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Qualitative properties of solutions of fractional order boundary value problems

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Abstract

In this article, we discuss two boundary value problems for fractional-order differential equations. We show unique solutions exist and some data continuous dependence, with aim of proving some characteristics for these solutions of a coupled system of conjugate orders. These coupled systems are equivalent to coupled systems of second-order differential equations. Therefore, the analysis of the spectra of these problems is a consequence of that of second-order differential equations.

Keywords: Boundary value problem, Caputo fractional derivative, Data continuous dependence, Eigenvalues and Eigenfunctions.
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1. Introduction

The development of positive and multiple positive solutions of multi-point, two-point boundary value problems (BVPs) identified together integer and fractional order differential equations (FODE) [1] has gained higher concerned

On the other hand, in differential equation theories, the problems of eigenvalue are among the most active areas, and some authors have been concerned about the cases of eigenvalue of nonlinear fractional differential equations [1] and [3].

Recently,[21] and [17] have examined the problems of integral boundary conditions. The eigenvalues of a Fredholm integral equation system are determined in [2] in such a way that a constant sign solution is available to the system. Also, it will present explicit intervals for the eigenvalues.

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The eigenvalues of a Fredholm integral equation system are determined in [2] in such a way the system will have a solution of constant signs. Explicit intervals would be provided for values of eigenvalues. Applications to many well-known BVPs show the generalization of the obtained results.

The fact that it exists of multiple positive solutions to a combined system of FODE has recently been developed by Prasad and Krushna,satisfying certain two-point boundary conditions of Sturm-Liouville. These findings are also applied to the iterative fractional order systems BVPs. In mention papers [4], [5]-[8], [14]-[16] and [18], Coupled systems of differential and integral equations are studied.

On a cone, the Guo-Krasnosel'skii fixed point theorem is applied, Al-Hossain [19] defines the own value intervals of $\lambda_1, \lambda_2, ..., \lambda_n$ so that the iterative method of three point BVP for the nonlinear Liouville-Caputo fractional order does have solution which

Ying He [20], deals with fractional differential-integral conditions in a coupled scheme of singular p-Laplacian differential equations. Ying uses Schauder's fixed point theorem and the upper and lower solution processes to define an eigen value interval for the current state of positive solutions.

Henderson and Luca [10] discuss the development and multiplicity of positive solutions for a system of Riemann–Liouville fractional differential equations related to multipoint boundary conditions including fractional derivatives.

Jleli et al. [11] recognize a coupled system of nonlinear differential fractional equations within Dirichlet boundary conditions. Jleli et al. set up a Lyapunov-type inequality for the problem considered, using Perov's fixed point theorem.

Motivate by above results, we are interested in determining the eigenvalue intervals of λ for there are however solutions for the coupled systems of nonlinear Liouville-Caputo fractional order two points BVP.

$$\begin{aligned} D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) &= -\lambda\, \mathfrak{y}(\mathfrak{t}),\ \mathfrak{t} \in \mathfrak{J} = (0,1),\ \alpha \in (0,1],\ \lambda \in \mathbb{R}, \\ D^{1-\alpha}\mathfrak{y}(\mathfrak{t}) &= \lambda\, \mathfrak{x}(\mathfrak{t}),\ \mathfrak{t} \in \mathfrak{J}, \end{aligned} \tag{1.1}$$

as long as the two sets of boundary conditions are met

$$\mathfrak{x}(0) = 0, \quad \mathfrak{y}(0) = 0 \ \text{ and } \quad \mathfrak{x}(1) = 0, \tag{1.2}$$

$$\mathfrak{x}(0) = 0, \quad \mathfrak{y}(0) = 0 \ \text{ and } \quad \mathfrak{x}(\tau) = 0,\ \tau \in (0,1) \tag{1.3}$$

Here D^{α} is the Liouville-Caputo fractional-order derivative (via Riemann-Liouville fractional- order derivative see [13]). Moreover, we shall find their equivalent integral representation and determine the eigenvalues and eigenfunctions of the two coupled systems of BVPs (1.1)-(1.2) and (1.1)-(1.3).

These coupled systems are equivalent to coupled systems of second order differential equations. Therefore, the analysis of the spectra of such problems is a consequence of that of second order differential equations.

Firstly, we shall investigate an existence theorem of solutions for the fractional order differential-integral equation

$$D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) = \mathfrak{f}\left(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \lambda I^{1-\alpha}\mathfrak{x}(\mathfrak{t})\right),\ \alpha \in (0,1],\ \mathfrak{t} \in \mathfrak{J} \tag{1.4}$$

subject to the two sets of boundary conditions

$$\mathfrak{x}(0) = 0,\ \mathfrak{x}(1) = 0, \tag{1.5}$$

$$\mathfrak{x}(0) = 0,\ \mathfrak{x}(\tau) = 0,\ \tau \in (0,1) \tag{1.6}$$

Where $D^{\alpha+1}$ and $I^{1-\alpha}$ the Liouville-Caputo fractional derivative of order $\alpha+1$ and Riemman Liouville fractional integral of order $1-\alpha$ respectively.

2. Solvability of the BVP of fractional order

Consider the BVP (1.4)-(1.5) under the indicate assumptions:

(i) $\mathfrak{f} : \mathfrak{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function.

(ii) There is K a positive constant such that

$$|\mathfrak{f}(\mathfrak{t}, \mathfrak{x}, \mathfrak{y}) - \mathfrak{f}(\mathfrak{t}, \mathfrak{u}, \mathfrak{v})| \leq K(|\mathfrak{x} - \mathfrak{u}| + |\mathfrak{y} - \mathfrak{v}|), \quad \text{for all } (\mathfrak{t}, \mathfrak{x}, \mathfrak{y}),\ (\mathfrak{t}, \mathfrak{u}, \mathfrak{v}) \in \mathfrak{J} \times \mathbb{R} \times \mathbb{R}.$$

The equivalence of the BVP (1.4)-(1.5) and the fractional order integral equation can be easily proved by direct calculations

$$\begin{aligned} \mathfrak{x}(\mathfrak{t}) &= \int_0^{\mathfrak{t}} \frac{(\mathfrak{t} - \mathfrak{s})^{\alpha}}{\Gamma(\alpha + 1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \theta)^{-\alpha}}{\Gamma(1 - \alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s} \\ &- \mathfrak{t} \int_0^{1} \frac{(1 - \mathfrak{s})^{\alpha}}{\Gamma(\alpha + 1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \theta)^{-\alpha}}{\Gamma(1 - \alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s},\ \mathfrak{t} \in \mathfrak{J}. \end{aligned} \tag{2.1}$$

The following theorem can be proven using the Banach contraction mapping theorem.

Theorem 2.1. *Assume that (i)-(ii) hold. If* $\frac{2K}{\Gamma(\alpha+2)} + K\lambda < 1,$ *then the BVP of fractional order integral equation (1.4)-(1.5) has a unique solution.*

Proof . Let the subset $\mathfrak{Q}_{\mathfrak{r}}$ of $C(\mathfrak{J}, \mathbb{R})$ be defined as

$$\mathfrak{Q}_{\mathfrak{r}} = \{\, \mathfrak{x} \in C(\mathfrak{J}, \mathbb{R}) :\ |\mathfrak{x}| \leq r,\ r > 0 \,\}.$$

Characterizing the operator $\mathfrak{A}$ by

$$\begin{aligned} \mathfrak{A}\mathfrak{x}(\mathfrak{t}) &= \int_0^{\mathfrak{t}} \frac{(\mathfrak{t} - \mathfrak{s})^{\alpha}}{\Gamma(\alpha + 1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \theta)^{-\alpha}}{\Gamma(1 - \alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s} \\ &- \mathfrak{t} \int_0^{1} \frac{(1 - \mathfrak{s})^{\alpha}}{\Gamma(\alpha + 1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s} - \theta)^{-\alpha}}{\Gamma(1 - \alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s},\ \mathfrak{t} \in \mathfrak{J}. \end{aligned} \tag{2.2}$$

In the light of assumptions (i)-(ii), then $\mathfrak{A}$ is continuous operator on $\mathfrak{Q}_{\mathfrak{r}}$.
From assumption (ii), we obtain

$$\begin{aligned} |\mathfrak{f}(\mathfrak{t}, \mathfrak{x}, \mathfrak{y}) - \mathfrak{f}(\mathfrak{t}, 0, 0)| &\leq K \mid \mathfrak{x} \mid + K \mid \mathfrak{y} \mid \\ |\mathfrak{f}(\mathfrak{t}, \mathfrak{x}, \mathfrak{y})| &\leq |\mathfrak{f}(\mathfrak{t}, 0, 0)| + K \mid \mathfrak{x} \mid + K \mid \mathfrak{y} \mid \\ &\leq \mathfrak{F} + K \mid \mathfrak{x} \mid + K \mid \mathfrak{y} \mid,\ \mathfrak{F} = \sup_{\mathfrak{t} \in \mathfrak{J}} |\mathfrak{f}(\mathfrak{t}, 0, 0)|. \end{aligned}$$

For each $\mathfrak{x} \in \mathfrak{Q}_{\mathfrak{r}}$, and $\mathfrak{t} \in \mathfrak{J}$,

$$\begin{aligned}
|\mathfrak{A}\mathfrak{x}(\mathfrak{t})| &= \Big| \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta\Big) \, d\mathfrak{s} \\
&- \mathfrak{t} \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta\Big) \, d\mathfrak{s} \Big| \\
&\leq \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \Big|\mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta\Big)\Big| \, d\mathfrak{s} \\
&+ \mathfrak{t} \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \Big|\mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{s} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta\Big)\Big| \, d\mathfrak{s} \\
&\leq \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, [\mathfrak{F} + K(|\mathfrak{x}(\mathfrak{s})| + |\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta|)] \, d\mathfrak{s} \\
&+ \mathfrak{t} \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, [\mathfrak{F} + K(|\mathfrak{x}(\mathfrak{s})| + |\lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}(\theta) \, d\theta \, |)] \, d\mathfrak{s} \\
&\leq \frac{2\mathfrak{F}}{\Gamma(\alpha+2)} + K \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, |\mathfrak{x}(\mathfrak{s})| \, d\mathfrak{s} \\
&+ \lambda \, K \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, |\mathfrak{x}(\theta)| \, d\theta \, d\mathfrak{s} \\
&+ \mathfrak{t} \, K \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, |\mathfrak{x}(\mathfrak{s})| \, d\mathfrak{s} \\
&+ \mathfrak{t} \, \lambda \, K \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, |\mathfrak{x}(\theta)| \, d\theta \, d\mathfrak{s} \\
&\leq \frac{2\mathfrak{F}}{\Gamma(\alpha+2)} + \frac{2 \, K \, \mathfrak{r}}{\Gamma(\alpha+2)} + \lambda \, K \, \mathfrak{r} \leq \mathfrak{r},
\end{aligned}$$

where

$$\mathfrak{r} \geq \frac{2\mathfrak{F}}{\Gamma(\alpha+2)} \Big(1 - \frac{2K}{\Gamma(\alpha+2)} - \lambda \, K \Big)^{-1}. \tag{2.3}$$

This proves that the operator $\mathfrak{A} : \mathfrak{Q}_{\mathfrak{r}} \to \mathfrak{Q}_{\mathfrak{r}}$. Let $\{\mathfrak{x}_n\} \subset \mathfrak{Q}_{\mathfrak{r}}$, $\mathfrak{x}_n \to \mathfrak{x}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \mathfrak{A}\mathfrak{x}_n(\mathfrak{t}) &= \lim_{n\to\infty} \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}_n(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}_n(\theta) \, d\theta\Big) \, d\mathfrak{s} \\
&- \lim_{n\to\infty} \mathfrak{t} \int_0^{1} \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)} \, \mathfrak{f}\Big(\mathfrak{s}, \mathfrak{x}_n(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)} \, \mathfrak{x}_n(\theta) \, d\theta\Big) \, d\mathfrak{s},
\end{aligned}$$

using the Lebesgue dominated convergent theorem (see [9]), and from assumption (i), we deduce that $\mathfrak{A}\mathfrak{x}_n(\mathfrak{t}) \to \mathfrak{A}\mathfrak{x}(\mathfrak{t})$ and $\mathfrak{A}$ is continuous.

Second, we apply Banach fixed point theorem to show that $\mathfrak{A}$ has a fixed point. In fact, proving $\mathfrak{A}$ is a contraction is sufficient.

Let $\mathfrak{t} \in \mathfrak{J}$ and for $\mathfrak{x},\ \mathfrak{y} \in C(\mathfrak{J}, \mathbb{R})$

$$\begin{aligned}
& |\mathfrak{A}\mathfrak{x}(\mathfrak{t}) - \mathfrak{A}\mathfrak{y}(\mathfrak{t})| \\
= & \Big| \int_0^t \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right)\ d\mathfrak{s} \\
- & \ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right)\ d\mathfrak{s} \\
- & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{y}(\theta)\ d\theta\right)\ d\mathfrak{s} \\
+ & \ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{y}(\theta)\ d\theta\right)\ d\mathfrak{s}\Big| \\
\leq & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)} \Big| \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right) \\
- & \ \mathfrak{f}\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{y}(\theta)\ d\theta\right) \Big|\ d\mathfrak{s} \\
+ & \ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)} \Big| \mathfrak{f}\left(\mathfrak{s}, \mathfrak{y}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right) \\
- & \ \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{y}(\theta)\ d\theta\right) \Big|\ d\mathfrak{s} \\
\leq & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ K[|\mathfrak{x}(\mathfrak{s}) - \mathfrak{y}(\mathfrak{s})| + \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ |\mathfrak{x}(\theta) - \mathfrak{y}(\theta)|\ d\theta]\ d\mathfrak{s} \\
+ & \ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ K\ [|\mathfrak{y}(\mathfrak{s}) - \mathfrak{x}(\mathfrak{s})| + \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ |\mathfrak{y}(\theta) - \mathfrak{x}(\theta)|\ d\theta]\ d\mathfrak{s} \\
\leq & \ \frac{K\ ||\mathfrak{x}-\mathfrak{y}||}{\Gamma(\alpha+2)} + K\ \lambda||\mathfrak{x}-\mathfrak{y}|| \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \frac{\mathfrak{s}^{1-\alpha}}{\Gamma(2-\alpha)}\ d\mathfrak{s} \\
+ & \ \frac{\mathfrak{t}\ K\ ||\mathfrak{x}-\mathfrak{y}||}{\Gamma(\alpha+2)} + \mathfrak{t}\ K\lambda||\mathfrak{x}-\mathfrak{y}|| \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \frac{\mathfrak{s}^{1-\alpha}}{\Gamma(2-\alpha)}\ d\mathfrak{s} \\
\leq & \ \frac{2K\ ||\mathfrak{x}-\mathfrak{y}||}{\Gamma(\alpha+2)} + K\lambda||\mathfrak{x}-\mathfrak{y}||.
\end{aligned}$$

Since $\frac{2K}{\Gamma(\alpha+2)} + K\lambda < 1$, this proves contraction of operator $\mathfrak{A}$. Therefore, according to the contraction theory of Banach, $\mathfrak{A}$ has a unique fixed point $\mathfrak{x} \in \mathfrak{Q}_{\mathfrak{r}}$ which is a solution of BVP (1.4)-(1.5) on $\mathfrak{J}$. □ Next, we consider the BVP (1.4)-(1.6), by direct calculations, we can easily prove the equality between the BVP (1.4)-(1.6) with the following FOIE

$$\begin{aligned}
\mathfrak{x}(\mathfrak{t}) \ = & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right) d\mathfrak{s} \\
- & \ \mathfrak{t} \int_0^{\tau} \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{\alpha}}{\Gamma(1+\alpha)}\ \mathfrak{x}(\theta)\ d\theta\right)\ d\mathfrak{s},\ \mathfrak{t} \in \mathfrak{J}.
\end{aligned} \tag{2.4}$$

Using the same method of calculation as Theorem 2.1, we will verify the corresponding theorem.

Theorem 2.2. *Assume that (i)-(ii) hold; if* $\frac{2K}{\Gamma(\alpha+2)} + K\lambda < 1$. *Then the BVP of fractional order integral equation (1.4)-(1.6) has a unique continuous solution.*

2.1. Continuous dependence of solutions of (1.4)-(1.5) on λ

Here, we will investigate continuous dependence of solutions of (1.4)-(1.5) on λ

Definition 2.3. *The solutions* $\mathfrak{x} \in C(\mathfrak{J}, \mathbb{R})$ *of (1.4)-(1.5) is continuously dependent on* λ *if* $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ *such that* $|\lambda - \tilde{\lambda}| \leq \epsilon$ *indicates that* $|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| \leq \delta$, *in which* $\tilde{\mathfrak{x}}$ *is a solution of the BVP (1.4)-(1.5).*

Theorem 2.4. *Assume that assumptions of Theorem 2.1 hold. The solutions* $\mathfrak{x} \in C(\mathfrak{J}, \mathbb{R})$ *of the BVP (1.4)-(1.5) are then continuously dependent on* λ.

Proof . For solutions $\mathfrak{x}$ and $\tilde{\mathfrak{x}}$ of the BVP (1.4)-(1.5)

$$
\begin{aligned}
|\mathfrak{x}(t) - \tilde{\mathfrak{x}}(t)| &= \Bigg| \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s} \\
&- \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \mathfrak{x}(\mathfrak{s}), \lambda \int_0^{s} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \mathfrak{x}(\theta)\, d\theta\right) d\mathfrak{s} \\
&- \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \tilde{\lambda} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \\
&+ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \tilde{\lambda} \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \\
&+ \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \\
&- \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \\
&+ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \\
&- \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, \mathfrak{f}\left(\mathfrak{s}, \tilde{\mathfrak{x}}(\mathfrak{s}), \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, \tilde{\mathfrak{x}}(\theta)\, d\theta\right) d\mathfrak{s} \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq\ & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, K[|\mathfrak{x}(\mathfrak{s}) - \tilde{\mathfrak{x}}(\mathfrak{s})| + \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, |\mathfrak{x}(\theta) - \tilde{\mathfrak{x}}(\theta)|\, d\theta]\, d\mathfrak{s} \\
+\ & \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, K[|\tilde{\mathfrak{x}}(\mathfrak{s}) - \mathfrak{x}(\mathfrak{s})| + \lambda \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, |\tilde{\mathfrak{x}}(\theta) - \mathfrak{x}(\theta)|\, d\theta]\, d\mathfrak{s} \\
+\ & \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, K[|\lambda - \tilde{\lambda}| \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, |\tilde{\mathfrak{x}}(\theta)|\, d\theta]\, d\mathfrak{s} \\
+\ & \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\, K[|\lambda - \tilde{\lambda}| \int_0^{\mathfrak{s}} \frac{(\mathfrak{s}-\theta)^{-\alpha}}{\Gamma(1-\alpha)}\, |\tilde{\mathfrak{x}}(\theta)|\, d\theta]\, d\mathfrak{s} \\
\leq\ & \frac{2K||\mathfrak{x} - \tilde{\mathfrak{x}}||}{\Gamma(\alpha+2)} + K\ \lambda||\mathfrak{x} - \tilde{\mathfrak{x}}|| + K\ \mathfrak{r}\ |\lambda - \tilde{\lambda}|.
\end{aligned}
$$

Using (2.3), then we have

$$\begin{aligned}(1-(\frac{2K}{\Gamma(\alpha+2)}+K\ \lambda)||\mathfrak{x}-\tilde{\mathfrak{x}}|| &\leq K\ \mathfrak{r}\ \epsilon\\ ||\mathfrak{x}-\tilde{\mathfrak{x}}|| &\leq \frac{K\ \mathfrak{r}\ \epsilon}{\Gamma(\alpha+2)-2K-K\ \Gamma(\alpha+2)\lambda}\\ &\leq \frac{2K\ \mathfrak{F}\ \epsilon}{\Gamma(\alpha+2)}=\delta.\end{aligned}$$

Then the solutions $\mathfrak{x}\in C(\mathfrak{J},\mathbb{R})$ of (1.4)-(1.5) is continuously dependent on λ. □

3. Remarks and particular cases

Remark 3.1. *Taking* $\mathfrak{y}(\mathfrak{t})=\lambda I^{1-\alpha}\mathfrak{x}(\mathfrak{t})$, *then the BVP (1.4)-(1.5) can be written as a fractional order coupled system.*

$$\begin{aligned}D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) &= \mathfrak{f}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{y}(\mathfrak{t})),\ \alpha\in(0,1],\ \mathfrak{t}\in\mathfrak{J}\\ \mathfrak{y}(\mathfrak{t}) &= \lambda I^{1-\alpha}\mathfrak{x}(\mathfrak{t})\\ \mathfrak{x}(0) &= \mathfrak{x}(1)=0,\ \mathfrak{y}(0)=0,\end{aligned}$$

and may be reduced to:

$$\begin{aligned}D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) &= \mathfrak{f}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{y}(\mathfrak{t})),\ \alpha\in(0,1],\ \mathfrak{t}\in\mathfrak{J}\\ D^{1-\alpha}\mathfrak{y}(\mathfrak{t}) &= \lambda\mathfrak{x}(\mathfrak{t})\\ \mathfrak{x}(0) &= \mathfrak{x}(1)=0,\ \mathfrak{y}(0)=0,\end{aligned}\tag{3.1}$$

that indicate the amounting to fractional order coupled system of integral equations order

$$\begin{aligned}\mathfrak{x}(\mathfrak{t}) &= I^{\alpha+1}\mathfrak{f}(\mathfrak{t},\mathfrak{x}(\mathfrak{t}),\mathfrak{y}(\mathfrak{t}))-\mathfrak{t}\int_0^1\frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{f}(\mathfrak{s},\mathfrak{x}(\mathfrak{s}),\mathfrak{y}(\mathfrak{s}))\ d\mathfrak{s},\ \alpha\in(0,1],\ \mathfrak{t}\in\mathfrak{J}\\ \mathfrak{y}(\mathfrak{t}) &= \lambda I^{1-\alpha}\mathfrak{x}(\mathfrak{t})\\ \mathfrak{x}(0) &= \mathfrak{x}(1)=0,\ \mathfrak{y}(0)=0.\end{aligned}$$

The problem of showing the presence of solutions to the BVP (1.4)-(1.5) is obviously equivalent of showing the existence of solutions to Eq.(2.1) or to showing that there are solutions to a coupled system (3.1).

Remark 3.2. *As a particular case, we take* $\mathfrak{f}(\mathfrak{t},\mathfrak{x},\mathfrak{y})=-\lambda\mathfrak{y}$, *then we have the coupled system of BVP (1.1),(1.2).*
The eigenvalue intervals of $\lambda_{\mathfrak{x}},\lambda_{\mathfrak{y}}$ *for which non trivial solutions of the coupled system of BVP (1.1),(1.2) exist will be determined.*
From a solution of (1.1),(1.2), we assume that $(\mathfrak{x}(\mathfrak{t}),\ \mathfrak{y}(\mathfrak{t}))\in C^2(\mathfrak{J},\mathbb{R})\times C^2(\mathfrak{J},\mathbb{R})$ *satisfying (1.1),(1.2).*

Lemma 3.3. *Let* $\alpha\in(0,1)$. *The pair of functions* $(\mathfrak{x}(\mathfrak{t}),\ \mathfrak{y}(\mathfrak{t})),\ \mathfrak{x},\ \mathfrak{y}\in C^2(\mathfrak{J},\mathbb{R})$ *is said to be a solution of the coupled system of fractional order BVPs (1.1),(1.2), iff* $(\mathfrak{x}(\mathfrak{t}),\ \mathfrak{y}(\mathfrak{t}))$ *is a solution of the coupled system of fractional integral equations*

$$\begin{aligned}\mathfrak{y}(\mathfrak{t}) &= \lambda^2\frac{\mathfrak{t}^{2-\alpha}}{\Gamma(3-\alpha)}\int_0^1\frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}-\lambda^2\ I^2\mathfrak{y}(\mathfrak{t}),\\ \mathfrak{x}(\mathfrak{t}) &= \lambda\ \mathfrak{t}\int_0^1\frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}-\lambda\int_0^{\mathfrak{t}}\frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}.\end{aligned}\tag{3.2}$$

Moreover,

$$I^{\alpha}\mathfrak{y}'(\mathfrak{t})|_{\mathfrak{t}=1} = 0.$$

Proof . From the first equation in the coupled system (1.1)-(1.2), we obtain

$$\begin{aligned} I^{2-(1+\alpha)}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ \mathfrak{y}(\mathfrak{t}),\\ I^{1-\alpha}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ \mathfrak{y}(\mathfrak{t}),\\ I^{1}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ I^{\alpha}\mathfrak{y}(\mathfrak{t}),\\ \mathfrak{x}'(\mathfrak{t})-\mathfrak{x}'(0) &= -\lambda\ I^{\alpha}\mathfrak{y}(\mathfrak{t}), \end{aligned}$$

by integrating both sides, then

$$\begin{aligned} \mathfrak{x}(\mathfrak{t})-\mathfrak{x}(0)-\mathfrak{t}\mathfrak{x}'(0) &= -\lambda\ I^{\alpha+1}\mathfrak{y}(\mathfrak{t}),\\ \mathfrak{x}(\mathfrak{t})-\mathfrak{t}\mathfrak{x}'(0) &= -\lambda\ I^{\alpha+1}\mathfrak{y}(\mathfrak{t}), \end{aligned}$$

at $\mathfrak{t}=1$, we get

$$\mathfrak{x}'(0) = \lambda \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}.$$

Then

$$\mathfrak{x}(\mathfrak{t}) = \lambda\ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}. \tag{3.3}$$

From second equation in coupled system (1.1), we have

$$\begin{aligned} I^{1-(1-\alpha)}\mathfrak{y}'(\mathfrak{t}) &= \lambda\ \mathfrak{x}(\mathfrak{t}),\\ I^{\alpha}\mathfrak{y}'(\mathfrak{t}) &= \lambda\ \mathfrak{x}(\mathfrak{t}),\\ I^{1}\mathfrak{y}'(\mathfrak{t}) &= \lambda\ I^{1-\alpha}\mathfrak{x}(\mathfrak{t}),\\ \mathfrak{y}(\mathfrak{t})-\mathfrak{y}(0) &= \lambda\ I^{1-\alpha}\mathfrak{x}(\mathfrak{t}),\\ \mathfrak{y}(\mathfrak{t}) &= \lambda\ I^{1-\alpha}\mathfrak{x}(\mathfrak{t}), \end{aligned} \tag{3.4}$$

substituting from (3.3) in (3.4) , we get

$$\mathfrak{y}(\mathfrak{t}) = \lambda^2 \frac{\mathfrak{t}^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2\ I^2\mathfrak{y}(\mathfrak{t}),$$

$$\mathfrak{y}(\mathfrak{t}) = \lambda^2 \frac{\mathfrak{t}^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^1 \frac{(1-\mathfrak{s})^{\alpha}}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2 \int_0^{\mathfrak{t}} (\mathfrak{t}-\mathfrak{s})\ \mathfrak{y}(\mathfrak{s})d\mathfrak{s} \tag{3.5}$$

which implies that $\mathfrak{y}\in C^2(\mathfrak{J},\mathbb{R})$ and therefore $\mathfrak{x}\in C^2(\mathfrak{J},\mathbb{R})$.
To prove the converse, differentiate the system (3.2), then

$$\begin{aligned} \mathfrak{x}''(t) &= -\lambda \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^{\alpha-2}}{\Gamma(\alpha-1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s},\\ I^{2-(1+\alpha)}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ I^{1-\alpha}\ I^{\alpha-1}\ \mathfrak{y}(\mathfrak{t}),\\ D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) &= -\lambda\ \mathfrak{y}(\mathfrak{t}), \end{aligned}$$

and

$$\begin{aligned} \mathfrak{y}'(\mathfrak{t}) &= \lambda^2 \frac{\mathfrak{t}^{1-\alpha}}{\Gamma(2-\alpha)} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2 \int_0^{\mathfrak{t}} \mathfrak{y}(\mathfrak{s}) d\mathfrak{s}, \\ I^\alpha \mathfrak{y}'(t) &= \lambda^2\ \mathfrak{t} \int_0^1 \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2 \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}, \\ D^{1-\alpha}\mathfrak{y}(\mathfrak{t}) &= \lambda\ \mathfrak{x}(\mathfrak{t}). \end{aligned}$$

Obviously, we can verify that $\mathfrak{x}(1) = 0$ and since $\mathfrak{x},\ \mathfrak{y} \in C^2(\mathfrak{J}, \mathbb{R})$, then can prove that $\mathfrak{x}(0) = \mathfrak{y}(0) = 0$.
Thus the equivalence between the coupled system (1.1), (1.2) and the system of fractional integral equations (3.2) is proved.
Now,

$$\begin{aligned} \mathfrak{y}(\mathfrak{t}) &= \lambda I^{1-\alpha}\mathfrak{x}(\mathfrak{t}) \\ \Rightarrow I^\alpha \mathfrak{y}(\mathfrak{t}) &= \lambda I^1 \mathfrak{x}(\mathfrak{t}) \\ \Rightarrow DI^\alpha \mathfrak{y}(\mathfrak{t}) &= \lambda D I^1 \mathfrak{x}(\mathfrak{t}) \\ \Rightarrow I^\alpha D \mathfrak{y}(\mathfrak{t}) &= \lambda \mathfrak{x}(\mathfrak{t}) \\ \Rightarrow I^\alpha \mathfrak{y}'(\mathfrak{t})|_{\mathfrak{t}=1} &= \lambda \mathfrak{x}(1) = 0. \end{aligned}$$

□ Now, from (3.5) we have

$$\mathfrak{y}(\mathfrak{t}) = \int_0^1 G(\mathfrak{t}, \mathfrak{s})\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s},$$

Here $G(\mathfrak{t}, \mathfrak{s})$ is the function of Green defined by the formula

$$G(\mathfrak{t}, \mathfrak{s}) = \begin{cases} \lambda^2\ \mathfrak{t}^{2-\alpha} \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)\Gamma(3-\alpha)} - \lambda^2\ (\mathfrak{t}-\mathfrak{s}) & ,\ 0 \le \mathfrak{s} \le \mathfrak{t} \le 1, \\ \\ \lambda^2\ \mathfrak{t}^{2-\alpha} \frac{(1-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)\Gamma(3-\alpha)} & ,\ 0 < \mathfrak{t} \le \mathfrak{s} \le 1. \end{cases} \tag{3.6}$$

Lemma 3.4. *Allow* $\alpha \in (0, 1)$. *The pair of functions* $(\mathfrak{x}(\mathfrak{t}),\ \mathfrak{y}(\mathfrak{t}))$, $\mathfrak{x},\ \mathfrak{y} \in C^2(\mathfrak{J}, \mathbb{R})$ *is said to be solution of the coupled system of fractional order BVPs (1.1), (1.3) iff* $(\mathfrak{x}(\mathfrak{t}),\ \mathfrak{y}(\mathfrak{t}))$ *is a solution of the fractional integral equations coupled system.*

$$\begin{aligned} \mathfrak{y}(\mathfrak{t}) &= \frac{\lambda^2}{\tau} \frac{\mathfrak{t}^{2-\alpha}}{\Gamma(3-\alpha)} \int_0^\tau \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2\ \mathfrak{J}^2 \mathfrak{y}(\mathfrak{t}), \\ \mathfrak{x}(\mathfrak{t}) &= \frac{\lambda\ \mathfrak{t}}{\tau} \int_0^\tau \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda \int_0^{\mathfrak{t}} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}. \end{aligned} \tag{3.7}$$

Moreover,

$$I^\alpha \mathfrak{y}'(\mathfrak{t})|_{\mathfrak{t}=\tau} = 0.$$

Proof . The proof straights forward as Lemma 3.3.
at $\mathfrak{t} = \tau$, we get

$$\mathfrak{x}'(0) = \frac{\lambda}{\tau} \int_0^\tau \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}.$$

Then

$$\mathfrak{x}(\mathfrak{t}) = \frac{\lambda\ \mathfrak{t}}{\tau}\ \int_0^\tau \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda\ \int_0^\mathfrak{t} \frac{(\mathfrak{t}-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}. \tag{3.8}$$

Then, we get

$$\mathfrak{y}(\mathfrak{t}) = \frac{\lambda^2}{\tau}\frac{\mathfrak{t}^{2-\alpha}}{\Gamma(3-\alpha)}\int_0^\tau \frac{(\tau-\mathfrak{s})^\alpha}{\Gamma(\alpha+1)}\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s} - \lambda^2\ \int_0^\mathfrak{t} (\mathfrak{t}-\mathfrak{s})\ \mathfrak{y}(\mathfrak{s})\ d\mathfrak{s}.$$

Clearly $\mathfrak{y} \in C^2(\mathfrak{J},\mathbb{R})$ and therefore $\mathfrak{x} \in C^2(\mathfrak{J},\mathbb{R})$ and the equivalence between coupled system of integral equations (3.7) and coupled system of BVPs (1.1) subject to (1.3) can easily be verified. Moreover,

$$I^\alpha \mathfrak{y}'(\mathfrak{t})|_{\mathfrak{t}=\tau} = \lambda\ \mathfrak{x}(\tau) = 0.$$

□

Theorem 3.5. *The coupled system of FODE (1.1) subjects to* $\mathfrak{y}(0) = 0$ *can be reduced to the system of second order differential equations*

$$\begin{aligned} \mathfrak{x}''(\mathfrak{t}) &= -\lambda^2\ \mathfrak{x}(\mathfrak{t}), \quad \mathfrak{t} \in \mathfrak{J} = (0,1), \\ \mathfrak{y}''(\mathfrak{t}) &= -\lambda^2\ \mathfrak{y}(\mathfrak{t}), \quad \mathfrak{t} \in \mathfrak{J}. \end{aligned} \tag{3.9}$$

Proof . Firstly,

$$D^{\alpha+1}\mathfrak{x}(\mathfrak{t}) = -\lambda\ \mathfrak{y}(\mathfrak{t}), \quad \mathfrak{t} \in \mathfrak{J} = (0,1),$$

$$\begin{aligned} I^{2-(1+\alpha)}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ \mathfrak{y}(\mathfrak{t}), \\ I^{1-\alpha}\mathfrak{x}''(\mathfrak{t}) &= -\lambda\ \mathfrak{y}(\mathfrak{t}). \end{aligned}$$

On both sides operating I^α , then

$$I^1\mathfrak{x}''(\mathfrak{t}) = -\lambda I^\alpha\ \mathfrak{y}(\mathfrak{t}),$$

by differentiating, we obtain

$$\begin{aligned} \mathfrak{x}''(\mathfrak{t}) &= -\frac{d}{d\mathfrak{t}}\lambda I^\alpha\ \mathfrak{y}(\mathfrak{t}), \\ &= -\lambda I^\alpha \frac{d}{d\mathfrak{t}}\ \mathfrak{y}(\mathfrak{t}), \quad \text{since } \mathfrak{y}(0) = 0, \\ &= -\lambda I^{1-(1-\alpha)}\frac{d}{d\mathfrak{t}}\ \mathfrak{y}(\mathfrak{t}), \\ &= -\lambda D^{1-\alpha}\ \mathfrak{y}(\mathfrak{t}), \\ &= -\lambda(\lambda\ \mathfrak{x}(\mathfrak{t})), \\ &= -\lambda^2\ \mathfrak{x}(\mathfrak{t}). \end{aligned} \tag{3.10}$$

Then,

$$D^{1-\alpha}\mathfrak{y}(\mathfrak{t}) = \lambda\ \mathfrak{x}(\mathfrak{t}), \quad \mathfrak{t} \in \mathfrak{J} = (0,1),$$

$$\begin{aligned} I^{1-(1-\alpha)}\mathfrak{y}'(\mathfrak{t}) &= \lambda\ \mathfrak{x}(\mathfrak{t}), \\ I^\alpha\mathfrak{y}'(\mathfrak{t}) &= \lambda\ \mathfrak{x}(\mathfrak{t}). \end{aligned}$$

Operating by D^{α} On both parts, we obtain

$$D^{\alpha} I^{\alpha} \mathfrak{y}'(\mathfrak{t}) = \lambda D^{\alpha} \mathfrak{x}(\mathfrak{t}),$$

and

$$\mathfrak{y}''(\mathfrak{t}) = \lambda D^{\alpha+1} \mathfrak{x}(\mathfrak{t}) = -\lambda^2 \mathfrak{y}(\mathfrak{t}). \tag{3.11}$$

Therefore, the fractional order BVPs (1.1) coupled system can be reduced to the system (3.9). □

Theorem 3.6. *The exact eigenvalues to (1.1) subjects to (1.2) are given as*

(i) *For* $\lambda > 0$

$$\lambda_{n,\mathfrak{x}} = n\pi, \quad \lambda_{n,\mathfrak{y}} = \frac{\pi(2n+1)}{2\zeta}, \quad n = 1, 2, ...,$$

The distinct eigenfunctions are as follows:

$$\mathfrak{x}_n(\mathfrak{t}) = d_n \ \sin(n\pi\mathfrak{t}), \quad \mathfrak{y}_n(\mathfrak{t}) = c_n \ \sin\left(\frac{\pi(2n+1)}{2\zeta} t\right), \ \mathfrak{t}, \ \zeta \in \mathfrak{J}$$

respectively.

(ii) *For* $\lambda < 0$

$$\mathfrak{x}_n(\mathfrak{t}) = d_n \sinh(\lambda_{n,\mathfrak{x}} \ \mathfrak{t}), \quad \mathfrak{y}_n(\mathfrak{t}) = c_n \sinh(\lambda_{n,\mathfrak{y}} \mathfrak{t}), \ \mathfrak{t}, \ \zeta \in \mathfrak{J}.$$

Proof . Due to the equivalence between coupled system of second order differential equations (3.9) and coupled system (1.1). Then the eigenvalues of coupled system of second order differential equations (3.9) are equal to those of the coupled system (1.1).

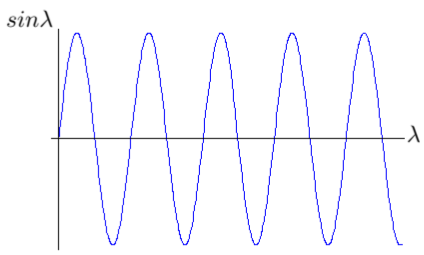

Figure.1b

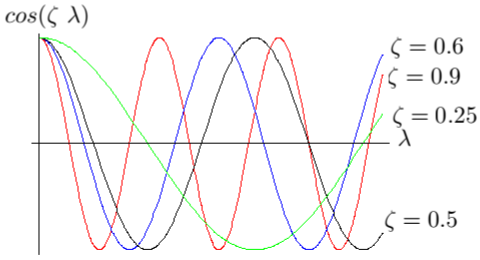

Figure. 1a

Figure.1a gives the graph of the eigenfunction $\cos(\zeta\lambda)$ for different amounts of ζ, $\zeta = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{y}} > 0$.
Figure.1b gives the graph of the eigenfunction $\sin(\lambda)$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{x}} > 0$ (The graphs shows ten values of $\lambda_{n,\mathfrak{x}}$).

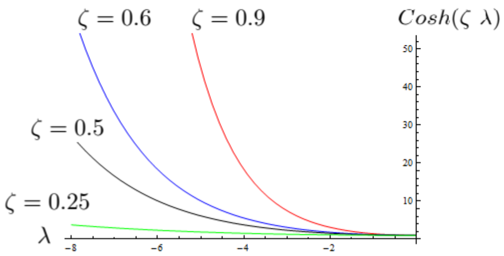

Figure. 2b

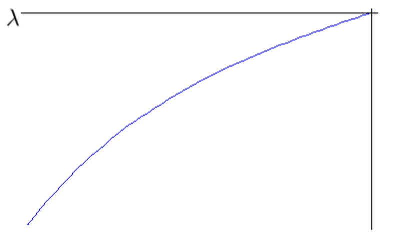

Figure. 2a

Figure.2a gives the graph of the eigenfunction $\sinh(\lambda)$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{x}} < 0$.
Figure.2b gives the graph of the eigenfunction $\cosh(\zeta\lambda)$ for different amounts of ζ, $\zeta = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of the function $\cosh(\zeta\lambda)$ are (complex numbers and equal to $\pm\,\frac{1.57008\ i}{\zeta}$) the eigenvalues $\lambda_{n,\mathfrak{y}} < 0$. $\square$

Theorem 3.7. *The exact eigenvalues to (1.1) subjects to (1.3) are given as*

(i) For $\lambda > 0$

$$\lambda_{n,\mathfrak{x}} = \frac{n\pi}{\tau}, \quad \lambda_{n,\mathfrak{y}} = \frac{\pi(2n+1)}{2\zeta}, \quad n = 1, 2, ...$$

where the corresponding distinct eigenfunctions are given as

$$\mathfrak{x}_n(\mathfrak{t}) = d_n\ \sin\left(\frac{n\pi}{\tau}\ \mathfrak{t}\right), \quad \mathfrak{y}_n(\mathfrak{t}) = c_n\ \sin\left(\frac{\pi(2n+1)}{2\zeta}\mathfrak{t}\right), \ \mathfrak{t},\ \zeta, \tau \in \mathfrak{J}$$

respectively.

(ii) For $\lambda < 0$

$$\mathfrak{x}_n(\mathfrak{t}) = d_n\ \sinh(\lambda_{n,\mathfrak{x}}\ \mathfrak{t}), \quad \mathfrak{y}_n(\mathfrak{t}) = c_n\ \sinh(\lambda_{n,\mathfrak{y}}\mathfrak{t}),\ \mathfrak{t},\ \zeta \in \mathfrak{J}.$$

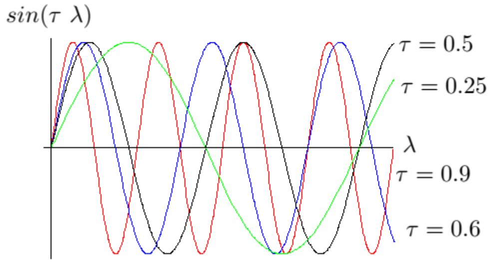

Figure. 3b

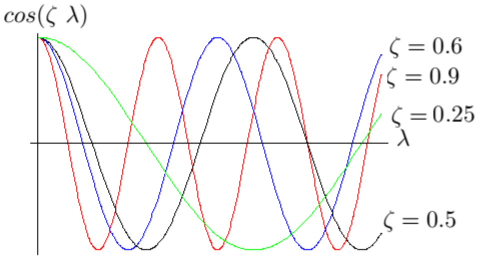

Figure. 3a

Figure. 3a gives the graph of the function $\cos(\zeta\lambda)$ for different values of $\zeta,\ \zeta = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{y}} > 0$.

Figure. 3b gives the graph of the function $\sin(\tau\lambda)$ for $\tau,\ \tau = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{x}} > 0$.

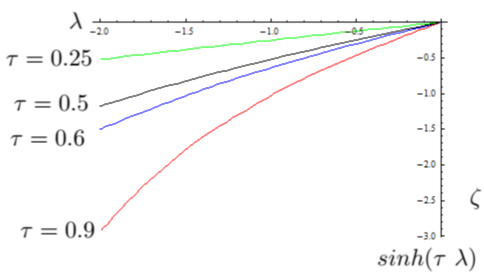

Figure. 4b

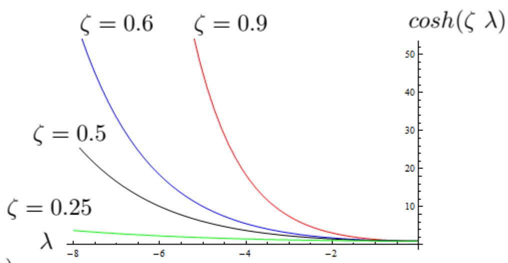

Figure. 4a

Figure. 4a gives the graph of the function $\cosh(\zeta\lambda)$ for different values of $\zeta,\ \zeta = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of this function are the eigenvalues $\lambda_{n,\mathfrak{y}} > 0$.
Figure.4b gives the graph of the function $\sinh(\tau\lambda)$ for $\tau,\ \tau = 0.9,\ 0.6,\ 0.5,\ 0.25$. Zeros of the function $\sinh(\tau\lambda)$ are the eigenvalues $\lambda_{n,\mathfrak{x}} < 0$. ■

Conclusion

In this work, two BVPs for FODE are discussed. The existence of unique solution and some data continuous dependence are proved, in aim of determining the eigenvalues and eigenfunctions of the two coupled systems of BVPs (1.1)-(1.2) and (1.1)-(1.3). These coupled systems are equivalent to coupled systems of second order differential equations. Therefore, analysis for the spectra of these problems is a consequence from that of second order differential equations.

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