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Hyers stability of lattice derivations

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Abstract

In this paper, the lattice derivatives and their properties are introduced and next to the generalized stability of lattice derivatives is investigated using the direct and fixed point method.

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1. Introduction

Functional equations and their stability are some of the classical and practical issues in the area of mathematical analysis. About half a century ago, the stability of functional equations was raised with the important question of Ulam [18]. It is said that a functional equation G is stable if each function g satisfying the equation $G-$approximately is near to the true solution of G. Hyers developed Ulam's question and theorem [5]. He posed the following theorem:
Suppose that U and V be Banach spaces and ρ be a function from U to V such that the following inequality satisfies for some $\delta > 0$ and for every $u, v \in U$,

$$\|\rho(u+v) - \rho(u) - \rho(v)\| \leq \delta.$$

Then there is only one additive function $T : U \to V$ so that

$$\|T(u) - \rho(u)\| \leq \delta$$

for any $u \in U$.
Mathematicians developed the results of the Hyers theorem. By changing space, norm, control

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function, and functional equation, they could prove more interesting theorems [1, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15]. For example, the Jenson functional equation or the integral and differential equations were used instead of the functional equation (in the theorem) and the validity of the theorem proved. Now, we change the functional equation to different lattice functional equation and replacing various control functions in the above theorem.

The first definition of Riesz spaces was done by Frigyes Riesz in 1930. In [16], Riesz introduced the vector lattice spaces and their properties. A Riesz space (vector lattice) is a vector space which is also a lattice, so that the two structures are compatible in a certain natural way. If, in addition, the space is a Banach space, it is a Banach lattice. We present some of the terms and concepts of the Riesz spaces used in this article, concisely. However, we relegate the reader to [2, 6, 17, 19], for the fundamental notions and theorems of Riesz spaces and Banach lattices.

Let V be real vector space that is supposed to be a partially orderly vector space or an ordered vector space, if it is equipped with a partial ordering " $\leq$ " that satisfies

1. $u \leq u$ for every $u \in V$.
2. $u \leq v$ and $v \leq u$ implies that $u = v$.
3. $u \leq v$ and $v \leq w$ implies that $u \leq w$.

A Riesz space is an ordered vector space in which for all $u, v \in U$ the infimum and supremum of $\{u, v\}$, denoted by $u \wedge v$ and $u \vee v$ respectively, exist in U . The negative part, the positive part, and the absolute value of $u \in U$, are defined by $u^- := -u \vee 0$, $u^+ := u \vee 0$, and $|u| := u \vee -u$, respectively. Let U be a Riesz space. U is named Archimedean if $\inf\{\frac{u}{n} : n \in \mathbb{N}\} = 0$ for every $u \in U^+$. If $|x| \leq |v|$ implies $\|u\| \leq \|v\|$ for all $u, v \in U$, then $\|.\|$ is called a lattice norm or Riesz norm on U.

1. $u + v = u \vee v + u \wedge v \quad , \quad -(u \vee v) = -u \wedge -v$.
2. $u + (v \vee w) = (u + v) \vee (u + w) \quad , \quad u + (v \wedge w) = (u + v) \wedge (u + w)$.
3. $|u| = u^+ + u^- \quad , \quad |u + v| \leq |u| + |v|$.
4. $u \leq v$ is equivalent to $u^+ \leq v^+$ and $v^- \leq u^-$.
5. $(u \vee v) \wedge w = (u \wedge v) \vee (v \wedge w) \quad , \quad (u \wedge v) \vee w = (u \vee v) \wedge (v \vee w)$.

Let U be a Riesz space. The sequence $\{u_n\}$ is called uniformly bounded if there exists an element $e \in U^+$ and sequence $\{a_n\} \in l^1$ such that $u_n \leq a_n.e$. A Riesz space U is named uniformly complete if $\sup\{\sum_{i=1}^{n} u_i \ : \ n \in \mathbb{N}\}$ exists for every uniformly bounded sequence $u_n \subset U^+$. Let U and Y be Banach lattices. Then the function $F : U \to V$ is called a cone-related function if $F(U^+) = \{F(|u|) : u \in U\} \subset V^+$.

Theorem 1.1. *[2] For a function $F : U \to V$ defined between Riesz spaces, the conditions stated are equivalent:*

1. *F is a lattice homomorphism.*
2. *$F(u)^+ = F(u^+)$ for all $u \in U$.*
3. *$F(u) \wedge F(v) = F(u \wedge v)$, for all $u, v \in U$.*
4. *If $u \wedge v = 0$ in U, then $F(u) \wedge F(v) = 0$ in V.*
5. *$F(|u|) = |F(u)|$, for every $u \in U$.*

Definition 1.2. *[14] A function $d : U \times U \to [0, \infty]$ is named a generalized metric on set U if d satisfies the succeeding conditions*

(i) for each $u, v \in U$, $d(u, v) = 0 \Leftrightarrow u = v$;
(ii) for all $u, v \in U$, $d(u, v) = d(v, u)$;
(iii) for all $u, v, w \in U$, $d(u, w) \leq d(u, v) + d(v, w)$.

Notice that the just generalized metric significant difference from the metric is that the generalized metric range contains the infinitude.

Theorem 1.3. *[14] Let (U, d) be a complete generalized metric space and $I : U \to U$ be a contractive mapping with Lipschitz constant $L < 1$. Then for every $u \in U$, either*

$$d(I^{n+1}u, I^n u) = \infty$$

for all non-negative integers n or there exists an integer $n_0 > 0$ so that
(i) For all $n \geq n_0$, $d(I^n u, I^{n+1}u) < \infty$;
(ii) $I^n(u) \to v^*$, where v^* is a fixed point of I;
(iii) v^* is the unique fixed point of I in the set $V = \{v \in U \ : \ d(I^{n_0}(u), v) < \infty\}$;
(iv) For each $v \in V$, $d(v^*, v) \leq \frac{1}{1-L} d(Iv, v)$.

2. Main Results

Definition 2.1. *Let L be a lattice. A derivation of L is a function $\rho : L \to L$ such that*

(D1) $\rho(u \vee v) = \rho(u) \vee \rho(v)$,

(D2) $\rho(u \wedge v) = (\rho(u) \wedge v) \vee (u \wedge \rho(v))$,

for all $u, v \in L$.

Theorem 2.2. *Let L be a Riesz space. A function $\rho : L \to L$ is a lattice derivation if and only if*

(D3) $\rho(u \vee v \vee w) = \rho(u) \vee \rho(v) \vee \rho(w)$,

(D4) $\rho(u \wedge v \wedge w) = (\rho(u) \wedge v \wedge w) \vee (u \wedge \rho(v) \wedge w) \vee (u \wedge v \wedge \rho(w))$,

for all $u, v, w \in L$.

Proof . *It is clear to see that $(D1) \Leftrightarrow (D3)$, we show that $(D2) \Leftrightarrow (D4)$. Let $(D2)$ is satisfied, so we have*

$$\begin{aligned} \rho(u \wedge v \wedge w) &= (\rho(u \wedge v) \wedge w) \vee (u \wedge v \wedge \rho(w)) \\ &= (\rho(u) \wedge v) \vee (u \wedge \rho(v) \wedge w) \vee (u \wedge v \wedge \rho(w)) \\ &= (\rho(u) \wedge v \wedge w) \vee (u \wedge \rho(v) \wedge w) \vee (u \wedge v \wedge \rho(w)), \end{aligned} \tag{2.1}$$

for all $u, v, w \in L$. Note that L is a Riesz space, so it has distributive low. For inverse, we substitute $w = v := w$ in (4), therefore we obtain

$$\rho(u) = (\rho(u) \wedge u),$$

this implies that $\rho(u) \leq u$, for all $u \in L$. Putting $w = u$ in (D4), we get

$$\rho(u \wedge v) = (\rho(u) \wedge v) \vee (u \wedge \rho(v)),$$

so, ρ satisfies in (D2). $\square$

Corollary 2.3. *Let L be a Riesz space. A function $\rho : L \to L$ is a lattice derivation if and only if*

$$(i)\ \rho\left(\bigvee_{i=1}^{n} u_i\right) = \bigvee_{i=1}^{n} \rho(u_i), \qquad (ii)\ \rho\left(\bigwedge_{i=1}^{n} u_i\right) = \bigvee_{i=1}^{n}\left(\rho(u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} u_j\right)$$

for all $u_1, u_2, ..., u_n \in L$.

Proof . *It is easy to show that above equalities is satisfied by Induction on n.* $\square$

Theorem 2.4. *Suppose that L be a Banach lattice and $\varphi, \psi : L^n \to [0, \infty)$ be two functions, so that*

$$\varphi(q_1u_1, q_2u_2, ..., q_nu_n) = (q_1q_2...q_n)^{\frac{\alpha}{n}}\varphi(u_1, u_2, ..., u_n) \tag{2.2}$$

and

$$\psi(q_1^nu_1, q_2^nu_2, ..., q_n^nu_n) = \mathcal{O}(q_1q_2...q_n) \tag{2.3}$$

for some $\alpha \in [0, 1)$ and for all $u_1, u_2, ...u_n \in L$ and $q_1, q_2, ..., q_n \geq 1$. Let $f : L \to L$ be a function such that

$$\left\|\bigvee_{i=1}^{n} f\left(\bigvee_{j=1}^{n} q_iu_j\right) + f\left(\bigvee_{i=1}^{n} q_iu_i\right) - 2\bigvee_{i=1}^{n} q_if\left(u_i\right)\right\| \leq \varphi(u_1, u_2, ..., u_n) \tag{2.4}$$

and

$$\left\|f\left(\bigwedge_{i=1}^{n} u_i\right) - \bigvee_{i=1}^{n}\left(f(u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} u_j\right)\right\| \leq \psi(u_1, u_2, ..., u_n) \tag{2.5}$$

for all $u, v, u_1, u_2, ...u_n \in L$ and $q_1, q_2, ..., q_n \geq 1$. Then there exists a unique lattice derivation $\rho : L \to L$ such that the following properties hold.

(1) For all $u \in L$ and $q_1 \geq 1$;

$$\|f(u) - \rho(u)\| \leq \frac{1}{2(q_1 - q_1^{\alpha})}\varphi(u, u, ..., u) \tag{2.6}$$

(2) $\rho^n = \rho$ which $\rho^n = \rho o \rho o ... o \rho$, furthermore $\rho(u) \leq u$ for all $u \in L$.

(3) ρ is increasing mapping. It means that if $v \leq u$ then $\rho(v) \leq \rho(u)$ for all $u, v \in L$.

(4) $Fix_{\rho}(L) = \{u \in L : \ \rho(u) = u\}$ is an ideal of L, moreover $Fix_{\rho}(L) \cap Ker(\rho) = \{0\}$

Proof . We define the set $\mathit{\Delta} := \{g : L \to L \mid g(0) = 0\}$ and the function $d : \mathit{\Delta} \times \mathit{\Delta} \to [0, \infty]$ such that

$$d(g, h) = \inf\{c \in \mathbb{R}^+ \ : \ \|g(u) - h(u)\| \leq c\varphi(u, u, .., u)\},$$

for all $u \in L$. It has been proven the function d is generalized metric on $\mathit{\Delta}$, moreover $(\mathit{\Delta}, d)$ is complete, [[3], theorem 2.5]. Next, we define the mapping $I : \mathit{\Delta} \to \mathit{\Delta}$, so that

$$Ig(u) := \frac{1}{q_1}g(q_1u) \tag{2.7}$$

for all $u \in L$ and $q_1 \geq 1$.
Putting $q_1 = q_2 = ... = q_n := q_1$ and $u_1 = u_2 = ... = u_n := u$ in (2.2), so we obtain

$$\begin{aligned} \left\|Ig(u) - Ih(u)\right\| &= \frac{1}{q_1}\left\|g(q_1u) - h(q_1u)\right\| \\ &\leq \frac{1}{q_1}c\varphi(q_1u, q_1u, ..., q_1u) \\ &\leq \frac{1}{q_1}cq_1^{\alpha}\varphi(u, u, ..., u) = cq_1^{\alpha-1}\varphi(u, u, ..., u), \end{aligned} \tag{2.8}$$

for all $u \in L$. It means that
$$d(Ig, Ih) \leq cq_1^{\alpha-1}.$$
Thus, we see that
$$d(Ig, Ih) \leq q_1^{\alpha-1} d(g, h), \tag{2.9}$$
therefore the mapping I is strictly contractive on Δ with $l = q_1^{\alpha-1} \leq 1$. Putting $u_1 = u_2 = ... = u_n := u$ and $q_1 = q_2 = ... = q_n := q_1$ in (2.4), so we have
$$\left\|2f(q_1 u) - 2q_1 f(u)\right\| \leq \varphi(u, u, ..., u), \tag{2.10}$$
therefore
$$\left\|\frac{1}{q_1} f(q_1 u) - f(u)\right\| \leq \frac{1}{2q_1} \varphi(u, u, ..., u).$$
Hence $d(If, f) \leq \dfrac{1}{2q_1}$. With theorem (1.3), there exists a mapping $\rho : L \to L$ so that
(1) the mapping ρ is a unique fixed point of I in $M = \{g \in \Delta : \ d(g, h) < \infty\}$, therefore
$$\rho(q_1 u) = q_1 \rho(u),$$
for all $u \in L$.
(2) $d(I^n f, \rho) \to 0$ as n tends to ∞. This alludes that
$$\lim_{n\to\infty} \frac{f(q_1^n u)}{q_1^n} = \rho(u) \tag{2.11}$$
for all $u \in L$.
(3) $d(f, \rho) \leq \frac{1}{1-l} d(f, If)$, therefore the following inequality holds.
$$d(f, \rho) \leq \frac{1}{2q_1} . \frac{1}{1 - q_1^{\alpha-1}} = \frac{1}{2(q_1 - q_1^{\alpha})}. \tag{2.12}$$
This alludes that
$$\left\|f(u) - \rho(u)\right\| \leq \frac{1}{2(q_1 - q_1^{\alpha})} \varphi(u, u, ..., u), \tag{2.13}$$
for all $u \in L$, so the inequality (2.6) is established. To prove ρ is lattice derivation, first, we show that ρ is lattice homomorphism. Putting $q_1 = q_2 = ... = q_n := q_1^n$ in (2.4), so we have
$$\left\| \bigvee_{i=1}^{n} f\left(\bigvee_{j=1}^{n} q_1^n u_j\right) + f\left(\bigvee_{i=1}^{n} q_1^n u_i\right) - 2\bigvee_{i=1}^{n} q_1^n f\left(u_i\right) \right\| \leq \varphi(u_1, u_2, ..., u_n), \tag{2.14}$$
therefore
$$\left\| f\left(\bigvee_{i=1}^{n} q_1^n u_i\right) - \bigvee_{i=1}^{n} q_1^n f\left(u_i\right) \right\| \leq \frac{1}{2} \varphi(u_1, u_2, ..., u_n). \tag{2.15}$$
In last inequality, we substitute $u_i := q_1^n u_i$ for $1 \leq i \leq n$, so
$$\begin{aligned} \left\| f\left(\bigvee_{i=1}^{n} q_1^{2n} u_i\right) - \bigvee_{i=1}^{n} q_1^n f\left(q_1^n u_i\right) \right\| &\leq \frac{1}{2} \varphi(q_1^n u_1, q_1^n u_2, ..., q_1^n u_n). \\ &\leq \frac{1}{2} q_1^{n\alpha} \varphi(u_1, u_2, ..., u_n). \end{aligned} \tag{2.16}$$

Then

$$\left\| \frac{1}{q_1^{2n}} f\left(\bigvee_{i=1}^{n} q_1^{2n} u_i\right) - \bigvee_{i=1}^{n} \frac{1}{q_1^n} f\left(q_1^n u_i\right) \right\| \leq \frac{1}{2} \frac{q_1^{n\alpha}}{q_1^{2n}} \varphi(u_1, u_2, ..., u_n). \tag{2.17}$$

Due to lattice operations are continuous and $\alpha \in [0, \frac{1}{2})$ therefore by (2.11), as $n \to \infty$, we have

$$\rho\left(\bigvee_{i=1}^{n} u_i\right) = \bigvee_{i=1}^{n} \rho(u_i). \tag{2.18}$$

Hence, ρ is lattice homomorphism. Second, we show that ρ satisfies in the part (ii) of corollary (2.3). Putting $u_i := q_1^n u_i$ for $i = 1, 2, ..., n$ in (2.6), so we have

$$\left\| f\left(\bigwedge_{i=1}^{n} q_1^n u_i\right) - \bigvee_{i=1}^{n} \left(f(q_1^n u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} q_1^n u_j \right) \right\| \leq \psi(q_1^n u_1, q_1^n u_2, ..., q_1^n u_n) \tag{2.19}$$

for all $u \in L$ and $q_1 \geq 1$. If we divide both sides in last inequality by q_1^n, we get

$$\left\| \frac{1}{q_1^n} f\left(\bigwedge_{i=1}^{n} q_1^n u_i\right) - \bigvee_{i=1}^{n} \left(\frac{1}{q_1^n} f(q_1^n u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} u_j \right) \right\| \leq \frac{1}{q_1^n} \psi(q_1^n u_1, q_1^n u_2, ..., q_1^n u_n) \tag{2.20}$$

for all $u, v \in L$. As $n \to \infty$, by (2.11) we have

$$\begin{aligned} \left\| \rho\left(\bigwedge_{i=1}^{n} u_i\right) - \bigvee_{i=1}^{n} \left(\rho(u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} u_j \right) \right\| &\leq \frac{1}{q_1^n} \psi(q_1^n u_1, q_1^n u_2, ..., q_1^n u_n), \\ &\to 0. \quad \text{as } n \to \infty \text{ (by 2.35)} \end{aligned} \tag{2.21}$$

Emphasizes that lattice operations are continuous. Therefore, we get

$$\rho\left(\bigwedge_{i=1}^{n} u_i\right) = \bigvee_{i=1}^{n} \left(\rho(u_i) \wedge \bigwedge_{j=1, j\neq i}^{n} u_j \right) \tag{2.22}$$

Then by (2.18) and (2.22) and corollary (2.3), ρ is a lattice derivation on L, therefore,

$$\rho(u) = \rho(u \wedge u \wedge ... \wedge u) = (\rho(u) \wedge u) \vee (u \wedge \rho(u)) = \rho(u) \wedge u, \tag{2.23}$$

It means that $\rho(u) \leq u$. Let $v \leq u$ then $v \vee u = u$, thus

$$\rho(u) = \rho(v \vee u) = \rho(v) \vee \rho(u) \quad \text{for all } u, v \in L$$

so $\rho(v) \leq \rho(u)$. Thus ρ is isotone derivation, as a result $\rho^2(u) \leq \rho(u) \leq u$, so

$$\begin{aligned} \rho^2(u) = \rho(\rho(u)) &= \rho(\rho(u) \wedge u) \\ &= (\rho(u) \wedge \rho(u)) \vee (u \wedge \rho^2(u)) = \rho(u) \vee \rho^2(u) = \rho(u), \end{aligned} \tag{2.24}$$

hence $\rho^n(u) = \rho(u)$, this can be easily shown with induction on n. To prove (4), assume that $u \in Fix_\rho(L)$ and $v \leq u$, then

$$\begin{aligned} \rho(v) = \rho(u \wedge v) &= (\rho(u) \wedge v) \vee (u \wedge \rho(v)) \\ &= (u \wedge v) \vee \rho(v) = v \wedge \rho(v) = v. \end{aligned} \tag{2.25}$$

It means that $v \in Fix_\rho(L)$. If $u, v \in Fix_\rho(L)$ then $\rho(u \vee v) = \rho(u) \vee \rho(v) = u \vee v$, therefore $u \vee v \in Fix_\rho(L)$, hence $Fix_\rho(L)$ is an ideal in L. Finally, if $u \in Fix_\rho(L) \cap Ker(\rho)$ then $0 = \rho(u) = u$, which complete the proof. □

Corollary 2.5. *Assume that L be a Banach lattice and $J : L \to L$ be a positive operator, such that*

$$\|J(\tau u \vee \eta v) - \tau J(u) \vee \eta J(v)\| \leq \Upsilon(\tau u \vee \eta v, \tau u \wedge \eta v) \tag{2.26}$$

and

$$\|J(u \wedge v) - ((J(u) \wedge v) \vee (u \wedge J(v)))\| \leq \theta(\|u\|^r \vee \|v\|^r) \tag{2.27}$$

Where $\Upsilon : L \times L \to [0, \infty)$ be a function such that

$$\Upsilon(u, v) \leq (\tau\eta)^{\frac{\alpha}{2}} \Upsilon\left(\frac{u}{\tau}, \frac{v}{\eta}\right)$$

for all $u, v \in L$, $\tau, \eta \geq 1$, $\theta > 0$ and for which there are number $\alpha \in [0, \frac{1}{2})$ and $r \in [0, 1)$. Then there is a unique isotone lattice derivation $H : L \to L$ satisfies in

$$\|H(u) - J(u)\| \leq \frac{\tau^{\alpha}}{\tau - \tau^{\alpha}} \Upsilon(u, u). \tag{2.28}$$

Proof . It has been proven that J is lattice homomorphism i.e.

$$J(u \vee v) = J(u) \vee J(v) \tag{2.29}$$

for all $u, v \in L$ and the following equality is satisfied,

$$\lim_{n\to\infty} \frac{1}{\tau^n} J(\tau^n u) = H(u) \tag{2.30}$$

for all $u \in L$. Also the inequality (2.28) holds, [[11] Theorem 1]. Replacing u, v by $\tau^n u, \tau^n v$, respectively in (2.6), so we have

$$\begin{aligned} &\left\|J(\tau^n (u \wedge v)) - (J(\tau^n u) \wedge \tau^n v) \vee (\tau^n u \wedge J(\tau^n v))\right\| \\ &\qquad\qquad \leq \theta(\|\tau^n u\|^r \vee \|\tau^n v\|^r) = \tau^{nr}\theta(\|u\|^r \vee \|v\|^r), \end{aligned} \tag{2.31}$$

for all $u, v \in L$ and $\tau \geq 1$. With dividing both sides in above inequality on τ^n, we get

$$\left\|\frac{1}{\tau^n} J(\tau^n (u \wedge v)) - \left((\frac{1}{\tau^n} J(\tau^n u) \wedge v) \vee (u \wedge \frac{1}{\tau^n} J(\tau^n v)) \right) \right\| \leq \tau^{n(r-1)}\theta(\|u\|^r \vee \|v\|^r), \tag{2.32}$$

for all $u, v \in L$. As $n \to \infty$, by (2.11) we have

$$\begin{aligned} \left\|H(u \wedge v) - ((H(u) \wedge v) \vee (u \wedge H(v)))\right\| &\leq \lim_{n\to\infty} \tau^{n(r-1)}\theta(\|u\|^r \vee \|v\|^r), \\ &\to 0. \end{aligned} \tag{2.33}$$

Emphasizes that lattice operations are continuous. Therefore, we get

$$H(u \wedge v) = (H(u) \wedge v) \vee (u \wedge H(v)). \tag{2.34}$$

Therefore H is a lattice derivation of L. □

Theorem 2.6. *Let L ba a Banach lattice. Conceive a cone-related functional $J : L \to L$ with $J(0) = 0$, such that*

$$\left\| J\left(\frac{\tau|u|+\nu|w|}{2} \vee \eta|v|\right) \vee 2J\big(\nu|w| - \tau|u| \wedge \eta|v|\big) - \nu J\big(|w|\big) \vee \eta J\big(|v|\big)\right\| \leq \varphi\big(\tau|u|, \eta|v|, \nu|w|\big) \tag{2.35}$$

and

$$\big\|J(|u| \wedge |v| \wedge |w|) - ((J(|u|) \wedge |v| \wedge |w|) \vee (|u| \wedge J(|v|) \wedge |w|) \vee (|u| \wedge |v| \wedge J(|w|)))\big\| \leq \theta\left((\|u\|^r \vee \|v\|^r) \wedge \|w\|^r\right) \tag{2.36}$$

for all $u, v, w \in L$ and $\tau, \eta, \nu \in [1, \infty)$, $\theta > 0$, and for some $r \in [0, 1)$. Suppose that $\varphi : L^3 \to [0, \infty)$ be a function satisfies in

$$\varphi(|u|, |v|, |w|) \leq (\tau\eta\nu)^{\frac{\alpha}{3}} \varphi\left(\frac{|u|}{\tau}, \frac{|v|}{\eta}, \frac{|w|}{\nu}\right), \tag{2.37}$$

for all $u, v, w \in L$ and for which there is number $\alpha \in [0, \frac{1}{3})$. Then there is an unique isotone cone-related lattice derivation $H : L \to L$ which satisfies in the following properties and inequality

$$\|H(|u|) - J(|u|)\| \leq \frac{\tau^\alpha}{\tau - \tau^\alpha}\varphi(|u|, |u|, |u|), \tag{2.38}$$

for all $u \in L$ and $\tau \in [1, \infty)$.

Proof . It has been shown the mapping $J : L \to L$ satisfies in cone-related lattice homomorphism. i.e.

$$J(|u| \vee |v| \vee |w|) = J(|u|) \vee J(|v|) \vee J(|w|),$$

for any $u, v, w \in L$. Also, it have proven that the limit

$$\lim_{n\to\infty} \frac{J(\tau^n|u|)}{\tau^n} = H(|u|) \tag{2.39}$$

exists, for every $u \in L$, furthermore, the inequality (2.38) is hold. We show that J is cone-related lattice derivation. Replacing u, v, w by $\tau^n u$ and $\tau^n v$ and $\tau^n w$, respectively in (2.36). So we obtain

$$\begin{aligned}\big\|J(|\tau^n u| \wedge |\tau^n v| \wedge |\tau^n w|) - ((J(|\tau^n u|) \wedge |\tau^n v| \wedge |\tau^n w|) \vee (|\tau^n u| \wedge J(|\tau^n v|) \wedge |\tau^n w|) \\ \vee (|\tau^n u| \wedge |\tau^n v| \wedge J(|\tau^n w|)))\big\| &\leq \theta\left((\|\tau^n u\|^r \vee \|\tau^n v\|^r) \wedge \|\tau^n w\|^r\right) \\ &= \theta\tau^{nr}\left((\|u\|^r \vee \|v\|^r) \wedge \|w\|^r\right),\end{aligned} \tag{2.40}$$

for all $u, v, w \in L$ and $\tau \geq 1$. With dividing both sides in above inequality on τ^n, we have

$$\left\|\frac{1}{\tau^n}J(\tau^n(|u| \wedge |v| \wedge |w|)) - ((\frac{1}{\tau^n}J(|\tau^n u|) \wedge |v| \wedge |w|) \vee (|u| \wedge \frac{1}{\tau^n}J(|\tau^n v|) \wedge |w|) \vee (|u| \wedge |v| \wedge \frac{1}{\tau^n}J(|\tau^n w|)))\right\| \leq \theta\tau^{n(r-1)}\left((\|u\|^r \vee \|v\|^r) \wedge \|w\|^r\right) \tag{2.41}$$

for all $u, v, w \in L$ and $\theta > 0$ for some $r \in [0, 1)$. As $n \to \infty$ by (2.39), we have

$$\begin{aligned}\big\|J(|u| \wedge |v| \wedge |w|) - ((J(|u|) \wedge |v| \wedge |w|) \vee (|u| \wedge J(|v|) \wedge |w|) \\ \vee(|u| \wedge |v| \wedge J(|w|)))\big\| &\leq \lim_{n\to\infty} \theta\tau^{n(r-1)}\left((\|u\|^r \vee \|v\|^r) \wedge \|w\|^r\right) \\ &\to 0.\end{aligned} \tag{2.42}$$

Therefore

$$J(|u| \wedge |v| \wedge |w|) = (J(|u|) \wedge |v| \wedge |w|) \vee (|u| \wedge J(|v|) \wedge |w|) \vee (|u| \wedge |v| \wedge J(|w|))$$

for every $u, v, w \in L$. $\square$

Theorem 2.7. *Let U be a Banach lattice and $\Lambda : U \to U$ be a cone-related functional for which there are numbers $\theta > 0$, $\delta \geq 0$ and $r < 2$ such that*

$$\|\Lambda(\tau|u| \vee \eta|v|) - (\tau^2\Lambda(|u|) \vee \eta^2\Lambda(|v|))\| \leq \delta + \theta(\|u\|^r + \|v\|^r) \tag{2.43}$$

and

$$\|\Lambda(\tau|u| \wedge \eta|v|) - ((\tau^2\Lambda(|u|) \wedge \eta^3|v|) \vee (\tau^3|u| \wedge \eta^2\Lambda(|v|)))\| \leq \theta(\|u\|^r \vee \|v\|^r) \tag{2.44}$$

for all $u, v \in U$ and $\tau, \eta > 1$. Then there exists a unique isotone cone related lattice derivation $\Pi : U \to U$, which satisfies properties and inequality.

$$\|\Pi(|u|) - \Lambda(|u|)\| \leq \frac{1}{\tau^2 - 1}\delta + \frac{2\theta}{\tau^2 - \tau^r}\|u\|^r \tag{2.45}$$

and

$$\Pi(|\tau u|) = \tau^2\Pi(|u|) \tag{2.46}$$

for all $u \in U$

Proof . It is easy to prove that the sequence $\tau^{-2n}\Lambda(\tau^n|u|)$ is a Cauchy sequence so, it's convergent to $\Pi(|u|) \in U$. Also, we can show that

$$\Pi(|u| \vee |v|) = \Pi(|u|) \vee \Pi(|v|) \tag{2.47}$$

for all $u, v \in U$. Moreover, we can see that (2.45) and (2.46) are satisfied. Next, by substituting τ and η with τ^n in (2.44) so we obtain

$$\|\Lambda(\tau^n|u| \wedge \tau^n|v|) - ((\tau^{2n}\Lambda(|u|) \wedge \tau^{3n}|v|) \vee (\tau^{3n}|u| \wedge \tau^{2n}\Lambda(|v|)))\| \leq \theta(\|u\|^r \vee \|v\|^r). \tag{2.48}$$

Replacing u with $\tau^n u$ and v with $\tau^n v$, we get

$$\|\Lambda(\tau^{2n}(|u| \wedge |v|)) - ((\tau^{2n}\Lambda(\tau^n|u|) \wedge \tau^{4n}|v|) \vee (\tau^{4n}|u| \wedge \tau^{2n}\Lambda(|\tau^n v|)))\| \leq \theta\tau^{nr}(\|u\|^r \vee \|v\|^r). \tag{2.49}$$

Thus

$$\left\|\frac{\Lambda(\tau^{2n}(|u| \wedge |v|))}{\tau^{4n}} - ((\frac{\Lambda(\tau^n|u|)}{\tau^{2n}}) \wedge |v|) \vee (|u| \wedge \frac{\Lambda(|\tau^n v|}{\tau^{2n}}))\right\| \leq \theta\frac{\tau^{nr}}{\tau^{4n}}(\|u\|^r \vee \|v\|^r), \tag{2.50}$$

by letting $n \to \infty$ and using the fact that lattice operators are continuous, we have

$$\begin{aligned} \|\Pi(|u| \wedge |v|) - ((\Pi(|u|) \vee |v|) \vee (|u| \wedge \Pi(|v|))\| &\leq \lim_{n\to\infty} \theta\tau^{n(r-4)}(\|u\|^r \vee \|v\|^r) \\ &= 0. \end{aligned} \tag{2.51}$$

Therefore

$$\Pi(|u| \wedge |v|) = (\Pi(|u|) \vee |v|) \vee (|u| \wedge \Pi(|v|) \tag{2.52}$$

for all $u, v \in U$. We come to the conclusion that Π is isotone cone-related lattice derivation by (2.47) and (2.52). $\square$

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