.$$

Then by Lemma 1.3, we have

$$\Re\left\{\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)}(1+ke^{i\theta}) - ke^{i\theta}\right\} \geq \gamma, \quad -\pi < \theta \leq \pi$$

or equivalently

$$\Re\left\{\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)} - \frac{ke^{i\theta}\left[\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)\right]}{\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)}\right\} \geq \gamma. \tag{2.2}$$

Let $H(z) = (1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta}) - ke^{i\theta}\left[\mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)\right]$
and $K(z) = \mathscr{I}_q^m u(z) - \mathscr{I}_q^m u(\varsigma z)$.
By Lemma 1.2, (2.2) is equivalent to

$$|H(z) + (1-\gamma)K(z)| \geq |H(z) - (1+\gamma)K(z)|, \quad \text{for } 0 \leq \gamma < 1.$$

But

$$\begin{aligned}
|H(z) + (1-\gamma)K(z)| = &\Big|(1-\varsigma)\big\{(2-\gamma)z - \sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m(n + u_n(1-\gamma))a_n z^n \\
&- ke^{i\theta}\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m(n-u_n)a_n z^n\big\}\Big| \\
\geq &|1-\varsigma|\Big\{(2-\gamma)|z| - \sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m|n+u_n(1-\gamma)|a_n|z^n| \\
&- k\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m|n-u_n|a_n|z^n|\Big\}.
\end{aligned}$$

Also

$$\begin{aligned}|H(z)-(1+\gamma)K(z)| = &\Big|(1-\varsigma)\big\{-\gamma z-\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m(n-u_n(1+\gamma))a_nz^n\\ &-ke^{i\theta}\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m(n-u_n)a_nz^n\big\}\Big|\\ \le&|1-\varsigma|\Big\{\gamma|z|+\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m|n-u_n(1+\gamma)||a_n|z^n|\\ &+k\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m|n-u_n|a_n|z^n|\Big\}.\end{aligned}$$

So

$$\begin{aligned}&|H(z)+(1-\gamma)K(z)|-|H(z)-(1+\gamma)K(z)|\\ &\ge|1-\varsigma|\Big\{2(1-\gamma)|z|-\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m\\ &\qquad\big[|n+u_n(1-\gamma)|+|n-u_n(1+\gamma)|+2k|n-u_n|\ \big]a_n|z^n|\Big\}\\ &\ge 2(1-\gamma)|z|-\sum_{n=2}^{\infty}2C(n,m)q^{n-1}(1-q)^m\big|n(k+1)-u_n(k+\gamma)\big|a_n|z^n|\ge 0\end{aligned}$$

or

$$\sum_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m\big|n(k+1)-u_n(k+\gamma)\big|a_n\le 1-\gamma.$$

Conversely, suppose that (2.1) holds. Then we must show

$$\Re\left\{\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})-ke^{i\theta}\left[\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right]}{\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)}\right\}\ge\gamma.$$

Upon choosing the values of z on the positive real axis where $0\le|z|=r<1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m[n(1+ke^{i\theta})-u_n(\gamma+ke^{i\theta})]a_nz^{n-1}}{1-\sum\limits_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m u_na_nz^{n-1}}\right\}\ge 0.$$

Since $\Re(-e^{i\theta})\ge-|e^{i\theta}|=-1$, the above inequality reduces to

$$\Re\left\{\frac{(1-\gamma)-\sum\limits_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m[n(1+k)-u_n(\gamma+k)]a_nr^{n-1}}{1-\sum\limits_{n=2}^{\infty}C(n,m)q^{n-1}(1-q)^m u_na_nr^{n-1}}\right\}\ge 0.$$

Letting $r\to 1^-$, we have desired conclusion. □

Corollary 2.2. *If* $u(z)\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ *then*

$$a_n\le\frac{1-\gamma}{C(n,m)q^{n-1}(1-q)^m|n(k+1)-u_n(k+\gamma)|}$$

where $k\ge 0, |\varsigma|\le 1, \varsigma\ne 1, 0\le\gamma<1$ *and* $u_n=1+\varsigma+\cdots+\varsigma^{n-1}$.

3. Neighborhood of the function class $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$

The notion of β-neighbourhood was introduced and studied by Goodman [5] and Ruscheweyh [13].

Definition 3.1. *Let $k\geq 0, |\varsigma|\leq 1, \varsigma\neq 1, 0\leq\gamma<1, \beta\geq 0$ and $u_n=1+\varsigma+\cdots+\varsigma^{n-1}$. We define the $\beta-$neighborhood of a function $u\in A$ and denote by $N_\beta(u)$ consisting of all functions $g(z)=z-\sum\limits_{n=2}^{\infty}b_nz^n\in S$, $(b_n\geq 0, n\in\mathbb{N})$ satisfying*

$$\sum_{n=2}^{\infty}\frac{C(n,m)q^{n-1}(1-q)^m|n(k+1)-u_n(k+\gamma)|}{1-\gamma}|a_n-b_n|\leq 1-\beta.$$

Theorem 3.2. *Let $u(z)\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ and for all real θ we have $\gamma(e^{i\theta}-1)-2e^{i\theta}\neq 0$. For any complex number ϵ with $|\epsilon|<\beta(\beta\geq 0)$, if u satisfies the following condition:*

$$\frac{u(z)+\epsilon z}{1+\epsilon}\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$$

then $N_\beta(u)\subset k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$.

Proof . It is obvious that $u\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ if and only if

$$\left|\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)\left(\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right)}{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)\left(\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right)}\right|<1,\ (-\pi\leq\theta\leq\pi),$$

for any complex number s with $|s|=1$, we have

$$\frac{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma)\left(\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right)}{(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})+(1-ke^{i\theta}-\gamma)\left(\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right)}\neq s.$$

In other words, we must have

$$(1-s)(1-\varsigma)z\left(\mathscr{I}_q^m u(z)\right)'(1+ke^{i\theta})-(ke^{i\theta}+1+\gamma+s(-1+ke^{i\theta}+\gamma))\left(\mathscr{I}_q^m u(z)-\mathscr{I}_q^m u(\varsigma z)\right)\neq 0$$

which is equivalent to

$$z-\sum_{n=2}^{\infty}\frac{C(n,m)q^{n-1}(1-q)^m\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}z^n\neq 0.$$

However, $u\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ if and only if $\frac{(u*h)}{z}\neq 0, z\in U-\{0\}$, where $h(z)=z-\sum\limits_{n=2}^{\infty}c_nz^n$ and

$$c_n=\frac{C(n,m)q^{n-1}(1-q)^m\left((n-u_n)(1+ke^{i\theta}-ske^{i\theta})-s(n+u_n)-u_n\gamma(1-s)\right)}{\gamma(s-1)-2s}$$

we note that

$$|c_n|\leq\frac{C(n,m)q^{n-1}(1-q)^m\left|n(1+k)-u_n(k+\gamma)\right|}{1-\gamma}$$

since $\frac{u(z)+\epsilon z}{1+\epsilon} \in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$, therefore $z^{-1}\left(\frac{u(z)+\epsilon z}{1+\epsilon} * h(z)\right) \neq 0$, which is equivalent to

$$\frac{(u*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon} \neq 0. \tag{3.1}$$

Now suppose that $\left|\frac{(u*h)(z)}{z}\right| < \beta$. Then by (3.1), we must have

$$\begin{aligned}\left|\frac{(u*h)(z)}{(1+\epsilon)z} + \frac{\epsilon}{1+\epsilon}\right| &\geq \frac{|\epsilon|}{|1+\epsilon|} - \frac{1}{|1+\epsilon|}\left|\frac{(u*h)(z)}{z}\right| \\ &> \frac{|\epsilon|-\beta}{|1+\epsilon|} \geq 0,\end{aligned}$$

this is a contradiction by $|\epsilon| < \beta$ and however, we have $\left|\frac{(u*h)(z)}{z}\right| \geq \beta$. If $g(z) = z - \sum\limits_{n=2}^{\infty} b_n z^n \in N_\beta(u)$, then

$$\begin{aligned}\beta - \left|\frac{(g*h)(z)}{z}\right| &\leq \left|\frac{((u-g)*h)(z)}{z}\right| \leq \sum_{n=2}^{\infty} |a_n - b_n||c_n||z^n| \\ &< \sum_{n=2}^{\infty} \frac{C(n,m)q^{n-1}(1-q)^m|n(1+k)-u_n(k+\gamma)|}{1-\gamma}|a_n-b_n| \leq \beta.\end{aligned}$$

□

4. Partial sums of the function class $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$

In this section, applying methods used by Silverman [16] and Silvia [17], we investigate the ratio of a function of the form (1.7) to its sequence of partial sums $u_j(z) = z + \sum\limits_{n=2}^{j} a_n z^n$.

Theorem 4.1. *If u of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left\{\frac{u(z)}{u_j(z)}\right\} \geq 1 - \frac{1}{\delta_{j+1}} \tag{4.1}$$

and

$$\delta_n = \begin{cases} 1, & \text{if } n = 2,3,\cdots,j; \\ \delta_{j+1}, & \text{if } n = j+1, j+2, \cdots, \end{cases} \tag{4.2}$$

where

$$\delta_n = \frac{C(n,m)q^{n-1}(1-q)^j|n(1+k)-u_n(k+\gamma)|}{1-\gamma}. \tag{4.3}$$

The result in (4.1) is sharp for every j, with the extremal function

$$u(z) = z + \frac{z^{j+1}}{\delta_{j+1}}. \tag{4.4}$$

Proof . Define the function w, we may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} =&\delta_{j+1}\left\{\frac{u(z)}{u_j(z)}-\left(1-\frac{1}{\delta_{j+1}}\right)\right\} \\ =&\left\{\frac{1+\sum\limits_{n=2}^{j}a_n z^{n-1}+\delta_{j+1}\sum\limits_{n=j+1}^{\infty}a_n z^{n-1}}{1+\sum\limits_{n=2}^{j}a_n z^{n-1}}\right\}.\end{aligned} \tag{4.5}$$

Then, from (4.5), we can obtain

$$w(z)=\frac{\delta_{j+1}\sum\limits_{n=j+1}^{\infty}a_n z^{n-1}}{2+2\sum\limits_{n=2}^{j}a_n z^{n-1}+\delta_{j+1}\sum\limits_{n=j+1}^{\infty}a_n z^{n-1}}$$

and

$$|w(z)|\leq\frac{\delta_{j+1}\sum\limits_{n=j+1}^{\infty}a_n}{2-2\sum\limits_{n=2}^{j}a_n-\delta_{j+1}\sum\limits_{n=j+1}^{\infty}a_n}.$$

Now $|w(z)|\leq 1$ if

$$2\delta_{j+1}\sum_{n=j+1}^{\infty}a_n\leq 2-2\sum_{n=2}^{j}a_n,$$

which is equivalent to

$$\sum_{n=2}^{j}a_n+\delta_{j+1}\sum_{n=j+1}^{\infty}a_n\leq 1. \tag{4.6}$$

It is suffices to show that the left hand side of (4.6) is bounded above by $\sum\limits_{n=2}^{\infty}\delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{j}(\delta_n-1)a_n+\sum_{n=j+1}^{\infty}(\delta_n-\delta_{j+1})a_n\geq 0.$$

To see that the function given by (4.4) gives the sharp result, we observe that for $z=re^{i\pi/n}$,

$$\frac{u(z)}{u_j(z)}=1+\frac{z^j}{\delta_{j+1}}. \tag{4.7}$$

Taking $z\to 1^-$, we have

$$\frac{u(z)}{u_j(z)}=1-\frac{1}{\delta_{j+1}}.$$

This completes the proof of Theorem 4.1. □

We next determine bounds for $\frac{u_j(z)}{u(z)}$.

Theorem 4.2. *If u of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left\{\frac{u_j(z)}{u(z)}\right\} \geq \frac{\delta_{j+1}}{1+\delta_{j+1}}. \tag{4.8}$$

The result is sharp with the function given by (4.4).

Proof . We may write

$$\begin{aligned}\frac{1+w(z)}{1-w(z)} =&(1+\delta_{j+1})\left\{\frac{u_j(z)}{u(z)} - \frac{\delta_{j+1}}{1+\delta_{j+1}}\right\}\\ =&\left\{\frac{1+\sum\limits_{n=2}^{j} a_n z^{n-1} - \delta_{j+1}\sum\limits_{n=j+1}^{\infty} a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty} a_n z^{n-1}}\right\},\end{aligned} \tag{4.9}$$

where

$$w(z) = \frac{(1+\delta_{j+1})\sum\limits_{n=j+1}^{\infty} a_n z^{n-1}}{-\left(2+2\sum\limits_{n=2}^{j} a_n z^{n-1} - (1-\delta_{j+1})\sum\limits_{n=j+1}^{\infty} a_n z^{n-1}\right)}$$

and

$$|w(z)| \leq \frac{(1+\delta_{j+1})\sum\limits_{n=j+1}^{\infty} a_n}{2-2\sum\limits_{n=2}^{j} a_n + (1-\delta_{j+1})\sum\limits_{n=j+1}^{\infty} a_n} \leq 1. \tag{4.10}$$

This last inequality is equivalent to

$$\sum_{n=2}^{j} a_n + \delta_{j+1}\sum_{n=j+1}^{\infty} a_n \leq 1. \tag{4.11}$$

It is suffices to show that the left hand side of (4.11) is bounded above by $\sum\limits_{n=2}^{\infty}\delta_n a_n$, which is equivalent to

$$\sum_{n=2}^{j}(\delta_n - 1)a_n + \sum_{n=j+1}^{\infty}(\delta_n - \delta_{j+1})a_n \geq 0.$$

This completes the proof of Theorem 4.2. □

We next turn to ratios involving derivatives.

Theorem 4.3. *If u of the form (1.1) satisfies the condition (2.1) then*

$$\Re\left\{\frac{u'(z)}{u'_j(z)}\right\} \geq 1 - \frac{j+1}{\delta_{j+1}} \tag{4.12}$$

$$\textit{and }\Re\left\{\frac{u'_j(z)}{u'(z)}\right\} \geq \frac{\delta_{j+1}}{1+j+\delta_{j+1}} \tag{4.13}$$

where

$$\delta_n \geq \begin{cases} 1, & \text{if } n = 1, 2, 3, \cdots, j; \\ n\frac{\delta_{j+1}}{j+1}, & \text{if } n = j+1, j+2, \cdots, \end{cases}$$

and δ_n *is defined by* (4.3). *The estimates in* (4.12) *and* (4.13) *are sharp with the extremal function given by* (4.4).

Proof . Firstly, we will give proof ou (4.12). We write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= \delta_{j+1}\left\{\frac{u'(z)}{u'_j(z)} - \left(1 - \frac{j+1}{\delta_{j+1}}\right)\right\} \\ &= \left\{\frac{1 + \sum\limits_{n=2}^{j} na_n z^{n-1} + \frac{\delta_{j+1}}{j+1}\sum\limits_{n=j+1}^{\infty} na_n z^{n-1}}{1 + \sum\limits_{n=2}^{j} a_n z^{n-1}}\right\}, \end{aligned}$$

where

$$w(z) = \frac{\frac{\delta_{j+1}}{j+1}\sum\limits_{n=j+1}^{\infty} na_n z^{n-1}}{2 + 2\sum\limits_{n=2}^{j} na_n z^{n-1} + \frac{\delta_{j+1}}{j+1}\sum\limits_{n=j+1}^{\infty} na_n z^{n-1}}$$

and

$$|w(z)| \leq \frac{\frac{\delta_{j+1}}{j+1}\sum\limits_{n=j+1}^{\infty} na_n}{2 - 2\sum\limits_{n=2}^{j} na_n + \frac{\delta_{j+1}}{j+1}\sum\limits_{n=j+1}^{\infty} na_n}.$$

Now $|w(z)| \leq 1$ if and only if

$$\sum_{n=2}^{j} na_n + \frac{\delta_{j+1}}{j+1}\sum_{n=j+1}^{\infty} na_n \leq 1, \tag{4.14}$$

since the left hand side ou (4.14) is bounded above by $\sum\limits_{n=2}^{\infty} \delta_n a_n$.

This completes the proof of Theorem. □

Theorem 4.4. $k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$ *is convex and compact subset of* T.

Proof . Suppose $u_j \in k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$,

$$u_j(z) = z - \sum_{n=2}^{\infty} |a_{j,n}| z^n. \tag{4.15}$$

Then for $0 \leq \psi < 1$, let $u_1, u_2 \in k - \widetilde{U}ST_s(q, m, \gamma, \varsigma)$ be defined by (4.15). Then

$$\begin{aligned} \xi(z) &= \psi u_1(z) + (1-\psi) u_2(z) \\ &= \psi\left(z - \sum_{n=2}^{\infty} |a_{1,n}| z^n\right) + (1-\psi)\left(z - \sum_{n=2}^{\infty} |a_{2,n}| z^n\right) \\ &= z - \sum_{n=2}^{\infty} \left(\psi |a_{1,n}| + (1-\psi)|a_{2,n}|\right) \end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)\left(\psi|a_{1,n}|+(1-\psi)|a_{2,n}|\right)\\
=&\psi\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)|a_{1,n}|\\
&+(1-\psi)\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)|a_{2,n}|\\
\le&\psi(1-\gamma)+(1-\psi)(1-\gamma)=1-\gamma.
\end{aligned}$$

Then $\xi(z)=\psi u_1(z)+(1-\psi)u_2(z)\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$.
Therefore $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ is convex.

Now we have to show $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ is convex.

For $u_j\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma),\varsigma\in\mathbb{N}$ and $|z|<r\quad(0<r<1)$, then we have

$$\begin{aligned}
|u_j(z)|&\le r+\sum_{n=2}^{\infty}|a_{j,n}|r^n\\
&\le r+\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)|a_{j,n}|r^n\\
&\le r+(1+r)r^n.
\end{aligned}$$

Therefore $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ is uniformly bounded.

Let $u_j(z)=z-\sum\limits_{n=2}^{\infty}|a_{j,n}|,\quad z\in U,\quad j\in\mathbb{N}$.

Also let $u(z)=z-\sum\limits_{n=2}^{\infty}a_nz^n$. Then, by Theorem 2.1, we get

$$\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)|a_n|\le 1-\gamma. \tag{4.16}$$

Assuming $u_j\to u$, then we have that $a_{j,n}\to a_n$ as $n\to\infty,\ (j\in\mathbb{N})$.

Let $\{\rho_n\}$ be the sequence of partial sums of the series

$$\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left(n(k+1)-u_n(k+\gamma)\right)|a_n|.$$

Then $\{\rho_n\}$ is non decreasing sequence and by (4.16) it is bounded above by $1-\gamma$. Thus it is convergent and

$$\sum_{n=2}^{\infty} C(n,m)q^{n-1}(1-q)^m \left|n(k+1)-u_n(k+\gamma)\right||a_{j,n}|=\lim_{n\to\infty}\rho_n\le 1-\gamma.$$

Therefore $u\in k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ and the class $k-\widetilde{U}ST_s(q,m,\gamma,\varsigma)$ is closed. □

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