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Fuzzy visible submodules with some results

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Abstract

In this paper, the concept of fuzzy visible submodules which is a new type of fuzzy submodules has been introduced. Some results and characterizations of fuzzy visible are established namely the homomorphic image of the fuzzy visible submodule, the sum of two fuzzy visible submodules. The relation between fuzzy visible submodule and its submodules. Also, the fuzzy quotient modules in sense of fuzzy visible have been presented. We prove that the intersection of a collection of fuzzy visible submodules are visible submodules and the converse is not true. Also, we define the strong cancellation fuzzy modules and we established some results of it with respect to fuzzy visible submodules. Many other properties we study in fuzzy visible submodules.

Keywords: Fuzzy visible submodule, Pure fuzzy submodule, T-pure fuzzy submodule, Fully cancellation fuzzy module, Strongly cancellation fuzzy module

1. Introduction

In1965,Zadeh presented the concept of fuzzy subsets in the paper fuzzy sets [22].The theory of fuzzy sets has been made enormous improvement .Many extensions were introduced such as L-fuzzy sets presented by Goguen in 1969 [3], also introduced another extensions is called interval valued fuzzy sets in [2], [23] the rough sets introduced in 1981 by Z. Pawlak [19] as a type of extensions to the concepts of fuzzy sets .

Through the paper M is a unitary $\mathbb{R}$-module and $\mathbb{R}$ is a c.r.w.1 and for every fuzzy ideal K of $\mathbb{R}$, $K(0) = 1$. In this paper our aim is to introduce a new type of fuzzy submodules which are called fuzzy visible submodules. Fuzzy visible submodule is characterized in terms of its level set. We investigate many of properties of fuzzy visible submodule and explain the relationship between it and

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other type of fuzzy submodule like pure, T-pure,divisible. Also we define fuzzy strongly cancellation submodule and use it to explain the relation between the nonzero fuzzy ideal of a ring R and fuzzy visible submodule on fuzzy $\mathbb{R}$-module.

In section 2,we give some preliminaries which are necessary to our work. In section 3, we define fuzzy visible submodule and study it by give some examples and prove some results.

2. Basic concepts

This section has provided definitions for fuzzy sets as well as their facets. We also provide fuzzy modules with specific properties.

Definition 2.1. *[22] Let $\mathbb{R}$ be a non-empty set and I be the closed interval $[0,1]$ of the real line (real numbers). A fuzzy set $\mathfrak{W}$ in R (a fuzzy subset of $\mathbb{R}$) is a function from $\mathbb{R}$ in to I.*

Definition 2.2. *[24, 4] Let $a_t : \mathbb{R} \to [0,1]$ be a fuzzy set in $\mathbb{R}$,where $a \in \mathbb{R}, t \in [0,1]$, define*

$$a_t(y) = \begin{cases} t & if \quad a = y \\ 0 & if \quad a \neq y \end{cases}$$

a_t is called a fuzzy singleton in $\mathbb{R}$.

Definition 2.3. *[15] Let a_t, x_k be two fuzzy singletons of $\mathbb{R}$. if $a_t = x_k$, then $a = x$ and $t = k$, where $t, k \in (0,1]$.*

Definition 2.4. *[17, 12] Let X be a non-empty set, and $\mathbb{A}$ be a fuzzy subset of X, for all, $t \in [0,1]$, the set $\mathbb{A}_t = \{x \in X : \mathbb{A}(x) \geq t\}$ is called a level set of X with respect to $\mathbb{A}$.*

Definition 2.5. *[8] Let f be a mapping from a set M into a set N. Let $\mathbb{A}$ be a fuzzy set in M and $\mathbb{B}$ be a fuzzy set in N. The image of $\mathbb{A}$ denoted by $f(\mathbb{A})$ is the fuzzy set in N defined by :*

$$f(\mathbb{A})(y) = \begin{cases} sup\{\mathbb{A}(z) : z \in f^{-1}(y), if \ f^{-1}(y) \neq 0, for \ all \ y \in N, \} \\ 0 \qquad\qquad\qquad\qquad o.w. \end{cases}$$

and the inverse image of $\mathbb{B}$ denoted by $f^{-1}(\mathbb{B})$ fuzzy set in M defined by:

$$f^{-1}(\mathbb{B})(x) = \mathbb{B}(f(x)), \ for \ allx \in M$$

Remark 2.6. *Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy sets in $\mathbb{R}$, then*

1. *$\mathbb{A} = \mathbb{B}$ if and only if $\mathbb{A}(x) = \mathbb{B}(x)$, $\forall x \in \mathbb{R}$,*
2. *$\mathbb{A} \subseteq \mathbb{B}$ if and only if $\mathbb{A}(x) \leqslant \mathbb{B}(x)$, $\forall x \in \mathbb{R}$,*
3. *$(\mathbb{A} \cap \mathbb{B})_t = \mathbb{A}_t \cap \mathbb{B}_t$ and $(\mathbb{A} \cup \mathbb{B})_t = \mathbb{A}_t \cup \mathbb{B}_t$, for all $t \in [0,1], [22]$,*
4. *$\mathbb{A} = \mathbb{B}$ if and only if $\mathbb{A}_t = \mathbb{B}_t$, for all $t \in [0,1]$,[11],*

Definition 2.7. *[24] Let M be an $\mathbb{R}$-module. A fuzzy set $\mathbb{P}$ of M is called a fuzzy module of an $\mathbb{R}$-module M if,*

1. *$\mathbb{P}(x - y \geq) min\{\mathbb{P}(x), \mathbb{P}(y)\}$, for all $x, y \in M$.*

2. $\mathbb{P}(rx) \geq \mathbb{P}(x)$, *for all* $x \in M, r \in \mathbb{R}$.

3. $\mathbb{P}(0) = 1$.

Let $FM(M)$ *denote the set of all fuzzy modules of an* $\mathbb{R}$*-module* M.

Definition 2.8. *[14] Let* $\mathbb{A}, \mathbb{P} \in FM(M)$. $\mathbb{A}$ *is called a fuzzy submodule of* $\mathbb{P}$ *if* $\mathbb{A} \subseteq \mathbb{P}$.
Let $FS(\mathbb{P})$ *symbolize the set of all fuzzy submodule of a fuzzy module* $\mathbb{P}$ *of an* $\mathbb{R}$*-module* M.

Definition 2.9. *[14]* $\mathbb{A} \in FS(\mathbb{P})$ *if and only if,* $\mathbb{A}_t$ *is a submodule of* $\mathbb{P}_t$*,for each* $t \in [0, 1]$.

Definition 2.10. *[9] Let* M, N *be any sets and* $f : M \to N$ *be any function. A fuzzy subset* $\mathbb{A}$ *of* M *is called f-invariant if* $f(x) = f(y)$ *implies* $\mathbb{A}(x) = \mathbb{A}(y)$ *where* $x, y \in M$.

Proposition 2.11. *[15] Let* f *be a function from a set* M *into set* N. $\mathbb{A}$ *and* $\mathbb{B}$ *are fuzzy subsets of* N *then*

$$f^{-1}(\mathbb{A} \cap \mathbb{B}) = f^{-1}(\mathbb{A}) \cap f^{-1}(\mathbb{B})$$

Proposition 2.12. *[9] If* f *is a function defined on a set* M, $\mathbb{A}$ *and* $\mathbb{B}$ *are fuzzy subsets of* M, $\mathbb{C}$ *and* $\mathbb{D}$ *are fuzzy subset of* $f(M)$. *The following are true:*

1. $A \subseteq f^{-1}(f(\mathbb{A}))$.

2. $A = f^{-1}(f(\mathbb{A}))$ *whenever* $\mathbb{A}$ *is f-invariant*

3. $f(f^{-1}(\mathbb{D})) = D$.

4. *If* $\mathbb{A} \subseteq \mathbb{B}$, *then* $f(\mathbb{A}) \subseteq f(\mathbb{B})$.

5. *If* $\mathbb{C} \subseteq \mathbb{D}$, *then* $f^{-1}(\mathbb{C}) \subseteq f^{-1}(\mathbb{D})$.

Proposition 2.13. *[25] Let* $X \in FM(M_1)$ *and* $Y \in FM(M_2)$. *Suppose that* $f : X \to Y$ *be a fuzzy homomorphism. If* $A \in FS(X)$ *and* $B \in FS(Y)$, *then:*

1. $f(A) \in FS(Y)$.

2. $f^{-1}(B) \in FS(X)$.

Definition 2.14. *[9, 1] A fuzzy subset* K *of a ring* R *is called a fuzzy ideal of* $\mathbb{R}$, *if for each* $x, y \in \mathbb{R}$:

1. $K(x - y) \geq min\{K(x), K(y)\}$.

2. $K(x, y) \geq max\{K(x), K(y)\}$.

We will use $FI(\mathbb{R})$ *to symbolize the set of all fuzzy ideals of* $\mathbb{R}$.

Corollary 2.15. *[15] Let* $(x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n}$ *be a fuzzy singleton of* $\mathbb{R}$.
Let $((x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n})(w) : \mathbb{R} \to [0, 1]$*,defined by :*

$$((x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n})(w) = \begin{cases} min\{t_1, \ldots, t_n\} & if w \in (x_1, \ldots, x_n) \\ 0 & o.w. \end{cases}$$

, then $(x_1, \ldots, x_n)$ *is a fuzzy ideal of* $\mathbb{R}$.
The fuzzy ideal $((x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n})$ *is called a finitely generated fuzzy ideal generated by* $(x_1)_{t_1}, (x_2)_{t_2}, \ldots, (x_n)_{t_n}$.

Remark 2.16. *[15] Let $A \in FI(R)$. Then A is a finitely generated ideal if and only if A_t is a finitely generated ideal of $\mathbb{R}$, $\forall t \in [0,1]$.*

Proposition 2.17. *[14] $K \in FI(R)$ if and only if $K_t, t \in [0,1]$ is an ideal of $\mathbb{R}$.*

Definition 2.18. *Let $\mathbb{P} \in FM(M)$, $\mathbb{A} \in FS(\mathbb{P})$ and $K \in FI(\mathbb{R})$. The product*

$$KA = \begin{cases} sup\{\inf\{K(r_1), \ldots, K(r_n), A(x_1), \ldots, A(x_n)\}\} & for\ some\ r_i \in \mathbb{R}, x_i \in M, n \in N \\ 0 & o.w. \end{cases}$$

Note that $KA \in FS(\mathbb{P})$ if $K(0) = 1$, [24]. And $(KA)_t = K_t A_t$ for each $t \in [0,1]$, [8].

Definition 2.19. *[24] Let $\mathbb{P} \in FM(M)$ and $A, B \in FS(\mathbb{P})$. The residual quotient ideal of A and B denoted by $(A : B)$ is the fuzzy subset of $\mathbb{R}$ defined by:*

$$(A : B)r = \sup\{t \in [0,1] : r_t B \subseteq A\}, for\ all r \in \mathbb{R}$$

.

Definition 2.20. *[24] If $A, B \in FS(\mathbb{P})$, then $(A : B) \in FI(\mathbb{R})$.*

Definition 2.21. *Let $A \in FS(\mathbb{P})$ be non- empty. The fuzzy annihilator of A denoted by F-$annA$ is defined by:*

$$(F - annA)(r) = sup\{t : t \in [0,1], r_t A \subseteq 0_1\}$$

Note that that $F - annA = (0_1 : A)$, [20]. Hence $(F - annA)_t \subseteq F - annA_t$, [7].

Definition 2.22. *[5] Let $\mathbb{P} \in FM(M)$, then $F - ann\mathbb{P} \in FI(\mathbb{R})$.*

Definition 2.23. *[24] Let $A, B \in FS(\mathbb{P})$. The $A + B$ defined by*

$$((A + B)(x) = sup\{inf\{A(y), B(z) : x = y + z\ for\ all\ x, y, z \in M\}\}$$

Remark 2.24. *[14] If $\mathbb{P} \in FM(M)$ and $x_t \in \mathbb{P}$, then for all fuzzy singleton r_k of $\mathbb{R}$, $r_k x_t = (rx)_\lambda$, where $\lambda = min\{k, t\}$.*

Definition 2.25. *[7] Let $J \in FI(\mathbb{R})$. J is called a cancellation fuzzy ideal if $AJ = BJ$, where $A, B \in FI(R)$. Then $A = B$.*

Proposition 2.26. *[14] Let $A \in FM(M)$,then we define $A_* = \{x \in M : A(x) = 1\}$.*

Proposition 2.27. *[14] Let $A \in FM(M)$, then A_* is a submodule of M.*

Remark 2.28. *[14] It is that $A_* = A_t \ni t = 1$.*

Definition 2.29. *[16] Let $\mathbb{P} \in FM(M)$ and $A \in FS(\mathbb{P})$. Define $\mathbb{P}/A : M/(A_* \to [0,1]$ by:*

$$\mathbb{P}/A(a + A_*) = \begin{cases} 1 & if a \in A_* \\ Sup\{\mathbb{P}(a + b)\} & if b \in A_*, a \neq A_* \end{cases}$$

for all coset $a + A_* \in M/A_*$.

Definition 2.30. *[16] If $\mathbb{P} \in FM(M)$ and $A \in FS(\mathbb{P})$, then $\mathbb{P}/A \in FM(M/A_*)$.*

Definition 2.31. *[16] Let $\mathbb{P} \in FM(M)$ such that $\mathbb{P}(x) = 1 \forall x \in M$. Then, $(P/A)_* = P_*/A_*$ for each $A \leq P$.*

Definition 2.32. *[13, 21] Let $H \in FI(\mathbb{R})$. H is called a principle fuzzy ideal if there exist $x_t \subseteq H$ such that $H = (x_t)$ then for each $n_s \subseteq H$,there exists a fuzzy singleton a_l of R such that $n_s = a_l x_t$, where $s, l, t \in [0, 1]$, that is $H = (x_t) = \{n_s \subseteq H : n_s = a_l x_t\}$ for some fuzzy singleton a_l of $\mathbb{R}$.*

Definition 2.33. *[15] Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W} \in FS(\mathbb{P})$.Then $\mathfrak{W}$ is called a pure fuzzy submodule if for each $H \in FI(\mathbb{R})$, $H\mathbb{P} \cap \mathfrak{W} = H\mathfrak{W}$.*

Definition 2.34. *[15] Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W} \in FS(\mathbb{P})$. Then $\mathfrak{V}$ is a pure fuzzy submodule if only if V_t is a pure submodule of $\mathbb{P}_t, \forall t \in (0, 1]$ ".*

Definition 2.35. *[21] Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W} \in FS(\mathbb{P})$. W is called T-pure fuzzy submodule of $\mathbb{P}$ if for each $J \in FI(\mathbb{R})$ such that $J^2P \cap \mathfrak{W} = J^2W$.*

Definition 2.36. *[20] Let $\mathbb{P} \in FM(M)$. $\mathbb{P}$ is called divisible if for every fuzzy singleton $\mathfrak{r}_1$ of $\mathbb{R}\mathfrak{r}_1 \neq 0$, $\mathfrak{r}_1\mathbb{P} = \mathbb{P} \forall l \in (0, 1])$.*

Definition 2.37.

1. *Let $\mathbb{P} \in FM(M)$. $\mathbb{P}$ is called fully cancellation fuzzy module if for every non-empty $J \in FI(\mathbb{R})$ and for every $\mathfrak{V}, \mathfrak{W} \in FS(\mathbb{P})$ such that $J\mathfrak{W} = J\mathfrak{V}$ implies $\mathfrak{W} = \mathfrak{V}$, [6].*
2. *Let $J \in FI(\mathbb{R})$ is non-empty, J is called fuzzy idempotent if$J = J^2$, [21].*

3. Fuzzy visible submodules with basic results

This section included the study of a new type of fuzzy submodules, which are called fuzzy visible submodules ,where the basic definition was given for our next work. Many examples ,characterization ,various and important properties have been proven.

Definition 3.1. *Let $\mathbb{P} \in FM(M)$ and $P \neq \mathfrak{W} \in FS(\mathbb{P})$. Then $\mathfrak{W}$ is referred to as fuzzy visible submodule whenever $\mathfrak{W} = W\mathfrak{W}$ for every non-empty $W \in FI(\mathbb{R})$.*

A fuzzy ideal of a ring $\mathbb{R}$ is called visible if it is visible of a fuzzy $\mathbb{R}$-module $\mathbb{R}$.

Proposition 3.2. *Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W}(\neq P) \in FS(\mathbb{P})$.Then $\mathfrak{W}$ fuzzy visible submodule if and only if $\mathfrak{W}_t$ is visible submodule of $\mathbb{P}_t$, $\forall t \in (0, 1]$.*

Proof . Let $H \neq 0$ be an ideal of a ring $\mathbb{R}$ and $\mathbb{P} \in FM(M)$.

Define $\mathfrak{J} : \mathbb{R} \to [0, 1]$ by $\mathfrak{J}(x) = \begin{cases} 1 & if x \in H \\ 0 & o.w. \end{cases}$.

It is obvious that $\mathfrak{J} \in FI(\mathbb{R})$ and $\mathfrak{J}_t = H$. Let $\mathfrak{W}$ be a fuzzy visible submodule of $\mathbb{P}$ then $\mathfrak{W}_t$ is a submodule of $\mathbb{P}_t$, $\forall t \in (0, 1]$. $\mathfrak{J}_t\mathfrak{W}_t = (\mathfrak{J}\mathfrak{W})_t$ by [8], $= \mathfrak{W}_t$ since $\mathfrak{W}$ is a fuzzy visible submodule of $\mathbb{P}$. Therefore $H\mathfrak{W}_t = \mathfrak{W}_t$ and hence $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t$.

Conversely, let $W \in FI(\mathbb{R})$ is non-empty and $\mathfrak{W} \in FS(\mathbb{P})$. Since $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t, \forall t \in (0, 1]$, then $\mathfrak{W}_t = W_t\mathfrak{W}_t = (W\mathfrak{W})_t$ by [8]. Hence $\mathbb{W} = W\mathfrak{W}$ by [11], that is $\mathfrak{W}$ a fuzzy visible submodule of $\mathbb{P}$. □

3.1. Remark and examples

1. Let $\mathbb{P} \in FM(M)$. Then 0_1 is a fuzzy visible submodule of $\mathbb{P}$.

   Proof . Let $\mathbb{P} \in FM(M)$.

   Define $0_1 : M \to [0,1]$ by $0_1(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & o.w. \end{cases}$.

   It is true that $0_1 \in FS(\mathbb{P})$ and $(0_1)_t = \{0\}, \forall t \in (0,1]$. But we have from [[10], remark and examples 1.2.2], $\{0\}$ is visible submodule of $\mathbb{P}_t$ and hence 0_1 be a fuzzy visible submodule of $\mathbb{P}$.

2. Let $M = Z_4$ as Z-module and $\mathbb{A} = (\overline{2})$ is a proper submodule of M. Let $J = (2)$ be an ideal of Z. Then, define $\mathbb{P} : M \to [0,1]$ by $\mathbb{P}(x) = \begin{cases} 1 & if\ x \in M \\ 0 & o.w. \end{cases}$ and

   define $\mathfrak{W} : M \to [0,1]$ by $\mathfrak{W}(x) = \begin{cases} 1 & if\ x \in \mathbb{A} \\ 0 & o.w. \end{cases}$

   $W : Z \to [0,1]$ by $W(x) = \begin{cases} 1 & if\ x \in J \\ 0 & o.w. \end{cases}$.

   It is clear that $\mathbb{P} \in FM(M)$, $\mathfrak{W} \in FS(\mathbb{P})$ and $W \in FI(\mathbb{R})$. Then $\mathbb{P}_t = Z_4$, $\mathfrak{W}_t = A$ and $W_t = J$ which implies that $\mathfrak{W}_t$ be not visible submodule of $\mathbb{P}_t$ by [[10], remark and examples [1.2.2]]. Thus $\mathfrak{W}$ be not fuzzy visible submodule of $\mathbb{P}$ by proposition 3.2. □

3. Let $M = Z_6$ as Z-module and $\mathbb{A} = (\overline{2})$ is a proper submodule of M. Let $J = (6)$ be an ideal of Z. Then, define $\mathbb{P} : M \to [0,1]$ by $\mathbb{P}(a) = \begin{cases} 1 & if\ a \in M \\ 0 & o.w. \end{cases}$ and

   define $\mathfrak{W} : M \to [0,1]$ by $\mathfrak{W}(a) = \begin{cases} 1 & if\ a \in (\overline{2}) \\ 0 & o.w. \end{cases}$ and

   $W : Z \to [0,1]$ by $W(a) = \begin{cases} 1 & if\ a \in (6) \\ 0 & o.w. \end{cases}$.

   It is true that $\mathbb{P} \in FM(M)$, $\mathfrak{W} \in FS(\mathbb{P})$ and $W \in FI(Z)$. Now,

   $\mathbb{P}_t = M$, $\mathfrak{W}_t = A = (\overline{2})$ and $W_t = J$, $(\overline{2})$ is not visible submodule of $\mathbb{P}_t$ by [13], [remark and examples 1 2.2]. Therefore $\mathfrak{W}$ is not fuzzy visible submodule of $\mathbb{P}$.

4. Let $M = Z_6$ as Z-module and $A = (\overline{3})$ is a proper submodule of M. Let $J = (2)$ be an ideal of Z. In a similar way above and by [[10], remark and examples 1.2.2], we have

   $$\mathfrak{W}(x) = \begin{cases} 1 & if\ x \in (\overline{3}) \\ 0 & o.w. \end{cases} \text{ is not fuzzy visible submodule of } \mathbb{P}.$$

5. If K is a non –empty fuzzy cyclic submodule of a fuzzy module Q as a Z-module, then it is not visible.

   Proof . Let $K = (x_t), x = \dfrac{a}{b}, a, b \in Z, a \neq 0, b \neq 0, t \in (0,1]$, be a fuzzy cyclic submodule of Q as Z-module and $L = (s_n) \in FI(Z), n \in (0,1])$, then $(s_n)(x_t) \neq (x_t)$. Hence K be not fuzzy visible. □

6. Let $\mathbb{P}_1, \mathbb{P}_2 \in FM(M)$ and $f : \mathbb{P}_1 \to \mathbb{P}_2$ be a fuzzy epimorphism . If $\mathfrak{W}$ is a fuzzy f-invariant visible submodule of $\mathbb{P}_1$,then $f(\mathfrak{W})$ be a fuzzy visible of $\mathbb{P}_2$.

   Proof . Let $f : \mathbb{P}_1 \to \mathbb{P}_2$ be a fuzzy epimorphism since $\mathfrak{W}$ is a fuzzy visible of $\mathbb{P}_1$ so $\mathfrak{W} = A\mathfrak{W}$ for every non-empty $A \in FI(\mathbb{R})$. Then $f(\mathfrak{W}) = f(A\mathfrak{W})$. Hence $f(\mathfrak{W}) = Af(\mathfrak{W})$ by [[15], lemma 2.3.1], then $f(\mathfrak{W})$ is a fuzzy visible of $\mathbb{P}$. □

7. Let $\mathbb{P}_1, \mathbb{P}_2 \in FM(M)$ and $f : \mathbb{P}_1 \to \mathbb{P}_2$ be a fuzzy epimorphism. If K is a fuzzy visible submodule of $\mathbb{P}_2$, then $f^{-1}(K)$ be a fuzzy visible of P_1.

   Proof . Let $f : \mathbb{P}_1 \to \mathbb{P}_2$ be a fuzzy epimorphism and K is a fuzzy visible submodule of $\mathbb{P}_2$, then for every nonzero $L \in FI(\mathbb{R})$, we have $K = LK$ then $f^{-1}(K) = f^{-1}(LK)$. Then $f^{-1}(K) = Lf^{-1}(K)$ by [15]. Therefore $f^{-1}(K)$ be a fuzzy visible. □

Proposition 3.3. *Let $\mathfrak{W}(\neq \mathbb{P}) \in FS(\mathbb{P})$. Then the following are equivalent:*

1. *$\mathfrak{W}$ is a fuzzy visible submodule.*

2. *$\mathfrak{W} = J\mathfrak{W}$ for each nonzero finitely generated fuzzy ideal Jof $\mathbb{R}$.*

Proof . 1 ⇒ 2 Clear.

2 ⇒ 1 : Let $\mathfrak{W} = J\mathfrak{W}$ for a nonzero finitely generated fuzzy ideal J of $\mathbb{R}$, then J_t is a finitely generated ideal of $\mathbb{R} \forall t \in (0, 1]$ by [[15],remark 2.1.9]. Since $\mathfrak{W}_t = (J\mathfrak{W})_t = J_t\mathfrak{W}_t$, then by [[10],proposition 1.1.3] ,we have $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t$, $\forall t \in (0, 1]$. Therefore $\mathfrak{W}$ be fuzzy visible. □

Proposition 3.4. *Let $\mathbb{P} \in FM(M)$, such that $\mathbb{P}(x) = 1$,for each $x \in M$, let $\mathfrak{W}$ be a fuzzy visible submodule of $\mathbb{P}$. Then $\mathfrak{W}/\mathfrak{V}$ is a fuzzy visible of $\mathbb{P}/\mathfrak{V}$ where $\mathfrak{V} \in FS(\mathfrak{W})$.*

Proof . *It is clear that $\mathfrak{W}/\mathfrak{V} \in FS(\mathbb{P}/\mathfrak{V})$. Since $\mathbb{P} \in FM(M)$ then $\mathbb{P}_t$ is a module and $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}, \forall t \in (0, 1]$ by proposition 3.2, then $\mathbb{P}_*$ is a module and $\mathfrak{W}_*$ is a visible submodule of it. Now by [[10], proposition 1.1.4], we have $\mathfrak{W}_*/\mathfrak{V}_*$ is a fuzzy visible of $\mathbb{P}_*/\mathfrak{V}_*$, then by proposition 2.31, we have $(\mathfrak{W}/\mathfrak{V})_* = \mathfrak{W}_*/\mathfrak{V}_*$ and hence $(\mathfrak{W}/\mathfrak{V})_*$ is a visible submodule of $(\mathbb{P}/\mathfrak{V})_*$,therefore $(\mathfrak{W}/\mathfrak{V})$ be a fuzzy visible submodule of $(\mathbb{P}/\mathfrak{V})$.* □

Proposition 3.5. *Assume that $\mathbb{P} \in FM(M)$.Then the sum of two fuzzy visible submodules ,also is a fuzzy visible.*

Proof . *Let $J \in FI(\mathbb{P})$ is non-empty and $\mathfrak{W}, \mathfrak{V} \in FS(\mathbb{P})$. Then $J(\mathfrak{W} + \mathfrak{V}) = J\mathfrak{W} + J\mathfrak{V} = \mathfrak{W} + \mathfrak{V}$ (since $\mathfrak{W}$ and $\mathfrak{V}$ are fuzzy visible submodules). Therefore $\mathfrak{W} + \mathfrak{V}$ be a fuzzy visible submodule of $\mathbb{P}$.* □

Proposition 3.6. *Every fuzzy submodule of a fuzzy visile submodule $\mathfrak{W}$ is also fuzzy visible .*

Proof . *Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W}$ be a fuzzy visible submodule of $\mathbb{P}$, let $\mathfrak{W}(\neq \mathbb{P}) \in FS(\mathbb{P})$ to prove that $\mathfrak{V}$ is a fuzzy visible. Since $\mathfrak{W}$ is a fuzzy visible by hypothesis. Then $\mathfrak{W}_t$ is visible submodule by proposition 3.2,since $\mathfrak{V} \in FS(\mathfrak{W})$, we have $\mathfrak{V}_t$ is a submodule of $\mathbb{P}_t, \forall t \in (0, 1]$ by [18], that is $\mathfrak{V}_t \subseteq W_t$. Therefore $\mathfrak{V}_t$ is a visible submodule of $\mathfrak{W}_t$ by [[10], proposition 1.1.7]. Hence $\mathfrak{V}$ be a fuzzy visible submodule of $\mathfrak{W}$ by proposition 3.2.* □

Proposition 3.7. *Let $\mathfrak{W}, \mathfrak{V} \in FS(\mathbb{P})$. If either $\mathfrak{W}$ or $\mathfrak{V}$ is a fuzzy visible submodule of $\mathbb{P}$, then $\mathfrak{W} \cap \mathfrak{V}$ is also fuzzy visible .*

Proof . *Let$\mathfrak{W}, \mathfrak{V} \in FS(\mathbb{P})$, then $\mathfrak{W} \cap \mathfrak{V} \in FS(\mathbb{P})$ [24], since either $\mathfrak{W}$ or $\mathfrak{V}$ is a fuzzy visible then $\mathfrak{W}_t$ or $\mathfrak{V}_t$ is a visible submodule and hence $\mathfrak{W}_t \cap \mathfrak{V}_t$ is a visible submodule by [[10], proposition 1.1.8].But $(\mathfrak{W}_t \cap \mathfrak{V}_t) = (\mathfrak{W} \cap \mathfrak{V})_t$ by [22]. Therefore $\mathfrak{W} \cap \mathfrak{V}$ be a fuzzy visible by proposition 3.2.* □

Corollary 3.8. *Let $\{\mathfrak{W}_i\}_{i\in I} \in FS(\mathbb{P})$ such that at least one of them is visible, then $\cap_{i=1}^{n}\mathfrak{W}_i$ is a fuzzy visible.*

Proof . *Since $\cap_{i=1}^{n}\mathfrak{W}_i \subseteq \mathfrak{W}_i, \forall i$, then by proposition 3.6 we have the result.*

The converse of proposition 3.6: Let $M = Z_{12}$ as Z_{36}–module, $H = (\overline{6})$ submodule of M.

define $\mathbb{P} : M \to [0,1]$ *by* $\mathbb{P}(x) = \begin{cases} 1 & if\ x \in M \\ 0 & o.w. \end{cases}$

define $0_1(x) : M \to [0,1]$ *by* $0_1(x) = \begin{cases} 1 & if\ x = 0 \\ 0 & o.w. \end{cases}$

It is apparent that $\mathbb{P} \in FM(M)$ and $0_1 \in FS(\mathbb{P})$ by remark and examples 3.1.

define $\mathfrak{W} : M \to [0,1]$ *by* $\mathfrak{W}(x) = \begin{cases} 1 & if\ x \in (\overline{6}) \\ 0 & o.w. \end{cases}$

Then $\mathfrak{V} \in FS(\mathbb{P}), 0_1 \subseteq \mathfrak{V}$ and $\mathfrak{V}$ is not fuzzy visible of $\mathbb{P}$ since $\mathfrak{V}_t = (\overline{6})\forall t \in [0,1]$ be not visible submodule of Z_{12} by [[10], p15]. □

Remark 3.9. *In easy way we can prove that ,if $J \in FI(\mathbb{R})$, then J is a fuzzy idempotent if and only if J_t is an idempotent $\forall t \in (0,1]$.*

Proposition 3.10. *Let $\mathbb{R}$ be a ring in which every nonzero fuzzy ideals of $\mathbb{R}$ are idempotent. Let $\mathfrak{W}$ be a fuzzy visible submodule of fully cancellation fuzzy $\mathbb{R}$-module $\mathbb{P}$. If $\mathfrak{V}(\neq \mathbb{P}) \in FS(\mathbb{P})$ such that $\mathfrak{W} \subseteq \mathfrak{V}$,then $\mathfrak{V}$ is a fuzzy visible submodule of $\mathbb{P}$.*

Proof . *Since $\mathfrak{W} \in FS(\mathbb{P})$, then $\mathfrak{W}_t$ is a visible submodule by proposition 3.2, let $\mathfrak{V} \in FS(\mathbb{P})$ such that $\mathfrak{W} \subseteq \mathfrak{V}$. But $\mathfrak{W}_t \subseteq \mathfrak{V}_t$ by [14] Since in $\mathbb{R}$ every non zero fuzzy ideals are idempotent then by remark 3.9 every non zero ideals of $\mathbb{R}$ is idempotent. Also since $\mathbb{P}$ is a fully cancellation fuzzy ,we have $\mathbb{P}_t$ is a fully cancellation $\forall t \in (0,1]$ by[[6], proposition 1.2.2] then $\mathfrak{V}_t$ is a visible of $\mathbb{P}_t$ by [[10], proposition 1.1.11],therefore $\mathfrak{V}$ be a fuzzy visible submodule of $\mathbb{P}$.* □

Definition 3.11. *An $\mathbb{R}$-module M is called strongly cancellation module if for each ideals L and J of $\mathbb{R}$ such that $LW = JW$, then $L = J$ for every submodule W of M, [6].*

This notion will be fuzzified into a strongly cancellation fuzzy module as follows:

Definition 3.12. *Let $\mathbb{P} \in FS(\mathbb{P})$ and $H\mathfrak{W} = K\mathfrak{W}$ for every $H, K \in FI(\mathbb{R})$.Then $\mathbb{P}$ is called strongly cancellation fuzzy module if H=K for every $\mathfrak{W} \in FS(\mathbb{P})$.*

Proposition 3.13. *Let$\mathbb{P} \in FM(M)$. Then $\mathbb{P}$ is strongly cancellation fuzzy module if and only if $\mathbb{P}_t$ is strongly cancellation module, $\forall t \in (0,1]$.*

Proof . *Let H and K are ideals of $\mathbb{R}$ and N be a submodule of M. Let $HN = KN$.*

define $\mathfrak{W} : M \to [0,1]$ *by* $\mathfrak{W}(x) = \begin{cases} 1 & if\ a \in N \\ 0 & o.w. \end{cases}$ *and*

$A : M \to [0,1]$ *by* $A(x) = \begin{cases} 1 & if\ x \in H \\ 0 & o.w. \end{cases}$ *and* $C : \mathbb{R} \to [0,1]$ *by* $C(x) = \begin{cases} 1 & if\ x \in K \\ 0 & o.w. \end{cases}$

Clear that $A, C \in FI(\mathbb{R})$ and $\mathfrak{W} \in FS(\mathbb{P})$ and hence $A_t = H, C_t = K$ and $\mathfrak{W}_t = N, \forall t \in (0,1]$. Hence $A_t\mathfrak{W}_t = C_t\mathfrak{W}_t$,which implies $(A\mathfrak{W})_t = (C\mathfrak{W})_t$ therefore $(A\mathfrak{W}) = (C\mathfrak{W})$, since $\mathbb{P}$ be a strongly cancellation fuzzy, then $A = C$. Therefore $H = K$.

Conversely, let $\mathbb{P}_t$ be a strongly cancellation where $t \in (0,1]$ and let $\mathfrak{W} \in FS(\mathbb{P})$ such that $A\mathfrak{W} = C\mathfrak{W}$, where $A, C \in FI(\mathbb{R})$. Then $(A\mathfrak{W})_t = (C\mathfrak{W})_t$ and hence $A_t\mathfrak{W}_t = C_t\mathfrak{W}_t$,but $\mathfrak{W}_t$ is a submodule of $\mathbb{P}_t$ and $\mathbb{P}_t$ is a strongly cancellation module by hypothesis ,then $A_t = C_t$. Hence $A = C$, therefore P be a strongly cancellation fuzzy module. □

Proposition 3.14. *Let $\mathbb{P}$ be a strongly cancellation fuzzy module $\mathbb{P}$ and $\mathfrak{W} \in FS(\mathbb{P})$. If $\mathfrak{W}$ is fuzzy visible, then $F - ann(H\mathfrak{W}) = F - ann(H)$, for every nonzero $H \in FI(\mathbb{P})$.*

Proof . *Let $r_t \subseteq F-ann(H)$, then $r_tH \subseteq 0_1$, $r \in \mathbb{R}$ and hence $r_tH\mathfrak{W} \subseteq 0_1$, then $r_t \subseteq F-ann(H\mathfrak{W})$. Therefore $F - ann(H) \subseteq F - ann(H\mathfrak{W})$. Let $s_t \subseteq F - ann(H\mathfrak{W})$. Then $s_tH\mathfrak{W} \subseteq 0_1$, but $\mathfrak{W}$ is a fuzzy visible, then $s_t\mathfrak{W} \subseteq 0_1$ and $s_t\mathfrak{W} \subseteq 0_1\mathfrak{W}$ but $\mathbb{P}$ is a strongly cancellation fuzzy module, then we have $s_t \subseteq 0_1$. Thus $s_tH \subseteq 0_1$ and hence $s_t \subseteq F - ann(H)$. Therefore $F - ann(H\mathfrak{W}) = F - ann(H)$.* □

Proposition 3.15. *Let $\mathbb{P}$ be a strongly cancellation fuzzy $\mathbb{R}$-module with a fuzzy visible submodule $\mathfrak{W}$.Then every nonzero $H \in FI(\mathbb{R})$ is cancellation.*

Proof . *Let $\mathfrak{W} \in FS(\mathbb{P})$ and $H \in FI(\mathbb{R})$ is a nonzero. Hence $\mathfrak{W} = H\mathfrak{W}$ for each nonzero $H \in FI(\mathbb{R})$. Let $AH = BH$ such that $A, B \in FI(\mathbb{R})$. Then $AH\mathfrak{W} = BH\mathfrak{W}$, hence $A\mathfrak{W} = B\mathfrak{W}$ (since $\mathfrak{W}$ is a fuzzy visible).Therefore $A = B$(since $\mathbb{P}$ is a strongly cancellation fuzzy module).* □

Proposition 3.16. *For each nonzero $H \in FI(\mathbb{R})$ and for each non-empty collection $\{\mathfrak{W}_j\}$ of fuzzy visible submodules of an $\mathbb{R}$-module M. We have $H(\cap_j\mathfrak{W}_j) = \cap_j H\mathfrak{W}_j$.*

Proof . *We Know that $\cap_j\mathfrak{W}_j \subseteq \mathfrak{W}_j$ for each j, but $\mathfrak{W}_j$ is a fuzzy visible for each j and hence $H\mathfrak{W}_j = \mathfrak{W}_j$ for each j, by proposition 3.7, we have $\cap_j\mathfrak{W}_j$ is a fuzzy visible submodule of $\mathbb{P}$. Therefore $\cap_j H\mathfrak{W}_j = \cap_j\mathfrak{W}_j = H(\cap_j\mathfrak{W}_j)$* □

Proposition 3.17. *Let $\mathbb{P} \in FM(M)$ and $\mathfrak{W} \in FS(\mathbb{P})$. Then $\mathfrak{W}$ is a pure fuzzy of $\mathbb{P}$.*

Proof . *Let $\mathfrak{W}$ be a fuzzy visible submodule of $\mathbb{P}$,then $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t \forall t \in (0, 1]$, and hence $\mathfrak{W}_t$ is a pure submodule of $\mathbb{P}_t$ by [[10], proposition 1.1.17]. Then $\mathfrak{W}$ be a pure fuzzy by proposition 2.34.* □

Remark 3.18. *If $\mathfrak{V}$ is a pure fuzzy submodule of $\mathbb{P}$, then it is not necessary will be fuzzy visible, for example:*

Consider $M = Z_6$ as Z-module. Define $\mathbb{P} : M \to [0, 1]$ by $\mathbb{P}(a) = \begin{cases} 1 & if\ x \in Z_6 \\ 0 & o.w. \end{cases}$

It is true that $\mathbb{P} \in FM(M)$. Then 0_1 and $\mathbb{P}$ are always pure submodule. But $\mathbb{P}$ is not fuzzy visible.

Proposition 3.19. *Let Y be a principle fuzzy ring on a ring $\mathbb{R}$ and $\mathbb{P}$ be a fuzzy divisible module. Then each proper fuzzy pure submodule of $\mathbb{P}$ is fuzzy visible.*

Proof . *Let $H \in FI(Y)$ and $V \neq (\mathbb{P}) \in FS(\mathbb{P})$. Since Y is a principle fuzzy ring, then $H = (r_l)$ for some some $r \in \mathbb{R}$, $r_l \neq 0_1$. We know that $H\mathfrak{W} \subseteq \mathfrak{W}$ to prove that $\mathfrak{W} \subseteq H\mathfrak{W}$. let $x_t \subseteq \mathfrak{W}$, then $x_t \subseteq \mathbb{P}$. Then $x_t \subseteq \mathfrak{W} \cap \mathbb{P}$. But $\mathbb{P}$ Is a fuzzy divisible module, then $\mathbb{P} = r_l\mathbb{P}$ for each fuzzy singleton r_l of $\mathbb{R}$, $r_l \neq 0_1$. Therefore $\mathbb{P} = (r_1)\mathbb{P}$, hence $x_t \subseteq \mathfrak{W} \cap (r_l)\mathbb{P} = (r_l)\mathfrak{W} = H\mathfrak{W}$, we have $\mathfrak{W} \subseteq H\mathfrak{W}$. Hence $\mathfrak{W} = H\mathfrak{W}$.* □

Proposition 3.20. *A direct summand of a fuzzy divisible module $\mathbb{P}$ over fuzzy P.I.D. ring $\mathbb{R}$ is a fuzzy visible .*

Proof . *Let $\mathfrak{W}$ be a direct summand of a fuzzy divisible module $\mathbb{P}$, then $\mathfrak{W}$ is a fuzzy pure by [[15], proposition 2.2.7] and hence $\mathfrak{W}$ is a visible submodule by proposition 3.19.* □

Proposition 3.21. *Let $\mathbb{R}$ be a P.I.R. and P be a fuzzy divisible $\mathbb{R}$-module and $\mathfrak{W} \leq \mathfrak{V} \leq \mathbb{P}$, then*

1. *If $\mathfrak{W}$ is a fuzzy visible in $\mathfrak{V}$ and $\mathfrak{V}$ is a fuzzy visible in $\mathbb{P}$, then $\mathfrak{W}$ is a fuzzy visible in $\mathbb{P}$.*
2. *If $\mathfrak{W}$ is a fuzzy visible in $\mathbb{P}$, then $\mathfrak{W}$ is a fuzzy visible in $\mathfrak{V}$.*

Proof .

1. *Since $\mathfrak{W}$ is a fuzzy visible in $\mathfrak{V}$,then by proposition 3.2 $\mathfrak{W}_t$ is a visible submodule in $\mathfrak{V}_t$. Also since $\mathfrak{V}$ is a fuzzy visible in $\mathbb{P}$, then $\mathfrak{V}_t$ is a visible submodule in $\mathbb{P}_t$. Since $\mathbb{P}$ is a fuzzy divisible we have $\mathbb{P}_t$ is a divisible by [[5], proposition 2.2.12]. And hence $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t$ by [[10], proposition 1.1.22 (2)]. Therefore $\mathfrak{W}$ is a fuzzy visible in $\mathbb{P}$.*

2. *Since $\mathfrak{W}$ is a fuzzy visible in $\mathbb{P}$, then $\mathfrak{W}_t$ is a visible submodule in $\mathbb{P}_t$ by proposition 3.2. But $\mathfrak{W} \leq \mathfrak{V} \leq \mathbb{P}$, then $\mathfrak{W}_t \leq \mathfrak{V}_t \leq \mathbb{P}_t$ by [22] and hence $\mathfrak{W}_t$ is a visible submodule of $\mathfrak{V}_t$ by [[10] , proposition 1.1.22, (2)], where $\mathbb{P}_t$ is a divisible $\mathbb{R}$-module ,and hence W be a fuzzy visible in $\mathfrak{V}$.*

□

Corollary 3.22. *A fuzzy visible submodule $\mathfrak{W}$ of $\mathbb{P}$ is T-Pure fuzzy submodule.*
Proof . *By [[21],remark and examples] and proposition 3.17, the result is true. In The following example we show that the converse of corollary 3.23 not satisfied :* □

Example 3.23. *Consider $M = Z_4$ as Z-module and $L = (\overline{2})$ is a submodule of Z_4 and $J = nZ$ is an ideal of Z where n is a positive integer.*

Define $\mathbb{P} : M \to [0,1]$ *by* $\mathbb{P}(a) = \begin{cases} 1 & if\ x \in M \\ 0 & o.w. \end{cases}$

Define $Y : M \to [0,1]$ *by* $Y(a) = \begin{cases} 1 & if\ x \in (\overline{2}) \\ 0 & o.w. \end{cases}$

Define $\mathfrak{V} : M \to [0,1]$ *by* $\mathfrak{V}(x) = \begin{cases} 1 & if\ x \in nZ \\ 0 & o.w. \end{cases}$

$\mathbb{P}_t = M, Y_t = (\overline{2}), V_t = nZ = J, Y_t$ *is a T-pure of Z_4, then Y is a fuzzy T-pure of $\mathbb{P}$ by [[21], proposition 1.2.4], but Y be not fuzzy visible by remark and examples 3.1 .*

Proposition 3.24. *Let $\mathfrak{W} \in FS(\mathbb{P})$ such that $\mathfrak{W} \subseteq \mathfrak{V}$. Then $\mathfrak{W}$ is a pure fuzzy submodule of $\mathfrak{V}$ whenever $\mathfrak{W}$ is a visile of $\mathbb{P}$.*
Proof . *Since $\mathfrak{W} \subseteq \mathfrak{V}$ then $\mathfrak{W}_t \subseteq \mathfrak{V}_t$ where $t \in (0,1]$, but $\mathfrak{W}_t$ is a visible submodule of $\mathbb{P}_t$ by proposition 3.2, therefore $\mathfrak{W}_t$ is a pure submodule of $\mathfrak{V}_t$ by [[10], proposition 1.1.28].Hence $\mathfrak{W}$ is a pure fuzzy submodule of $\mathfrak{V}$.* □

Proposition 3.25. *Let $\mathbb{P} \in FM(M)$ and $\mathfrak{V}$ be a pure fuzzy submodule in $\mathbb{P}$. If $\mathfrak{W}$ is a fuzzy visible submodule of $\mathfrak{V}$, then $\mathfrak{W}$ is a pure fuzzy in $\mathbb{P}$.*
Proof . *Since $\mathfrak{W} \subseteq \mathfrak{V}$ then $\mathfrak{W}_t \subseteq \mathfrak{V}_t \forall t \in (0,1]$, $\mathfrak{W}_t$ is visible submodule of $\mathfrak{V}_t$ by proposition 3.2, and $\mathfrak{V}_t$ is a pure submodule of $\mathbb{P}_t$,therefore $\mathfrak{W}_t$ is a pure of $\mathbb{P}_t$ by [[10], proposition 1.1.29], and hence $\mathfrak{W}$ be a pure in $\mathbb{P}$.* □

Definition 3.26. *Let $\mathbb{P} \in FM(M)$. $\mathfrak{W} \in FS(\mathbb{P})$ is called idempotent in $\mathbb{P}$ if and only if $\mathfrak{W} = (\mathfrak{W} : \mathbb{P})\mathfrak{W}$.*

Proposition 3.27. *Every fuzzy visible submodule $\mathfrak{W}$ of $\mathbb{P}$ is an idempotent submodule.*
Proof . *Since $\mathfrak{W}$ is a fuzzy visible of $\mathbb{P}$, then $\mathfrak{W} = H\mathfrak{W}$, for every non-empty $H \in FI(\mathbb{R})$. Put $H = (\mathfrak{W} : \mathbb{P})$, thus $\mathfrak{W} = (\mathfrak{W} : \mathbb{P})\mathfrak{W}$. Therefore $\mathfrak{W}$ be a fuzzy idempotent.* □

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