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Weighted approximation using neural network in terms of fractional modulus of smoothness of fractional derivative

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Abstract

We introduce direct theorems for universal weighted approximation. This approximation in terms of weighted Ditzain-Totik modulus of smoothness for the fractional derivative of functions in L_p (quasi-normed spaces).

Keywords: Ditzain-Totik modulus, quasi-norm, weighted approximation

1. Introduction

Here at the beginning of this work we would like to know the space $W_p^k[a,b]$ Sobolev space, where $W_p^k[a,b]$ is the set of all functions from $L_p[a,b]$ and $f^{(k)} \in L_p[a,b]$ where $L_p[a,b]$ the space of all measurable functions and we can define the norm of $f \in L_p[a,b]$ as follows, $\|f\|_p = \left(\int_a^b |f|^p \, dx\right)^{1/p}$ and we can define the k-th symmetric difference of f is given by

$$\Delta_h^k(f,\ x\ [a,b]) = \Delta_h^k(f,\ x\ ) = \begin{cases} \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} f\left(x - \frac{rh}{2} + ih\right), & x \pm \frac{rh}{2} \in [a,b] \\ 0 & otherwhere \end{cases}$$

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Then the r-th usual modulus of smoothness of $f \in L_p[a,b]$ is defined by:

$$\omega_k(f,\delta,\ [a,b])_p = \sup_{0<|h|\leq\delta} \|\Delta_h^r(f,.)\|_{L_p[a,b]}\ ,\quad \delta \geq 0$$

and we can define Ditzain-Totik modulus of smoothness which defined for such an f as follows:

$$\omega_k^{\emptyset}(f,\delta,\ [a,b])_p = \sup_{0<|h|\leq\delta} \left\|\Delta_{h\emptyset(\cdot)}^r(f,.)\right\|_{L_p[a,b]} \quad \text{and} \quad \lim_{h\to 0}\omega_k^{\emptyset}(f,h)\ = 0.$$

In the applications the $\emptyset$ usually used $\emptyset(x) = (x(1-x))^{1/2}$ and $\emptyset(x) = \sqrt{x}(1-x)$ for $J = [0.1]$, $\emptyset(x) = (1-x^2)^{\frac{1}{2}}$ for $J = [-1,1]$, $\emptyset(x) = \sqrt{x}$ and $\emptyset(x) = (x(1+x))^{\frac{1}{2}}$and $\Phi(x) = x$ for $J = [0,\infty]$. [2] is the first outhouse studied neural network approximations and introduce neural network operators, continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of the sigmoidal and hyperbolic tangent types which resulted in [4]. Working in this paper, neural network approximations are introduced at the fractal level resulting in higher approximations. we include the left and right Caputo derivatives of the function under approximate fractional calculus and neural networks, all of which are necessary to expose our work. Applications It is presented at the end. feed-forward neural networks with a single hidden layer, are expressed mathematically as follows

$$N_n(x_1,\ldots,x_s) = \sum_{i=0}^{n} l_i\sigma(\sum_{k=1}^{s} a_{ik}x_i) + b_i$$

where $0 \leq i \leq n$, $a_i \in R^s$, $s \in N$ are connection weights, $l_i \in R$ is coefficients , $a_i, x,\ \sigma$ activation function of the network. the network can be found in [4, 5].

2. Auxiliary Results

Definition 2.1. *[5] Let $f \in W_p[a,b]$ and K-modulus smoothness of f at t is given by $\omega_k(f,h) = \sup_{0<|h|\leq\delta}\left\|\Delta_h^k\ \ f\right\|_p$, and $\lim_{h\to 0}\omega_k(f,h)\ = 0$. And the Ditzian – totik modulus of smoothness of f at t is given by $\omega_k^{\emptyset}(f,h) = \sup_{0<|h|\leq\delta}\left\|\Delta_{h\emptyset(\cdot)}^k\ \ f\right\|_p$ and $\lim_{h\to 0} omega_k^{\Phi}(f,h) = 0$.*

Definition 2.2. *[5, 6] Let $v \geq 0$, $n = \lceil v \rceil$ where $\lceil . \rceil$ (celling of number), $f \in W_p^n$ (where Sobolve space) and $f^{(n-1)} \in W_p$. we can define left Caputo fractional derivative as follow.*

$$D_{*a}^v\ f(x) = \frac{1}{\Gamma(n-v)}\int_a^x (x-t)^{n-v-1}\quad f^n(t)\ \ dt,\ \forall x \in\ [a,b]$$

where Γ is gamma function.

$$\Gamma(v) = \int_o^{\infty} e^{-t}\ t^{v-1}\ dt\ ,\ v > 0.$$

Note that $D_{*a}^v\ f\ \in L_p\ [a,b]$ and D_{*a}^v exist a.e on $[a,b]$. We set $D_{*a}^o\ f(\ x) = f(x)\,,\ \forall\ x\ \in\ [a,b]$.

Definition 2.3. *[3] Let $f \in W_p^m[a,b]$, $m = \lceil\propto\rceil$, $\propto > 0$, the right Caputo fractional derivative of order $\propto > 0$ is given as follows*

$$D_{b-}^v\ f(x) = \frac{(-1)^m}{\Gamma(m-\propto)}\int_x^b (\varepsilon-x)^{m-\propto-1}\quad f^m(\varepsilon)\ \ d\varepsilon,\ \forall x \in\ [a,b]\,,$$

we set $D_{b-}^o f(x) = f(x)\,,\forall x \in [a,b]$. Note that $D_{b-}^{\propto}\ f \in\ L_p\ [a,b]$ and $D_{b-}^{\propto}\ f$ exist a.e on $[a,b]$. We let that $D_{*x_0}^{\propto}\ f(x) = 0\ \ for x < x_0$ and $D_{x_0-}^{\propto}\ f(x) = 0$ for $x > x_0$ for all $x,\ x_0\ \in [a,b]$.

Remark 2.4. *Let* $f \in W_p^{n-1}[a,b]$, $f^n \in W_p[a,b], n = \lceil v \rceil$, $v > 0$, $v \notin N$, *then we have*

$$\|D_{*a}^{v} f(x)\|_{L_p[a,b]} \le c(p,k)\frac{\| f^n(x)\|_{L_p[a,b]}}{\Gamma(n-v+1)}(x-a)^{n-v}$$

thus, we observe that

$$\omega_k^{\emptyset}(D_{*a}^{v} f, \delta)_{L_p[a,b]} = \sup_{0<|h|\le\delta}\left\|\Delta_{h\emptyset}^{k} D_{*a}^{v} f(x)\right\|_{L_p[a,b]}$$
$$= \sup_{0<|h|\le\delta}\left\|\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}D_{*a}^{v}f\left(x-\frac{kh\emptyset}{2}+ih\emptyset\right)\right\|_{L_p[a,b]}$$

where $x \pm kh\emptyset \in [a,b]$

$$\begin{aligned}
\omega_k^{\emptyset}(D_{*a}^{v} f, \delta)_{L_p[a,b]} &\le c(p,k)\sup_{0<|h|\le\delta}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}\|D_{*a}^{v}f(x)\|_{L_p[a,b]}\\
&\le \|D_{*a}^{v}f(x)\|_{L_p[a,b]}\\
&\le \left\|\frac{1}{\lceil(n-v)}\int_a^b (b-t)^{n-v-1}f^n(x)\,dt\right\|_{L_p[a,b]}\\
&= \frac{\|f^{(n)}\|_{L_p[a,b]}}{\Gamma(n-v)}\frac{(b-t)^{n-v}}{n-v}\bigg|_a^b\\
&= \frac{\| f^n\|_{L_p[a,b]}}{(n-v)\,\Gamma(n-v)}\left\|(b-b)^{n-v}(b-a)^{n-v}\right\|_{L_p[a.b]}\\
&\le c(p,k)\frac{\|f^{(n)}\|_{L_p[a,b]}}{\Gamma(n-v+1)}(b-a)^{n-v}. \qquad (2.1)
\end{aligned}$$

Note that $(n-v)\,\Gamma(n-v) = \Gamma(n-v+1)$. *Similarly, let* $f \in W_p^{m-1}([a,b])$, $f^{(m)} \in W_p[a,b]$, $m = [\propto]$, $\propto > 0, \propto \notin N$, *then*

$$w_k^{\emptyset}\left(D_{b-}^{\propto} f, \delta\right) = \sup_{o<|h|\le\delta}\|\Delta_{h\emptyset}^{k}\ D_{b-}^{\propto} f(x)\|_{L_p[a,b]}$$
$$= \sup_{o<|h|\le\delta}\left\|\sum_{i=o}^{k}\binom{k}{i}(-1)^{k-i}\ D_{b-}^{\propto} f(x-\frac{kh\emptyset}{2}+ih\emptyset\right\|_{L_p\ [a,b]}.$$

where $x \mp \dfrac{kh\emptyset}{2} \in [a,b]$. *Then*

$$\begin{aligned}
\omega_k^{\emptyset}\left(D_{b-}^{\propto} f, \delta\right)_{L_P\ [a,b]} &\le c(p,k)\sup_{o<|h|\le\delta}\sum_{i=o}^{k}\binom{k}{i}(-1)^{k-i}\left\|D_{b-}^{\propto} f(x)\right\|_{L_p\ [a,b]}\\
&\le \left\|D_{b-}^{\propto} f(x)\right\|_{L_{p[a,b]}}\\
&\le \left\|\frac{1}{\Gamma(m-\propto)}\int_a^b (\mathcal{E}-t)^{m-\propto-1}\ f^{(m)}(\mathcal{E})\,d\mathcal{E}\right\|_{L_{p[a,b]}}\\
&\le \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)}\int_a^b (\mathcal{E}-t)^{m-\propto-1}\ d\mathcal{E}\\
&\le \frac{\| f^m \|_{L_{p[a,b]}}}{\Gamma(m-\propto)}\frac{(\mathcal{E}-t)^{(m-\propto-1)+1}}{(m-\propto-1)+1}\ \Big|_a^b.
\end{aligned}$$

Since $x \in [a,b]$, we have

$$\omega_k^{\emptyset}\left(D_{b-}^{\propto} f, \delta\right)_{L_p\ [a,b]} \leq c\,(p,k)\ \frac{\|f^m\|_{L_{p[a,b]}}}{\Gamma\,(m-\propto)}\ \frac{(b-a)^{m-\propto}-(a-a)^{m-\propto}}{m-\propto}$$

$$\omega_k^{\emptyset}\left(D_{b-}^{\propto} f, \delta\right)_{L_{p[a,b]}} \leq c\ (p,k)\,\frac{\|f^m\|_{L_{p[a,b]}}}{(m-\propto)\,\Gamma\,(m-\propto)}\ (b-a)^{m-\propto}. \tag{2.2}$$

Then from (2.1) and (2.2), we find that

$$\varsigma_1\ (\delta) = \omega_k^{\emptyset}\ (D_{*xo}^{\propto} f, \delta) \leq c\ (p,k)\ \frac{\|f^m\|_{L_{p[a,b]}}}{\Gamma\,(m-\propto)}(b-a)^{m-a} \tag{2.3}$$

and

$$\varsigma_2\ (\delta) = \omega_k^{\emptyset}\left(D_{xo-}^{\propto} f, \delta\right) \leq c\ (p,k)\ \frac{\|f^m\|_{L_{p[a,b]}}}{\Gamma\,(m-\propto)}(b-a)^{m-a}.$$

Then $D_{*x0}^{\propto} f \in L_p\ [x_0,b]$ and $D_{x_0-}^{\propto} f \in L_p\ ([a,x_0])$. Clearly, we have $\varsigma_1\ (\delta) \to 0$ and $\varsigma_2\,(\delta) \to 0$ as $\delta \to 0$.

Definition 2.5. *We define here the sigmoidal function of logarithmic type*

$$\varsigma\,(x) = \frac{1}{1+e^{-x}},\ x \in R$$

and

$$\lim_{x\to+\infty} \varsigma\,(x)\ = 1\ \textit{and}\ \lim_{x\to-\infty} \varsigma\,(x)\ = 0$$

This function plays the role of activation function in the hidden layer of networks we consider that

$$\emptyset\,(x) = \frac{1}{2}\ (\varsigma\,(x+1) - \varsigma\,(x-1)\,,\quad x \in R.$$

We note the following properties

(i) $\emptyset\,(x) > 0, \forall x \in R$,

(ii) $\displaystyle\sum_{k=-\infty}^{\infty} \emptyset\ (x-k) = 1,\ \forall x \in R$,

(iii) $\displaystyle\sum_{k=-\infty}^{\infty} \emptyset\ (nx-k) = 1,\ \forall x \in R;\ n \in N$,

(iv) $\displaystyle\int_{-\infty}^{\infty} \emptyset\ (x)\,dx = 1$,

(v) $\emptyset$ is density function,

(vi) $\emptyset$ is even i.e $\emptyset\,(-x) = \emptyset\,(x)\,,\quad x \geq 0$. In [5] we can find

$$\emptyset\,(x) = \frac{(e^2-1)}{2e}\ \frac{e^{-x}}{(1+e^{-x-1})\,(1+e^{-x+1})} = \left(\frac{e^2-1}{2e^2}\right)\frac{1}{(1+e^{x-1})(1+e^{-x-1}}$$

(vii) In [5] we can find $\emptyset$ is decreasing on R^{+} (positive real numbers) and increasing in R^{-}(negative real numbers).

(viii) In [5] we can find

$$\sum_{|nx-k|=-\infty}^{\infty} \emptyset(nx-k) < (\frac{e^2-1}{2})e^{-n^{(1-B)}} = 3.1992e^{-n^{(1-\beta)}}$$

Denote $\lceil . \rceil$ the ceiling of number and $\lfloor . \rfloor$ the integral part of number. t Let $x \in [a,b] \subset R, \;\; n \in N$ *such that* $\lceil na \rceil \leq \lfloor nb \rfloor$.

(ix) In [5] we can find integral part

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \emptyset(nx-k)} < \frac{1}{\emptyset(1)} 5.25, \quad \forall x \in [a,b] \,.$$

(x) In [5] we see that $\lim\limits_{x \longrightarrow \infty^{+}} \sum\limits_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \emptyset\,(nx-k) \neq 1,$ for at least some $x \in [a,b]$, let $f \in L_p\,[a,b]\,,\; n \in N$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

Here we study the fractional level the point wise and uniform convergence of $G_n\,(f,x)$ to $f\,(x) \in L_p$ in termes of $\omega_k^{\emptyset}\,(f,x)$ we define the positive linear neural network operator.

$$G_n\,(f,x) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^{k-i} f(x' - \frac{rh\emptyset}{2} + ih\emptyset)\Phi(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)}, \quad x, x' \in [a,b]\,.$$

where

$$G_n^*\,(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \sum_{i=0}^{k} \begin{pmatrix} k \\ i \end{pmatrix} (-1)^{k-i} f(x' - \frac{kh\emptyset}{2} + ih\emptyset \;\Phi(nx-k).$$

Thus

$$G_n\,(f,x) - f\,(x) = \frac{G_n^*\,(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)} - f\,(x) = \frac{G_n^*\,(f,x) - f\,(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)}$$

Note that $\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)} = \frac{1}{\Phi(1)} = 5.25$

$$\begin{aligned}\|G_n\,(f,x) - f\,(x)\,\|_{L_{p[a,b]}} &= \left\| \frac{G_n^*\,(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)} - f\,(x) \right\|_{L_{p[a,b]}} \\ &= \left\| \frac{G_n^*\,(f,x) - f\,(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi\,(nx-k)} \right\|_{L_{p[a,b]}}.\end{aligned}$$

By (ix) we get

$$\left\|G_n(f,x)-f(x)\right\|_{L_{p[a,b]}}=\frac{1}{\Phi(1)}\left\|G_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\right\|_{L_{p[a,b]}}.$$

By (ix) too we obtain

$$=5.25\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i}f\left(x-\frac{rh\emptyset}{2}+ih\emptyset\right)\Phi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\|_{L_{p[a,b]}}\right\|$$

$$=5.25\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\sum_{i=o}^{k}\binom{k}{i}(-1)^{k-i}f\left(x-\frac{kh\emptyset}{2}+ih\emptyset\right)\Phi(nx-k)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)\|_{L_{p[a,b]}}$$

$$\leq c\ (p,k)\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}f\left(x-\frac{kh\emptyset}{2}+ih\emptyset\right)\Phi(nx-k)\|_{L_{p[a,b]}}. \tag{2.4}$$

We will estimate the right hand side of (2.4) involving the right and left Caputo fractional derivatives of f.

3. Main Results

Theorem 3.1. *Let* $\propto>0$, $N=\lceil\propto\rceil$, $\propto\notin N$, $f\in W_p[a,b]$ *with* $f^{(N)}\in Lp[a,b]$, $0<B<1$, $x\in[a,b]$, $n\in N$ *then*

(i)

$$\left\|G_n(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n((.,x)^j)(x)-f(x)\right\|_{Lp[a,b]}$$
$$\leq\frac{10.4\ c(p)}{\Gamma(\propto+1)}[\frac{1}{n^B}(w_k^{\emptyset}(D_{x-}^{\propto}f,\frac{1}{n^B}\ )_{Lp[a,x]}+[w_k^{\emptyset}(D_{*x}^{\propto}f,\frac{1}{n^B}\ )_{Lp[x,b]})$$
$$+3.19\ e^{-n(1-B)}(\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}+\|D_{*x}^{\propto}f\|_{Lp[x,b]}(b-x)^{\propto}].$$

When $\propto>1$ *note here extremely high rate of convergence at* $n^{-(\propto+1)B}$. *If* $f^{(j)}(x)=0$ *for* $j=1,\ldots,N-1$, *we have*

$$\|G_n(f,x)-f(x)\|_{Lp[a,b]}\leq\frac{10.4\ c(p)}{\Gamma(\propto+1)}[w_k^{\emptyset}(D_{x-}^{\propto}f,\frac{1}{n^B}\ )_{Lp[a,x]}+w_k^{\emptyset}(D_{*x}^{\propto}f,\frac{1}{n^B}\ )_{Lp[x,b]}$$
$$+3.19\ e^{-n(1-B)}(\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}+\|D_{*x}^{\propto}f\|_{Lp[x,b]}(b-x)^{\propto}].$$

When $\propto>1$ *note here extremely high rate of convergence at* $n^{-(\propto+1)B}$.

(ii)

$$\|G_n(f,x)-f(x)\|_{Lp[a,b]} \leq \frac{10.4\ c(p)}{\Gamma(\propto+1)} \sum_{j=1}^{N-1} \frac{\|f^{(j)}\|}{j!} \left\{ \frac{1}{n^B} + (b-a)^j 3.19\ e^{-n(1-B)} \right\}$$
$$+ (\frac{10.4c(p)}{\Gamma(\propto+1)} [(w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B})_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]})$$
$$+ 3.19\ e^{-n(1-B)} (\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto})].$$

Proof . Let $x, x' \in [a,b]$ we have $D_{x-}^{\propto} f = D_{*x}^{\propto} f = 0$, we get by the left Caputo fractional weighted Taylor formula that

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^j$$
$$+ \frac{1}{\Gamma(\propto)} \int_x^{x+\frac{kh\emptyset}{2}++ih\emptyset} (\left(x' + \frac{kh\emptyset}{2} + \ ih\emptyset) - j\right)^{\propto-1} . (D_{*x}^{\propto} f(j-x)\, dj$$

where $x \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq b$. Using right Caputo fractional weighted Taylor formal, we get

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) = \sum_{j=0}^{N-1} \frac{f^j(x)}{j!} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right)^j$$
$$+ \frac{1}{\Gamma(\propto)} \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j - x' + \frac{kh\emptyset}{2} + ih\emptyset\right) \left(D_{x-}^{\propto} f(j+x)\, dj\right)^{\propto-1},$$

where $a \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq x$.

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) \Phi(nx-k)$$
$$= \sum_{j=0}^{N-1} \frac{f^j(x)}{j!} \Phi(nx-k) + \frac{\Phi(nx-k)}{\Gamma(\propto)} \int_x^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset - j\right)^{\propto-1} (D_{*x}^{\propto} f(j+x))dj.$$

where for all $x \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq b$ if and only if $\lceil nx \rceil \leq k \leq \lfloor nb \rfloor$ and

$$f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) \Phi(nx-k)$$
$$= \sum_{j=0}^{N-1} \frac{f^j(x)}{j!} \Phi(nx-k) (x' + \frac{kh\emptyset}{2} + ih\emptyset - x)^j$$
$$+ \frac{\Phi(nx-k)}{\Gamma(\propto)} \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j - (x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1} \left(D_{x-}^{\propto} f(j+x)\right) dj,$$

where $a \leq x' + \frac{kh\emptyset}{2} + ih\emptyset \leq x$ if and only if $\lceil na \rceil \leq k \leq \lfloor nx \rfloor$ we have $\lceil nx \rceil \leq \lfloor nx \rfloor + 1$, therefore

$$\sum_{k=\lceil nx \rceil+1}^{\lfloor nb \rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Phi(nx-k)$$
$$= \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k)\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{j} \tag{3.1}$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) - j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\, dj.$$

and

$$\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Phi(nx-k)$$
$$= \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)\left((x' + \frac{kh\emptyset}{2} + ih\Phi - x\right)^{j} \tag{3.2}$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(J - (x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\, dj.$$

By (3.1) and (3.2), we get

$$G_n^*(f,x) = \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right)\Phi(nx-k)$$

and

$$\frac{G_n^*(f,x)}{\sum_{i=0}^{k}\binom{k}{i}} = \sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)\left((x' + \frac{kh\emptyset}{2} + ih\Phi - x\right)^{j}$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(J - (x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\, dj$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) - j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\, dj$$

$$G_n^*(f,x) = c(p)\,[\sum_{j=0}^{N-1} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k))\left((x' + \frac{kh\emptyset}{2} + ih\Phi - x\right)^{j}$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(J - (x' + \frac{kh\emptyset}{2} + ih\emptyset\right)^{\propto-1} D_{x-}^{\propto} f(j+x)\, dj$$
$$+ \frac{1}{\Gamma(\propto)} \sum_{k=\lfloor nx \rfloor+1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(\left(x' + \frac{kh\emptyset}{2} + ih\emptyset\right) - j\right)^{\propto-1} D_{*x}^{\propto} f(j+x)\, dj]$$

Where assume $\sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i}$ estimation equal $c(p)$ and assume

$$\emptyset_n(x) = \frac{1}{\Gamma(\propto)} \Big(\sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j - (x' + \frac{kh\emptyset}{2} + ih\emptyset \right)^{\propto -1} D_{x-}^{\propto} f(j+x)\, dj$$

$$+ \sum_{k=\lceil nx \rceil +1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{x'+\frac{kh\emptyset}{2}+ih\emptyset} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset) - j \right)^{\propto -1} D_{*x}^{\propto} f(j-x)\, dj \Big).$$

Then

$$G_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k) \ = c(p) \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!}\ G_n^*(.-x)^j(x) + \Phi_n(x) \tag{3.3}$$

We put

$$\Phi_{1n}(x) = \frac{1}{\Gamma(\propto)} \sum_{k=\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k) \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} \left(j - \left(x' + \frac{kh\emptyset}{2} + ih\emptyset \right) \right) D_{x-}^{\propto} f(j-x)\, dj$$

$$\Phi_{2n}(x) = \frac{1}{\Gamma(\propto)} \sum_{k=\lceil nx \rceil +1}^{\lfloor nb \rfloor} \Phi(nx-k) \int_{x}^{x'+\frac{kh\Phi}{2}+ih\Phi} \left(x' + \frac{kh\emptyset}{2} + ih\emptyset - j \right)^{\propto -1} D_{*x}^{\propto} f(j+x))dj$$

We mean $\Phi_n(x) = \Phi_{1n}(x) + \Phi_{2n}(x)$. We assume that $b-a > \frac{1}{n^B,}$ $\quad 0 < B < 1$, Which always large enough $n \in N$ that is when $n > \left[(b-a)^{\frac{-1}{B}} \right]$ it is for $k = \lceil na \rceil, \ \ldots, \ \lfloor nx \rfloor$.

We consider

$$y_{1k} = \left| \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j - (x' + \frac{kh\emptyset}{2} + ih\emptyset)^{\propto -1} D_{x-}^{\propto} f(j+x)\, dj \right|,$$

$$y_{1k} = \left| \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j - (x' + \frac{kh\emptyset}{2} + ih\emptyset)^{\propto -1} D_{x-}^{\propto} f(j)\, dj \right|,$$

$$y_{1k} \le \int_{x'+\frac{kh\emptyset}{2}+ih\emptyset}^{x} (j - (x' + \frac{kh\emptyset}{2} + ih\emptyset)^{\propto -1} \left| D_{x-}^{\propto} f \right| dj,$$

$y_{1k} \le \left| D_{x-}^{\propto} f \right| \frac{(x-\ a)^{\propto}}{\propto}$, where $a \le x' + \frac{kh\emptyset}{2} + ih\emptyset \le x$ if and only if $\lceil na \rceil \le k \le \lfloor nx \rfloor$ we have $\lceil nx \rceil \le \lfloor nx \rfloor + 1$.

$$\|y_{1k}\|_{Lp[a,b]} \le \left\| D_{x-}^{\propto} f \right\|_{Lp[a,x].}$$

Also, we have in case of $\left| x' + \frac{kh\emptyset}{2} + ih\emptyset - x \right| \le \frac{1}{n^B},\ k\ = \lceil na \rceil, \ldots, \lfloor nx \rfloor$.

$$\|y_{1k}\|_{Lp[a,x]} \le \frac{w_k^{\emptyset}(D_{x-}^{\propto} f, \frac{1}{n^B}\ )_{Lp[a,x]}}{\propto n^{\propto} B},$$

$$\|\varphi_{1n}(x)\|_{Lp[a,x]} = \left\| \frac{1}{\Gamma(\propto)} \sum_{\lceil na \rceil}^{\lfloor nx \rfloor} \Phi(nx-k)\, y_{1k} \right\|_{Lp[a,x]}.$$

$$\|\varphi_{1n}(x)\|_{Lp[a,x]} = \frac{1}{\Gamma(\propto)}\left\| \sum_{\substack{k=\lceil na\rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right|\le\frac{1}{nB}}}^{\lfloor nx\rfloor} \Phi(nx-k)\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{\propto n^{\propto}B} + \sum_{\substack{k=\lceil na\rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right|>\frac{1}{nB}}}^{\lfloor nx\rfloor} \Phi(nx-k)\|D_{*x}^{\propto}f\|_{Lp[a,x]}\frac{(x-a)^{\propto}}{\propto}\right\|_{Lp[a,x]}.$$

Since Φ is increasing on bounded interval, we have

$$\|\varphi_{1n}(x)\|_{Lp[a,x]} \le \frac{c(p)}{\propto\Gamma(\propto)}\{(\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto}B} + \sum_{\substack{k=\lceil na\rceil \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right|>\frac{1}{nB}}}^{\lfloor nx\rfloor} \Phi(nx-k))\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}$$

$$\le \frac{c(p)}{\Gamma(\propto+1)}\left\{\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto}B} + \sum_{\substack{k=-\infty \\ \left|x'+\frac{kh\Phi}{2}+ih\Phi-x\right|>\frac{1}{nB}}}^{\infty} \Phi(nx-k))\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}\right\}$$

$$\le \frac{c(p)}{\Gamma(\propto+1)}\left\{\frac{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}}{n^{\propto}B} + 3,199\ e^{-n(1-B)}\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}(x-a)^{\propto}\right\},$$

where $k=\lfloor nx\rfloor+1,\ldots,\lfloor nb\rfloor$. We consider, when $\left|x'+\frac{kh\emptyset}{2}+ih\emptyset-x\right|>\frac{1}{n^B}$. Then

$$y_{2k} = \left|\int_x^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1}D_{*x}^{\propto}f(j+x)\,dj\right|,$$

$$y_{2k} \le \int_x^{x'+\frac{kh\emptyset}{2}+ih\emptyset}\left(\left(x'+\frac{kh\emptyset}{2}+ih\emptyset\right)-j\right)^{\propto-1}\left|D_{*x}^{\propto}f(j+x)\right|dj,$$

$$y_{2k} \le \left|D_{*x}^{\propto}f\right|\frac{(b-x)^{\propto}}{\propto},$$

and

$$\|y_{2k}\|_{Lp[a,x]} \le \|D_{*x}^{\propto}f\|_{Lp[a,x].}\frac{(b-x)^{\propto}}{\propto}.$$

Also, we have in case of $\left|x' + \frac{kh\emptyset}{2} + ih\emptyset - x\right| \le \frac{1}{n^B}, \ k \ = \lceil na \rceil \ , \ldots, \ \lfloor nx \rfloor$.

$$y_{2k} = \left| \int_{x''}^{x' + \ kh\emptyset/2 + ih\emptyset} ((x + \frac{kh\emptyset}{2} + ih\emptyset - j)^{\propto -1} D_{*x}^{\propto} f\,(j + x)\, dj \right|$$

where $x'' < x$,

$$\begin{aligned} \|y_{2k}\|_{Lp[x,b]} &= \int_{x''}^{x' + \ kh\emptyset/2 + ih\emptyset} ((x + \frac{kh\emptyset}{2} + ih\emptyset - j)^{\propto -1} \|D_{*x}^{\propto} f\,(x)\|_{Lp[x,b]} \\ &\le \frac{w_k^{\emptyset}(D_{*x}^{\propto} f, \frac{1}{n^B}\ )_{Lp[x,b]}}{\propto n^{\propto} B}, \qquad \delta \le \frac{1}{nB} \end{aligned}$$

As the previous cases we can find

$$\|\varphi_{2n}\,(x)\|_{Lp[x,b]} = \left\| \frac{1}{\Gamma\,(\propto)} \sum_{\lceil na \rceil}^{\lfloor nx \rfloor} \Phi\,(nx - k)\, y_{2k} \right\|_{Lp[x,b]}$$

$$\|\varphi_{2n}\,(x)\|_{Lp[x,b]} = \frac{1}{\Gamma\,(\propto)} \left\| \sum_{\substack{k\, =\, \lceil nx \rceil + 1 \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| \le \frac{1}{nB}}}^{\lfloor nb \rfloor} \Phi\,(nx - k)\, y_{2k} + \sum_{\substack{k\, =\, \lceil nx \rceil + 1 \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| > \frac{1}{nB}}}^{\lfloor nb \rfloor} \Phi\,(nx - k)\, y_{2k} \right\|_{Lp[x,b]} .$$

Since Φ is increasing on bounded interval, we get

$$\begin{aligned} &\le \frac{c\,(p)}{\propto \Gamma\,(\propto)} \{(\frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\ \right)_{Lp[x,b]}}{n^{\propto} B} + 3.199 e^{-2n^B} \|D_{*x}^{\propto}\ f\|\ _{Lp[x,b]} (b - x)^{\propto} \\ &\le \frac{c\,(p)}{\propto \Gamma\,(\propto)} \left\{ \frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\ \right)_{Lp[x,b]}}{n^{\propto} B} + \sum_{\substack{k\, =\, -\infty \\ \left|x' + \frac{kh\Phi}{2} + ih\Phi - x\right| > \frac{1}{nB}}}^{\infty} \Phi\,(nx - k) \left\| D_{x-}^{\propto}\ f \right\|\ _{Lp[x,b]} (b - x)^{\propto} \right\} \\ &\le \frac{c\,(p)}{\Gamma\,(\propto + 1)} \left\{ \frac{w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\ \right)_{Lp[a,x]}}{n^{\propto} B} + 3,199\ e^{-n(1-B)} \|D_{*x}^{\propto}\ f\|\ _{Lp[a,x]} (b - x)^{\propto} \right\} \end{aligned}$$

$$\|\varphi_n(x)\|_{Lp[a,x]} \le c(p)(\|\varphi_{1n}(x)\|_{Lp[a,x]} + \|\varphi_{2n}(x)\|_{Lp[x,b]})$$

$$\le \frac{2c(p)}{\Gamma(\propto+1)} \left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} \right.$$

$$\left. + 3.19\ e^{-n(1-B)} \left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto} \right) \right\}.$$

In [4], we have

$$\left| G_n^*((.-x)^j)(x) \right| \le \frac{1}{n^{B_j}} + (b-a)^j (3.1992)\, e^{-n^{(1-B)}} \tag{3.4}$$

for $j = 1, \ldots, N$, for all $x \in [a,b]$. Using (2.3) in (3.4) to get

$$\left\| G_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \Phi(nx-k)\right) \right\|_{Lp[a,b]}$$

$$\le c(p) \left[\sum_{j=1}^{N-1} \frac{\left|f^{(j)}(x)\right|}{j!}) \left[\frac{1}{n^{Bj}} + (b-a)^j (3.1992)\, e^{-n^{(1-B)}} \right] \right.$$

$$+ \frac{2c(p)}{\Gamma(\propto+1)} \left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} \right. \tag{3.5}$$

$$\left.\left. + 3.19\ e^{-n(1-B)} \left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto} \right) \right\} \right] = A_n, \quad \forall x \in [a,b].$$

$$\|G_n(f,x) - f(x)\|_{Lp[a,b]} \le (5.25)\, A_n(x),\ \forall x \in [a,b] \tag{3.6}$$

Now let as estimate

$$\|A_n\|_{Lp[a,b]} \le c(p)[(\sum_{J=1}^{N-1} \frac{\left\|f^{(j)}\right\| Lp_{[a,b]}}{j!}) \left[\frac{1}{n^{Bj}} + (b-a)^j (3.1992)\, e^{-n^{(1-B)}} \right]$$

$$+ 2c(p) \left\{ \frac{w_k^{\emptyset}\left(D_{x-}^{\propto} f, \frac{1}{n^B}\right)_{Lp[a,x]} + w_k^{\emptyset}\left(D_{*x}^{\propto} f, \frac{1}{n^B}\right)_{Lp[x,b]}}{n^{\propto} B} \right. \tag{3.7}$$

$$\left. + 3.19\ e^{-n(1-B)} \left(\left\|D_{x-}^{\propto} f\right\|_{Lp[a,x]} (x-a)^{\propto} + \|D_{*x}^{\propto} f\|_{Lp[x,b]} (b-x)^{\propto} \right) \right\}] = Bn.$$

Hence it holds

$$\|G_n(f,x) - f(x)\|_{Lp[a,b]} \le 5.25\ B_n.$$

This complete the prove of (iii) and $j=0$ we get (ii). We finally note that

$$G_n(f,x)-\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n((.-x)^j(x)-f(x)$$
$$=\frac{G_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}-\frac{\left(\sum_{J=1}^{N-1}\frac{f^{(j)}}{j!}G_n^*((.-x)^j(x)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}-f(x)$$
$$=\frac{G_n^*(f,x)-\left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*((.-x)^j(x)\right)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)f(x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)}$$
$$=\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)]}[G_n^*(f,x)-\left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*((.-x)^j(x)\right)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)f(x)].$$

Then we have

$$\left\|G_n(f,x)-\sum_{J=1}^{N-1}\frac{f^{(j)}}{j!}G_n((.-x)^j(x)-f(x)\right\|_{Lp[a,b]}$$
$$\leq 5.25\left\|G_n^*(f,x)-\left(\sum_{J=1}^{N-1}\frac{f^{(j)}(x)}{j!}G_n^*((.-x)^j(x)\right)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\Phi(nx-k)f(x)\right\|_{Lp[a,b]},$$

and by(3.5),(3.6),(3.7), for all $x\in[a,b]$ satisfy proof (i). □

4. Application

Corollary 4.1. *Let* $\propto>0,\ 0<\ B<1,\ f\in W_p^1[a,b],\ n\in N,$ *then*

$$\|G_nf-f\|_{Lp[a,b]}\leq\frac{10.4}{\Gamma(\propto+1)}(\left\{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}+w_k^{\emptyset}\left(D_{*x}^{\propto}f,\frac{1}{n^B}\right)_{Lp[x,b]}\right\}$$
$$+3.199\ e^{-n(1-B)}\left((b-a)^{\propto}(\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}+\|D_{*x}^{\propto}f\|_{Lp[x,b]}\right)$$

Proof . Like the proof of Theorem 3.1 when $N=1$, then $\sum_{j=1}^{N-1}.=0$. In the same way this theorem is proved. □

Corollary 4.2. *Let* $\propto=\frac{1}{2},\ \ 0<\ B<1,\ f\in W_p^1[a,b],\ n\in N,$ *then*

$$\|G_nf-f\|_{Lp[a,b]}\leq\frac{10.4}{\Gamma\left(\frac{3}{2}\right)}\left\{w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}+w_k^{\emptyset}\left(D_{*x}^{\propto}f,\frac{1}{n^B}\right)_{Lp[x,b]}\right\}$$
$$+3.199\ e^{-n(1-B)}\left((b-a)^{\propto}(\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}+.\|D_{*x}^{\propto}f\|_{Lp[x,b]}\right).$$

We can write as follows when $\Gamma(3/2)=1/2\sqrt{\pi}$. So,

$$\|G_nf-f\|_{Lp[a,b]}\leq\frac{10.4}{\sqrt{\pi}}(\left\{\frac{1}{n^{B/2}}w_k^{\emptyset}\left(D_{x-}^{\propto}f,\frac{1}{n^B}\right)_{Lp[a,x]}+w_k^{\emptyset}\left(D_{*x}^{\propto}f,\frac{1}{n^B}\right)_{Lp[x,b]}\right\}$$
$$+3.199\ e^{-n(1-B)}\left(\sqrt{(b-a)}(\left\|D_{x-}^{\propto}f\right\|_{Lp[a,x]}+\|D_{*x}^{\propto}f\|_{Lp[x,b]}\right).$$

5. Conclusion

We estimate the degree of best neural approximation of functions in L_p (quasi-norm spaces) in terms of kth order weighted modulus of smoothness of fractional derivative for functions in L_p.

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