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Laguerre-Chebyshev Petrov-Galerkin method for solving integral equations

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Abstract

This project aims to solve the second kind of Volterra, Fredholm integrodifferential equations, and mixed integral equations (VIDE, FIDEs and MIEs respectively) will be solved using the Laguerre-Chebyshev Petrov-Galerkin method (PGM). By solving three cases to show how the recommended technique works in this study, we established the PGM to find the approximate solution for linear VIDEs, FIDEs, and MIEs.

Keywords: Integral equation, Laguerre-Chebyshev, Petrov-Galerkin method, Orthogonal polynomial, Convergence analysis

1. Introduction

Many problems of mathematical physics can be started in the form of integral equations (IEs), especially VIDEs and FIDEs of the second kinds and MIEs, respectively. Several researchers have discussed and implemented these IEs in order to get the numerical solutions. Recently, there have been a study in interest in VIDEs, FIDEs and MIEs, because to their many of applications in mathematical physics (astrophysics, problem of contact, problem of heat transfer, and reactor theory). Most typical numerical IEs solvers have been invented and built since the debut of the digital computer a few years ago. In mathematical applied there are many ways for solving many problems to find the approximate solution of the second kind VIE and VIDE by Homotopy analysis method [4] solution of VIDEs used variational iteration method [2]. There are many numerical methods, Laplace-Adomain decomposition method [15], Chebyshev collocation method [9]. Solution of FIDEs by using a hybrid of block-pulse functions and Taylor polynomials [8], the modified decomposition

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method for solving MIDEs [13]. Approximate method for solving VFIDEs using normalization Bernstein polynomials [3]. In [1] using Touchard polynomials method of for solving linear VIDEs, where [17] used Legendre spectral Galerkin method for solving the second-kind VIEs. Provide general spectral and pseudo-spectral Jacobi-Petrov-Gauss-Legendre quadrature formula is used to approximate the integral operator and the inner product based on the Jacobi weight is implemented in the weak formulation in the numerical implementation. The spectral Jacobi PGM, rigorous error analysis in both $L^2_w(\alpha, \beta)$ and $L^\infty_w(\alpha, \beta)$ norms are given provided that both the kernel function and the function source are sufficiently smooth [16]. Solving the Singular Integro-Differential Equations using B-Spline Methods [6].

The current study attempts to implement the Laguerre-Chebyshev Petrov-Galerkin technique for VIDEs, FIDEs, and MIEs using the techniques described in [5]. The primary goals of this study are to design PGM for VIDEs, FIDEs, and MIEs, as well as to explore the convergence of the suggested method.

1.1. Integral equations

Definition 1.1. *[17] An integral equation is one in which the unknown function $u(x)$ appears both inside and outside the integral signs. The most common type of integral equation is one of the following:*

$$h(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt, \quad x \in [a,b] \tag{1.1}$$

An integro-differential equation (IDE) an equation involving derivative and integral together with unknown function $u(x)$, which is of the form:

$$u^{(j)}(x) + \sum_{j=0}^{k-1} p_j(x)u^{(j)}(x) = f(x) + \lambda \int_{\alpha(x)}^{b(x)} k(x,t)u(t)dt, \tag{1.2}$$

where, $u^{(j)}(x) = \dfrac{d^j u}{dx^j}$.

where, $\alpha(x)$ and $\beta(x)$ are integration limits, λ is a constant parameter, and $k(x,t)$ the kernel of the integral, which is a known function of the two variables x and t. The functions $f(x)$ and $k(x,t)$ are given in advance. It should be emphasized that the integral limits are dictated by the variables $\alpha(x)$, $\beta(x)$ and perhaps both variables, constants, or mixed.

1.2. Forms of the integral equations

An integral equation (1.1) is called

1. Non-linear IE, if the kernel $k(x,t)$ is given in the form $k(x,t,u(t))$.
2. Homogenous IE, if $f(x) = 0$, otherwise it is called non- homogenous.
3. Linear integral equation of the first kind, if $h(x) = 0$, while if $h(x) = 1$ it called linear IE of the second kind, otherwise it is called of the third kind.
4. VIEs, when $\alpha(x) = a$ and $\beta(x) = x$, where a is constant and x variable, which has a form:

$$h(x)u(x) = f(x) + \lambda \int_{\alpha(x)}^{\beta(x)} k(x,t)u(t)dt, \quad x \in [a,b] \tag{1.3}$$

5. FIEs, when $\alpha(x) = a$ and $\beta(x) = b$, where a and b are constants, which has a form:

$$h(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt, \quad x \in [a,b] \tag{1.4}$$

1.3. Mixed integral equations

Mixed integral equations (MIEs), have a form:

$$h(x)u(x) = f(x) + \lambda_1 \int_{\alpha(x)}^{\beta(x)} k_1(x,t)u(t)dt + \lambda_2 \int_{\delta(x)}^{\gamma(x)} k_2(x,t)u(t)dt \quad x \in [a,b] \tag{1.5}$$

Or

$$h(x)u(x) = f(x) + \lambda_1 \int_{\alpha(x)}^{\beta(x)} \int_{\delta(x)}^{\gamma(x)} k_1(x,t)u(t)dt \quad x \in [a,b] \tag{1.6}$$

when $\alpha(x) = a$ and $\beta(x) = x$, where a and x are constant and variable respectively and $\delta(x) = c$ and $\gamma(x) = d$, where c and b are constants.

1.4. Forms of the kernel of integral equations

If the kernel in integral equation (1.1) is called

(a) Difference kernel if it is depending on the difference (x-t), then the equation is called IE of convolution type. i.e

$$k(x,t) = k(x-t)$$

(b) Degenerate or (sparable) kernel, when the kernel may be decomposed as follows:

$$k(x,t) = \sum_{k=1}^{n} a_k(x)b_k(t)$$

2. Orthogonal polynomials

In the areas, orthogonal polynomials have piqued the interest of mathematicians. In recent years, with the finding of their relevance to inetegrable systems, this attention has often come from beyond the polynomials community [12]. Let

$$\int_a^b w(x)\varphi_i(x)\varphi_j(x) = \delta_{ij} \tag{2.1}$$

with the Kronecker delta δ_{ij} defined by

$$\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

where $w(x)$ is a continuous, positive weight function on $[a,b]$, and the moments areexist.

Then the inner product of polynomials φ_i and φ_j given by:

$$<\varphi_i, \varphi_j> = \int_a^b w(x)\varphi_i(x)\varphi_j(x)dx \tag{2.2}$$

for orthogonality

$$<\varphi_i, \varphi_j> = 0, i \neq j \tag{2.3}$$

The weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ in the interval $[a, b]$ is used in this investigation.

The weight function in the interval $[a, b]$ is used in this work, as well as the creation of φ_i, $i = 0, 1, 2, \ldots$ of the following approximant functions.

$$u_n(x) = \sum_{j=0}^{n-1} u_j(\varphi_j(x) + s_j\varphi_{j+1}(x)) \cong u(x) \tag{2.4}$$

where s_j is the constant and $j = 0, 1, 2, \ldots, n-1$ is the φ_j basis function.

The trail and test functions of a basis Laguerre and a basis Chebyshev polynomial with the following weight function will be discussed in this post [5, 7, 9, 10].

3. The convergence of spectral petrov-galerkin method (SPGM)

The SPGM and the conditions that lead to convergence are discussed in this section. If X is a Banach space with the norm $\|.\|$ and X^* is its dual space, then $U_n \in X$ and $V_n \in X^*$ are two separate sequences of finite-dimensional sub spaces that satisfy the following condition [11],

(H) $\forall u \in X$ and $v \in X^*$, $\exists u_n \in U_n$ and $v_n \in V_n$ such that

- $\|u_n - u\| \to 0 and \|v_n - v\| \to 0$ as $n \to \infty$.
- $dim U_n = dim V_n, n = 1, 2, \ldots.$

In the SPGM, that is a numerical method, we seek $u_n \in U_n$ so as each $v_n \in V_n$ be orthogonal on both sides of equation

$$u - ku = f \tag{3.1}$$

As define

$$ku^{(j)}(x) = \int_a^b k(x,t)u(t)dt, \quad FIDEs$$

$$ku^{(j)}(x) = \int_a^x k(x,t)u(t)dt, \quad VIDEs$$

$$<(\sum_{i=0}^{n} c_i D^i - k)u_n, v_n> = <f, v_n> \quad \forall v_n \in V_n \tag{3.2}$$

Furthermore, if an element $P_n u \in U$ satisfies the equation

$$<u - P_n u, v_n> = 0 \quad \forall v_n \in V_n \tag{3.3}$$

for $u \in U$, is referred to as a generalized best approximation (GBA) from u_n to u with respect to V_n .

As a result, the SPGM is a GBA projection method. In terms of the existence and uniqueness of the GBA, the following claim is valid:

or each $u \in U$, the GBA from u_n to u with respect to V_n exists uniquely if and only if

$$V_n \cap U_n^{\perp} = \{0\} \tag{3.4}$$

where $U_n^{\perp}$ denotes the annihilator of u_n in X^* that is the set all functions satisfying a given set of conditions which is zero on every member of a given set and say that $U_n \cap V_n$ if $V_n \cap U_n^{\perp} \neq \{0\}$. By this condition P_n is a projection, for the proof P_n is projection.

Since $dim\ U_n = dimV_n$, then assume that U_n and V_n have bases $\{\varphi_j\}_{j=1}^{n-1}$ and $\{\xi_i\}_{i=1}^{n-1}$ respectively. Let $u \in U$be given. To show that there is a unique $P_n u \in U_n$ satisfying (3.3), we show that the linear system

$$\sum_{j=0}^{n-1} c_j(\varphi_j + s_j\varphi_{j+1}, \xi_i) = (x, \xi_i) \quad i, j = 0, 1, \ldots, n-1 \tag{3.5}$$

has only one solution $\{c_j\}_{j=1}^{n-1}$. This is equivalent to showing that the coefficient matrix $A = (\varphi_j + s_j\varphi_{j+1}, \xi_i)$ is nonsingular.

To prove necessity, must be assume that there exist that $y_n \in Y_n \cap X_n^{\perp}$, since $y_n \in Y_n$, can be write $y_n = \sum_{i=0}^{n-1} c_i\xi_i$. By the fact that $y_n \in X_n^{\perp}$, we have right hand side of (3.5) equals zero.

$$\sum_{j=0}^{N-1} c_j(\varphi_j + s_j\varphi_{j+1}, \xi_i) = 0 \quad i, j = 0, 1, \ldots, n-1 \tag{3.6}$$

Since the matrix A is nonsingular, $c_j = 0$ for $j = 0, 1, \ldots, n-1$. Thus $V_n = 0$ and $V_n \cap U_n^{\perp} = \{0\}$.

Conversely, A are nonsingular. Then there exist $\{c_i\}_{i=1}^{n-1}$, not all zero, such that (3.6).

Let $v_n = \sum_{i=0}^{n-1} c_i\xi_i$. thus $y_n \neq 0$ and $v_n \in V_n \cap U_n^{\perp}$. this implies that $V_n \cap U_n^{\perp} \neq \{0\}$. A contradiction, it remains to show that P_n is a projection.

We have just proved that under condition (3.4), for each $u \in U$ there exists a unique $P_n u \in U_n$ that satisfies (3.3). For any $u \in U$, we have $P_n u \in U_n \subseteq X$, thus, by definition,

$$<P_n u - P_n^2 u, v_n> = 0 \quad \forall v_n \in V_n$$

From this equation and (3.3), we find that $P_n^2 u \in U_n$ satisfies

$$<u - P_n^2 u, v_n> = 0 \quad \forall v_n \in V_n$$

By the uniqueness, we conclude that $\forall u \in U$

$$P_n^2 u = P_n x$$

That is, P_n is a projection.

However, this is not a sufficient condition for insurance. Every $x \in X$ has a Petrov-Galerkin approximation that is unique. As a result, we must introduce a new idea known as the regular pair. If there exists a linear operator $\prod_n : X_n \longrightarrow Y_n$ with $\prod_n X_n = Y_n$ such that satisfying the condition.

$$(H-1)\ \|x_n\| \leq C_1 <x_n, \prod_n x_n>^{1/2} \quad \forall x_n \in X_n,$$

$$(H-2)\ \|\prod_n x_n\| \leq C_2\|x_n\| \quad \forall x_n \in X_n,$$

Where C_1 and C_2 are positive constants independent of n., if X_n and Y_n satisfy the condition (H) and $\{X_n, Y_n\}$ be a regular pair, we have the following statements:

1. $\|P_n x - x\| \to 0$ as $n \to \infty$, $\forall x \in X$.

2. $\|P_n x - x\| \leq C\|Q_n x - x\|$ for some constant $C > 0$ independent of n.

We must analyze the requirement (H), as well as the conditions (H-1) and (H-2) for each construction independently in order to ensure that the approximation solution $x \in X$ exists and is unique.

Definition 3.1. *Let H be a real Hilbert space, we suppose that for each $n \in \mathbb{N}$, X_n and Y_n are subspace of H of the same dimension. we begin with following definitions.*

Definition 3.2. *The spectral Petrov-Galerkin approximation to $u \in H$ satisfying (1.1) is an element $u_n \in X_n$ such that for any $V \in Y_n$.*

$$<u_n - ku_n, v> = <f, v>$$

A condition that ensures the existence of a unique Petrov-Galerkin approximation $u_n \in X_n$ is that $Y_n^{\perp} \cap X_n = \{0\}$.

4. The implement of the SPGM

The SPGM for (1.2) is numerical method for finding $u_n \in X_n$ such that [5].

$$<u^{'} + ku_n, v_n> = <f, v_n> \quad \forall v_n \in V_n \tag{4.1}$$

If $\{X_n, Y_n\}$ is a regular pair with a linear operator $\prod_n : X_n \longrightarrow Y_n$, then the equation (3.3) may be rewritten as

$$<u^{'} + ku_n, \prod_n x_n> = <f, \prod_n x_n> \quad \forall v_n \in V_n \tag{4.2}$$

Now, assume $u_n \in X_n$ and $\{\varphi_j + s_j\varphi_{j+1}\}_{j=0}^{N-1}$ is a basis for X_n (trail space) and $\xi_{i i=0}^{n-1}$ (test space) is a basis for Y_n.

Now apply SPGM, using a basis Laguerre polynomial is trail function and a basis Chebyshev polynomial with weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$, is test function define on interval $[a, b]$. Then (1.2) leads to determining $u_{i i=0}^{n-1}$ as the solution of the linear system

$$(u_N^{'}(x), v_N(x))_w + (ku_N, v_N)_w = (g, v_N)_w, \quad \forall v_n \in P_{N-1} \tag{4.3}$$

$$(ku, v)_w = \int_a^b \int_a^x k(x,t)u(t)v(x)w(x)dtdx, \quad VIEs.$$
$$(ku, v)_w = \int_a^b \int_a^b k(x,t)u(t)v(x)w(x)dtdx, \quad FIEs.$$

where

$$(u, v)_w = \int_a^b w(x)u(t)v(x)dx$$

is the continuous inner product. Set

$$U_N(x) = \sum_{j=0}^{n-1} u_j(\varphi_j(x) + s_j\varphi_{j+1}(x)),$$

where s_j is a constant chosen as the boundary condition.

when $\xi_i(x), i = 0, 1, \ldots, n-1$ is a Chebyshev polynomial test function with a weight function from space, Laguerre polynomial $\varphi_j(x), j = 0, 1, \ldots, n-1$ is used. We get the following result from (1.2)

$$\sum_{j=0}^{n-1}(\xi_i(x), (\varphi_j(x) + \varphi_{j+1}(x)))_w u_j + \sum_{j=0}^{N-1}(\xi_i(x), k(\varphi_j(x) + \varphi_{j+1}(x)))_w u_j = (\xi_i(x), f(x))_w \tag{4.4}$$

which leads to an equation of matrix form

$$(A + B)U_{n-1} = f_{N-1}, \tag{4.5}$$

$$U_{n-1} = [u_0, u_1, \ldots, u_{n-1}]^T \qquad A(i,j) = (\xi_i, \varphi_j' + s_j\varphi_{j+1})_w$$

$$B(i,j) = (\xi_i, k(\varphi_j + s_j\varphi_{j+1}))_w \qquad f_{n-1}(i) = (\xi_i, f)_w$$

5. Numerical examples

Because the exact solution to these problems is available in the literature, we employ several VIDEs, FIDEs, and MIEs to test the accurate recommendations. For all of the cases, the proposed method's answers are superior to exact solutions based on two polynomials: Laguerre polynomials for the trail function and Chebyshev polynomials for the test function. Convergence is determined for each VIE using the following formula

$$E = |U_{Ex} - U_{ap}| < \delta$$

where, U_{Ex} exact solution and U_{ap} approximation solution.

Example 1: Consider the following VIDEs of the second kind [8];

$$u'(x) = 1 - 2xsinx + \int_0^x u(t)dt, \quad u(0) = 0$$

with the exact solution $u(x) = xcos(x)$, for $0 \le x \le 1$.

By using the present method, we solve this problem, thus the absolute errors obtained are compared with those obtained in [7] and they are presented in Table 1 and Figure 1 which shows that the plots of the exact and approximate solutions for case $N = 4$ and obtained $L^\infty = 3.4183e - 05$ with $L^2 = 6.4873e - 05$.

Table 1: Comparison of absolute errors for Example 1

X	Exact solution	Approximate solution	Absolute Error of the present method	Absolute Error [8]
0	0	0	0	0
0.1	9.9500e-02	9.9466e-02	3.4183e-05	4.233e-04
0.2	1.9601e-01	1.9601e-01	1.2185e-06	6.317e-04
0.3	2.8660e-01	2.8663e-01	2.8805e-05	8.544e-04
0.4	3.6842e-01	3.6846e-01	3.1948e-05	1.161e-03
0.5	4.3879e-01	4.3880e-01	1.2699e-05	1.543e-03
0.6	4.9520e-01	4.9519e-01	1.1674e-05	1.971e-03
0.7	5.3539e-01	5.3537e-01	2.4094e-05	2.430e-03
0.8	5.5737e-01	5.5735e-01	1.7329e-05	2.951e-03
0.9	5.5945e-01	5.5945e-01	2.2173e-07	3.626e-03
1	5.4030e-01	5.4031e-01	3.1250e-06	4.629e-03

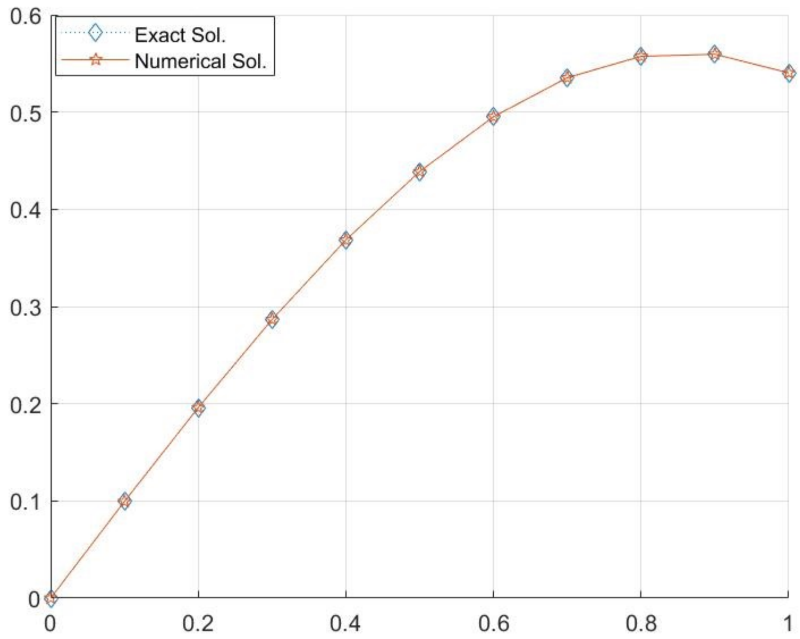

Figure 1: Plots of the exact solution and approximate solutions of Example 1 by SPGM for value of $N = 4$

Example 2: Consider the following linear Fredholm integro-differential equation [3].

$$u^{'}(x) = xe^{x} + e^{x} - x + \int_{0}^{1} xu(t)dt, \quad u(0) = 0$$

with the exact solution $u(x) = xe^{x}$, $0 \leq x \leq 1$.

We solve this problem by using the present method and the absolute errors obtained are compared with those obtained in [3] and they are presented in Table 2 and Figure 2 which shows that the plots of the exact and approximate solutions for case N=4 and obtained $L^2 = 2.0182e - 04$ with $L^\infty = 1.0368e - 04$.

Table 2: Comparison of absolute errors for Example 2

x	Exact solution	Approximate solution	Absolute Error of the present method	Absolute Error [3]
0	0	0	0	0
0.1	1.1052e-01	1.1042e-01	9.3420e-05	8.0000e-05
0.2	2.4428e-01	2.4426e-01	2.3682e-05	3.1000e-04
0.3	4.0496e-01	4.0503e-01	6.7818e-05	7.2000e-04
0.4	5.9673e-01	5.9683e-01	1.0368e-04	1.3600e-03
0.5	8.2436e-01	8.2443e-01	6.6769e-05	2.8500e-03
0.6	1.0933e+00	1.0933e+00	1.2063e-05	6.5000e-03
0.7	1.4096e+00	1.4096e+00	7.5809e-05	1.4300e-02
0.8	1.7804e+00	1.7804e+00	7.3915e-05	2.9200e-02
0.9	2.2136e+00	2.2136e+00	6.9374e-06	5.5000e-02
1	2.7183e+00	2.7183e+00	1.5257e-05	9.6600e-02

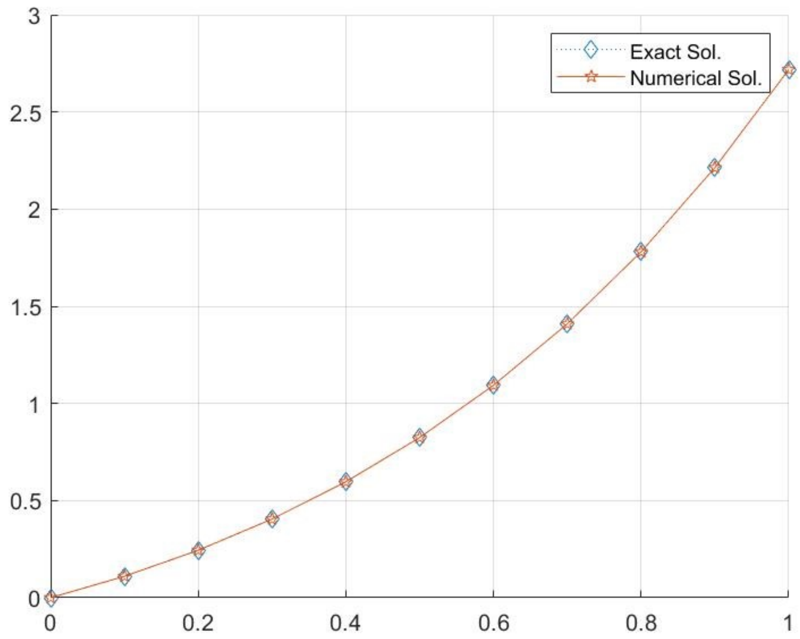

Figure 2: Plots of the exact solution and approximate solutions of Example 2 by SPGM for value of $N = 4$

Example 3 : Consider the following MIE of the second kind given in the form [14]

$$u(x) = (cosx - 1)x^2 + (2cos1 - cosx - sin1 - 1)x + 2sinx + \int_0^x (x^2 - t)u(t)dt + \int_0^1 (xt + x)u(t)dt$$
$$0 \leq x \leq 1$$

with exact solution $u(x) = sin(x)$.

We solve this problem by using the present method and the absolute errors obtained are compared with those obtained in [13] and they are presented in Table 3 and Figure 3 which shows that the plots of the exact and approximate solutions for case N=5 and obtained $L^2 = 4.0813e - 06$ with $L^\infty = 3.3274e - 06$.

Table 3: Comparison of absolute errors for Example 3

x	Exact solution	Approximate solution	Absolute Error of the present method	Absolute Error [14]
0	0	3.3274e-06	3.3274e-06	1.335e-05
0.2	1.9867e-01	1.9867e-01	1.1182e-06	1.155e-05
0.4	3.8942e-01	3.8942e-01	1.0806e-06	1.075e-05
0.6	5.6464e-01	5.6464e-01	5.8167e-08	1.283e-05
0.8	7.1736e-01	7.1736e-01	1.9189e-07	1.108e-05
1	8.4147e-01	8.4147e-01	4.4156e-07	1.309e-05

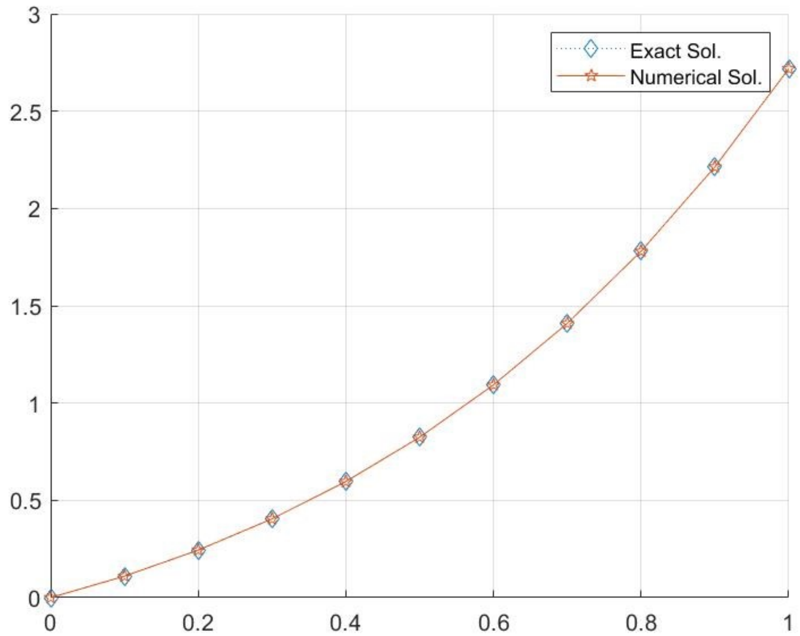

Figure 3: Plots of the exact solution and approximate solutions of Example 3 by SPGM for value of $N = 5$

6. Conclusions

In this research, we used the Laguerre-Chebyshev SPGM, which is based on the orthogonal polynomials basic tool and was designed to solve first and second-kind VIEs. The numerical results obtained using the suggested method indicate an excellent rate of convergence, as shown in Tables 1-3. Furthermore, the numerical and analytical solutions are identical even when using a small number of degrees of polynomials to determine an approximation solution. As a result of L^2 and L^∞-norms errors, the current method is effective, efficient, and dependable for solving various forms of integral equations.

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