Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3505-3516
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.6132

The Rayleigh Gompertz distribution: Theory and real applications

Nadia Hashim Al-Noor[a,*], Mundher A. Khaleel[b], Noor Kareem Assi[a]

[a]*Mathematics Department, College of Science, Mustansiriyah University, Baghdad, Iraq*
[b]*Mathematics Department, College of Computer Science and Mathematics, Tikrit University, Tikrit, Iraq*

(Communicated by Madjid Eshaghi Gordji)

Abstract

The generalization of distributions is an important topic in probability theory. Several distributions, whether symmetrical, semi-symmetrical or heavily skewed, are unsuitable for modelling modern data. In this paper, the Rayleigh Gompertz distribution as a new compound flexible distribution is introduced. Several important statistical properties of the new distribution have been examined and studied as well as its flexibility is proved through various real datasets with different information fitting criteria. The flexibility of this new distribution allows using it in various application areas.

Keywords: The Rayleigh Gompertz Distribution, exponential distribution, reliability function and hazard function.

1. Introduction

Many researchers have focused their attention on the class of generalized or expanded distributions. Different flexible new distributions and families are introduced, such as Beta-G [1], Marshall Olkin–G [2], generalized gamma–G [3], Kumaraswamy–G [4], Weibull–G [5], Rayleigh–G [6] and many others, through various procedures. In this article, the Rayleigh–G family is considered to introduce a new extension distribution of Gompertz named Rayleigh Gompertz.

The Gompertz distribution is a flexible distribution that can be skewed to both sides (right and left). This distribution is a generalization of exponential distribution and is widely used in many applied problems, particularly in modeling survival and human mortality. For more details about the *Go* distribution and its applications, see [7] and [8]. The remains of this article are established

*Corresponding author
Email addresses: nadialnoor@uomustansiriyah.edu.iq, drnadialnoor@gmail.com (Nadia Hashim Al-Noor), mun880088@tu.edu.iq (Mundher A. Khaleel)

Received: March 2021 *Accepted:* June 2021

as follows: In section 2, a brief detail about the new extension of Gompertz distribution is provided. Sections 3, 4, 5, and 6 respectively address the properties, entropies, reliability stress strength model, and the maximum likelihood estimators of the new distribution. Section 7 address the numerical illustration through the application of different real data sets. The concluding remarks are presented in section 8.

2. New compound of Gompertz Distribution

For any baseline distribution with the cumulative distribution function (cdf), say $G(x)$, and probability density function (pdf), say $g(x)$, Al-Noor and Assi [6] proposed a new flexible family of probability distributions named Rayleigh-G $(R-G)$ with the following general formulas of cdf and pdf

$$F(x)_{R-G}=e^{-\frac{\theta}{2}(-\ln G(x))^{2}};\ x\geq 0,\ \theta>0 \tag{2.1}$$

$$f(x)_{R-G}=\theta\frac{g(x)}{G(x)}\ (-\ln G(x))\, e^{-\frac{\theta}{2}(-\ln G(x))^{2}} \tag{2.2}$$

where θ is the scale parameter.

Now, assume that $G(x)$ and $g(x)$ be the Gompertz distribution with the cdf and its associated pdf given respectively by [9]

$$G(x)_{Go}=1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\quad ;\ x\geq 0,\ \alpha,\ \lambda>0 \tag{2.3}$$

$$g(x)_{Go}=\alpha\, e^{\lambda x}e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)} \tag{2.4}$$

where α and λ are the scale and shape parameters respectively.

After substituting (2.3) and (2.4) in (2.1) and (2.2) instead of $G(x)$ and $g(x)$, a new sub-model of the Rayleigh-G family named Rayleigh Gompertz (RGo) is proposed. The cdf and pdf of the new distribution will be as

$$F(x)_{RGo}=e^{-\frac{\theta}{2}\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)\right]^{2}} \tag{2.5}$$

$$f(x)_{RGo}=\frac{\theta\alpha\, e^{\lambda x}e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}}{1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}}\ \left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)\right] e^{-\frac{\theta}{2}\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)\right]^{2}} \tag{2.6}$$

Figures 1 and 2 demonstrate different shapes for the cdf and pdf of the RGo distribution with various parameter values. The pdf shape can be decreased and different skewed. The distribution is therefore highly flexible to model different positive data.

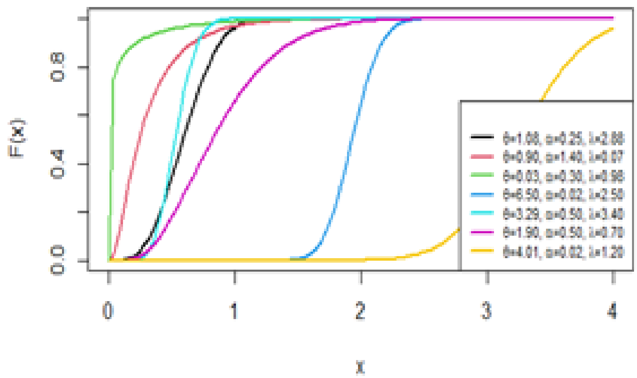

Figure 1: Plot of ***RGO*** cdf with various parameter values

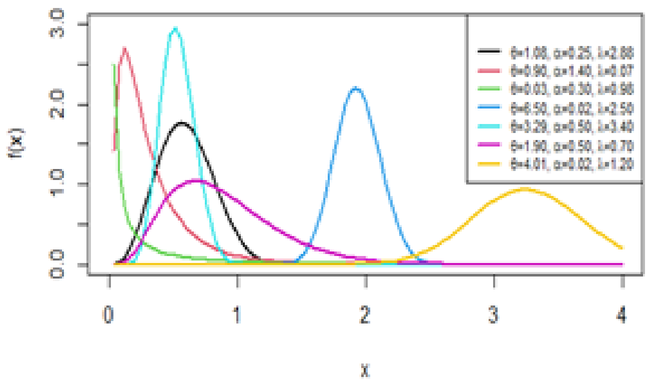

Figure 2: Plot of ***RGO*** pdf with various parameter values

3. Properties of the *RGo* Distribution

In this section, several important statistical properties of the new distribution have been examined and studied.

Property 1. The non-central r^{th} moment of RGo distribution can be found through

$$E\left(X^r\right)_{RGo} = \int_0^\infty x^r f(x)_{RGo}\, dx \tag{3.1}$$

where $f(x)_{RGo}$ represents the pdf as in (2.6). Since

$$e^{-z} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\, z^i \tag{$S1$}$$

Then

$$e^{-\frac{\theta}{2}\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^2} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\theta}{2}\right)^i \left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^{2i}$$

The pdf in (2.6), will be

$$f(x)_{RGo} = \frac{\theta\alpha\, e^{\lambda x} e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}}{1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}} \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{\theta}{2}\right)^i \left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^{2i+1} \tag{3.2}$$

Using the following especial formula (see [10])

$$[-\ln z\,]^a = \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{j+k+l}\, a}{a-j}\, \overset{k-a}{\underset{k}{C}}\; C_j^k C_l^{a+k}\, P_{j,k}\, z^l \tag{S2}$$

where $P_{j,\,0} = 1$ for $j \geq 0$ and $P_{j,k} = k^{-1} \sum_{m=1}^{k} \frac{(-1)^m [m(j+1)-k]}{m+1}\, P_{j,k-m}$ for $k = 1, 2, \ \ldots\ .$

The $\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)\right]^{2i+1}$ will be

$$\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)\right]^{2i+1}=\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}\ (2i+1)}{2i+1-j}C_k^{k-(2i+1)}C_j^k C_l^{2i+1+k}P_{j,k}\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)^l$$

Consequently, the pdf in (3.2), will be

$$f(x)_{RGo}=\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}\ (2i+1)}{i!(2i+1-j)}\ \frac{\theta^{i+1}}{2^i}\ C_k^{k-(2i+1)}\ \ C_j^k\ C_l^{(2i+1)+k}\ P_{j,k}$$
$$\alpha\ e^{\lambda x}\ e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\ \left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)^{l-1} \tag{3.3}$$

Let $A=\sum_{i,k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l}\ (2i+1)}{i!(2i+1-j)}\ \frac{\theta^{i+1}}{2^i}C_k^{k-(2i+1)}\ C_j^k\ C_l^{2i+1+k}P_{j,k}$, then

$$f(x)_{RGo}=A\ \alpha\ e^{\lambda x}\ e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)^{l-1} \tag{3.4}$$

Now, $E(X^r)_{RGo}$ in (3.1) will be

$$E(X^r)_{RGo}=A\int_0^\infty x^r\alpha\ e^{\lambda x}\ e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)^{l-1}\ dx$$
$$=A\frac{1}{l}\int_0^\infty x^r\alpha l\ e^{\lambda x}\ e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\left(1-e^{-\frac{\alpha}{\lambda}(e^{\lambda x}-1)}\right)^{l-1}\ dx$$

The result of the above integration is the r^{th} moment of the generalized Gompertz distribution (see [11]) with parameters α,λ and l , i.e.

$$\int_0^\infty x^r\alpha l\ e^{\lambda x}e^{-\ \frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\ \left(1-e^{-\ \frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)^{l-1}\ \ dx$$
$$=\alpha l\ \Gamma(r+1)\sum_{m,t=0}^{\infty}C_m^{l-1}\frac{(-1)^{m+t}}{\Gamma(t+1)}\left(\ \frac{\alpha}{\lambda}(m+1)\right)^t\left(-\frac{1}{\lambda(t+1)}\right)^{r+1}e^{\ \frac{\alpha}{\lambda}(m+1)}$$

Therefore, the r^{th} moment of the RGo distribution is

$$E(X^r)_{RGo}=A\alpha\Gamma(r+1)\sum_{m,t=0}^{\infty}\ C_m^{l-1}\frac{(-1)^{m+t}}{\Gamma(t+1)}\left(\ \frac{\alpha}{\lambda}(m+1)\right)^t\left(-\frac{1}{\lambda(t+1)}\right)^{r+1}e^{\ \frac{\alpha}{\lambda}(m+1)} \tag{3.5}$$

Property 2. The RGo characteristic function can be achieved by $C_X(t)_{RGo}=\sum_{r=0}^{\infty}\frac{(it)^r}{r!}E(X^r)_{RGo}$ as

$$C_X(t)_{RGo}=A\alpha\sum_{m,t,r=0}^{\infty}\frac{(it)^r}{r!}\Gamma(r+1)\,C_m^{l-1}\frac{(-1)^{m+t}}{\Gamma(t+1)}\left(\ \frac{\alpha}{\lambda}(m+1)\right)^t\left(-\frac{1}{\lambda(t+1)}\right)^{r+1}e^{\ \frac{\alpha}{\lambda}(m+1)} \tag{3.6}$$

Property 3. The RGo quantile function can be attained through inverting the cdf in (2.5) as

$$x_{(q)-RGo}\ =\frac{1}{\lambda}\ln\left[1-\frac{\lambda}{\alpha}\ln\left(1-e^{-\left(-\frac{2}{\theta}\ln(q)\ \right)^{\frac{1}{2}}}\right)\ \right] \tag{3.7}$$

The median of RGo random variable can be attained from (3.7) by setting $q = \frac{1}{2}$ as

$$Median_{RGo} = x_{\left(\frac{1}{2}\right)-RGo} = \frac{1}{\lambda}\ln\left[1 - \frac{\lambda}{\alpha}\ln\left(1 - e^{-\left(-\frac{2}{\theta}\ln\left(\frac{1}{2}\right)\right)^{\frac{1}{2}}}\right)\right] \tag{3.8}$$

Property 4. The RGo random sample can be simulated based on U that has standard uniform distribution as

$$x_{RGo} = \frac{1}{\lambda}\ln\left[1 - \frac{\lambda}{\alpha}\ln\left(1 - e^{-\left(-\frac{2}{\theta}\ln(u)\right)^{\frac{1}{2}}}\right)\right] \tag{3.9}$$

Property 5. The RGo reliability function and hazard function can be easily obtained from (2.5) and(2.6) as

$$R(x)_{RGo} = 1 - F(x)_{RGo} = 1 - e^{-\frac{\theta}{2}\left[-\ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^2} \tag{3.10}$$

$$h(x)_{RGo} = \frac{f(x)_{RGo}}{R(x)_{RGo}} = \frac{\theta\alpha\, e^{\lambda x}e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\left[-ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]e^{-\frac{\theta}{2}\left[-ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^2}}{\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\left(1-e^{-\frac{\theta}{2}\left[-ln\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)\right]^2}\right)} \tag{3.11}$$

4. Shannon and Relative Entropies of the *RGo* Distribution

The RGo Shannon entropy can be found through

$$Sh_{RGo} = -\int_0^{\infty}\ln\left(f(x)_{RGo}\right)\, f(x)_{RGo}\, dx \tag{4.1}$$

where $f(x)_{RGo}$ represents the pdf of RGo distribution.
Using especial formulas

$$(1-z)^b = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{\Gamma(b+1)}{\Gamma(b-i+1)}\, z^i = \sum_{i=0}^{\infty}(-1)^i\, C_i^b\, z^i\ ;\ |z|<1,\ b>0 \tag{S3}$$

$$(1-z)^{-b} = \sum_{i=0}^{\infty}\frac{\Gamma(b+i)}{i!\,\Gamma(b)}\, z^i\quad ;\ |z|<1,\ b>0 \tag{S4}$$

Now, according to (2.6) and then (3.3), $\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)^{l-1}$ have two cases as

$$\left(1-e^{-\frac{\alpha}{\lambda}\left(e^{\lambda x}-1\right)}\right)^{l-1} = \begin{cases}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^m}{m!}\frac{\Gamma(l)}{\Gamma(l-m)}\, e^{-\frac{\alpha m}{\lambda}\left(e^{\lambda x}-1\right)} & ;\ l-1>0\\ \displaystyle\sum_{m=0}^{\infty}\frac{\Gamma(l-1+m)}{m!\,\Gamma(l-1)}\, e^{-\frac{\alpha m}{\lambda}\left(e^{\lambda x}-1\right)} & ;\ l-1<0\end{cases}$$

For $l-1>0$, $f(x)_{RGo}$ in (3.3) will be

$$f(x)_{RGo} = \sum_{i,k,l,m=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+m}\,(2i+1)}{i!m!\,(2i+1-j)}\frac{\theta^{i+1}}{2^i}C_k^{k-(2i+1)}C_j^k C_l^{2i+1+k}$$
$$P_{j,k}\frac{\Gamma(l)}{\Gamma(l-m)}\alpha\, e^{\lambda x}e^{-\frac{\alpha(m+1)}{\lambda}\left(e^{\lambda x}-1\right)}$$

Using (S1)

$$e^{-\frac{\alpha(m+1)}{\lambda}\left(e^{\lambda x}-1\right)} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left(\frac{\alpha(m+1)}{\lambda}\right)^s \left(e^{\lambda x}-1\right)^s$$

and since

$$(a+b)^n = \sum_{k=0}^{\infty} C_k^n \ a^{n-k} \ b^k = \sum_{k=0}^{\infty} C_k^n \ a^k \ b^{n-k} \quad ; \ n \geq 0 \tag{S5}$$

Now

$$e^{-\frac{\alpha(m+1)}{\lambda}\left(e^{\lambda x}-1\right)} = \sum_{s,t=0}^{\infty} \frac{(-1)^{s+t}}{s!} \left(\frac{\alpha(m+1)}{\lambda}\right)^s C_t^s e^{\lambda t x}$$

Then

$$f(x)_{RGo} = \sum_{i,k,l,m,s,t=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k+l+m+s+t} \ (2i+1)}{i!m!s!\,(2i+1-j)} \frac{\theta^{i+1}}{2^i \lambda^s} C_k^{k-(2i+1)} C_j^k C_l^{2i+1+k} C_t^s P_{j,k} \frac{\Gamma(l)}{\Gamma(l-m)} \alpha^{s+1} (m+1)^s \ e^{\lambda(t+1)x}$$

Similarly for $(l-1) < 0$, $f(x)_{RGo}$ in (3.3) will be

$$f(x)_{RGo} = \sum_{i,k,l,m,s,t=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k+l+s+t} \ (2i+1)}{i!m!s!\,(2i+1-j)} \frac{\theta^{i+1}}{2^i \lambda^s} C_k^{k-(2i+1)} C_j^k C_l^{2i+1+k} C_t^s P_{j,k} \frac{\Gamma(l-1+m)}{\Gamma(l-1)} \alpha^{s+1} (m+1)^s \ e^{\lambda(t+1)x}$$

Let

$$B = \begin{cases} \displaystyle\sum_{i,k,l,m,s,t=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k+l+m+s+t} \ (2i+1)}{i!m!s!\,(2i+1-j)} \frac{\theta^{i+1}\alpha^{s+1}}{2^i \lambda^s} (m+1)^s \, C_k^{k-(2i+1)} C_j^k C_l^{2i+1+k} C_t^s P_{j,k} \ \frac{\mathit{\Gamma}(l)}{\mathit{\Gamma}(l-m)} & ; \ l-1 > 0 \\ \displaystyle\sum_{i,k,l,m,s,t=0}^{\infty} \sum_{j=0}^{k} \frac{(-1)^{i+j+k+l+s+t} \ (2i+1)}{i!m!s!\,(2i+1-j)} \frac{\theta^{i+1}\alpha^{s+1}}{2^i \lambda^s} (m+1)^s \, C_k^{k-(2i+1)} C_j^k C_l^{2i+1+k} C_t^s P_{j,k} \frac{\mathit{\Gamma}(l-1+m)}{\mathit{\Gamma}(l-1)} & ; \ l-1 < 0 \end{cases} \tag{4.2}$$

Then

$$f(x)_{RGo} = B \ e^{\lambda(t+1)x} \tag{4.3}$$

Now, $\ln(f(x)_{RGo})$ in (4.1) can be attained from (4.3) as

$$\ln(f(x)_{RGo}) = \ln(B) + \lambda(t+1)x$$

and then simply, the *RGo* Shannon entropy in (4.1) can be found as

$$Sh_{RGo} = -\ln(B) - \lambda(t+1)E(X)_{RGo} \tag{4.4}$$

where $E(X)_{RGo}$ as in (3.5) with $r = 1$.

Furthermore, the *RGo* relative entropy can be obtained from

$$RE_{RGo} = \int_0^\infty \ln\left(\frac{f(x)_{RGo}}{f_1(x)_{RGo}}\right) f(x)_{RGo}\, dx \tag{4.5}$$

where $f(x)_{RGo}$ and $f_1(x)_{RGo}$ represent the pdf with parameters $(\theta,\ \alpha,\ \lambda)$ and $(\theta_1,\ \alpha_1,\ \lambda_1)$ respectively. With recall (4.3)

$$\begin{aligned}\ln\left(\frac{f\,(x)_{RGo}}{f_1\,(x)_{RGo}}\right) &= \ln\left(\frac{B\ e^{\lambda(t+1)x}}{B_1\ e^{\lambda_1(t+1)x}}\right) = \ln\left(\frac{B}{B_1}\right) + \lambda\,(t+1)\,x - \lambda_1\,(t+1)\,x\\ &= \ln\left(\frac{B}{B_1}\right) + \ (\lambda\,(t+1) - \lambda_1\,(t+1))\,x\end{aligned}$$

The *RGo* relative entropy in (4.5) can be then obtained as

$$RE_{RGo} = \ln\left(\frac{B}{B_1}\right) + \ (\lambda\,(t+1) - \lambda_1\,(t+1))\,E(X)_{RGo} \tag{4.6}$$

Where $B,\ B_1$ as in (4.2) with specific parameters and $E(X)_{RGo}$ as in (3.5) with $r = 1$.

5. The Reliability Stress Strength Model of the *RGo* Distribution

Consider Y(stress) and X(strength) two independent random variables have *RGO* with parameters $(\theta_1,\ \alpha_1,\ \lambda_1)$ and $(\theta,\ \alpha,\ \lambda)$ respectively. The reliability stress strength model can be obtained as

$$SS_{RGo} = P\,(Y < X) = \int_0^\infty f_X(x)_{RGo}\ F_Y(x)_{RGo}\ \ dx \tag{5.1}$$

Recall the cdf in (2.5) with parameters $(\theta_1,\ \alpha_1,\ \lambda_1)$, and based on $(S1)\,,\ (S2)\,,\ (S3)$ we get

$$\begin{aligned}F_Y(x)_{RGo} &= e^{-\ \frac{\theta_1}{2}\ \left[-\ln\left(1-e^{-\ \frac{\alpha_1}{\lambda_1}(e^{\lambda_1 x}-1)}\right)\right]^2}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ \left(\frac{\theta_1}{2}\right)^i\ \left[-\ln\left(1-e^{-\ \frac{\alpha_1}{\lambda_1}(e^{\lambda_1 x}-1)}\right)\ \right]^{2i}\\ &= \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\ \left(\frac{\theta_1}{2}\right)^i\sum_{k,l=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{j+k+l}\ 2i}{2i-j}C_k^{k-2i}C_j^kC_l^{2i+k}P_{j,k}\left(1-e^{-\ \frac{\alpha_1}{\lambda_1}(e^{\lambda_1 x}-1)}\right)^l\\ &= \sum_{i,k,l,s=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+s}\ 2i}{i!\,(2i-j)}\left(\frac{\theta_1}{2}\right)^iC_k^{k-2i}C_j^kC_l^{2i+k}C_s^l\ P_{j,k}\ \mathrm{e}^{-\ \frac{\alpha_1 s}{\lambda_1}\ (e^{\lambda_1 x}-1)}\end{aligned}$$

and for $e^{-\ \frac{\alpha_1 s}{\lambda_1}\ (e^{\lambda_1 x}-1)}$follow the same steps in page 3510, then

$$F_Y(x)_{RGo} = \sum_{i,k,l,s,m,t,r=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+s+2m-t}\ \ (2i)}{i!m!r!\,(2i-j)}\left(\frac{\theta_1}{2}\right)^iC_k^{k-2i}\ C_j^kC_l^{2i+k}C_s^l\ C_t^mP_{j,k}\left(\frac{\alpha_1 s}{\lambda_1}\right)^m(\lambda_1 t)^r x^r$$

Therefore, the stress strength of RGo distribution in (5.1) can be obtained as

$$SS_{RGo}=\sum_{i,k,l,s,m,t,r=0}^{\infty}\sum_{j=0}^{k}\frac{(-1)^{i+j+k+l+s+2m-t}\ (2i)}{i!m!r!\,(2i-j)}\left(\frac{\theta_1}{2}\right)^{i}C_k^{k-2i}\ C_j^k C_l^{2i+k}$$
$$C_s^l\ C_t^m P_{j,k}\left(\frac{\alpha_1 s}{\lambda_1}\right)^{m}(\lambda_1 t)^r E\left(X^r\right)_{RGo} \qquad (5.2)$$

where $E\left(X^r\right)$ as in (3.5).

6. The Maximum Likelihood Estimators of the *RGo* Parameters

For the parameters vector $\tau=(\theta,\ \alpha,\lambda)^T$, the natural logarithm likelihood function of a complete random sample with size n, say $(x_1,x_2,\ldots,x_n)$, follow RGo distribution (recall (2.6)) is

$$\ell\left(\tau|\underline{x}\right)=n\ln\left(\theta\alpha\right)+\lambda\sum_{i=1}^{n}x_i\ -\frac{\alpha}{\lambda}\sum_{i=1}^{n}\left(e^{\lambda x_i}-1\right)-\sum_{i=1}^{n}\ln\left(1-e^{-\ \frac{\alpha}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)$$
$$+\ \sum_{i=1}^{n}\ln\left[-\ln\left(1-e^{-\ \frac{\alpha}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)\right]-\frac{\theta}{2}\sum_{i=1}^{n}\left[-\ln\left(1-e^{-\ \frac{\alpha}{\lambda}\left(e^{\lambda x_i}-1\right)}\right)\right]^2 \qquad (6.1)$$

The maximum likelihood estimators (MLE) of the RGo parameters can be obtained by solving the three nonlinear differential equations $\frac{\partial\ell(\tau|\underline{x})}{\partial\theta}=0,\ \ \frac{\partial\ell(\tau|\underline{x})}{\partial\alpha}=0,\ \ \frac{\partial\ell(\tau|\underline{x})}{\partial\lambda}=0$. The estimators are not in closed forms, so numerical method is used.

7. Real Applications

Three real data sets are considered here.

Data -1: The first data set represents the crude mortality rate among people who inject drugs with a sample size of 65 observations [12].

"2.01, 6.32, 3.52, 2.15, 5.42, 2.04, 2.77, 2.26, 1.95, 1.00, 2.45, 0.74, 0.98, 1.27, 2.77, 3.68, 1.18, 1.09, 1.60, 0.57, 3.33, 0.91, 7.14, 2.08, 3.85, 1.99, 7.76, 2.52, 1.57, 4.67, 4.22, 1.92, 1.59, 4.08, 2.02, 0.84, 6.85, 2.18, 2.04, 1.05, 2.91, 1.37, 2.43, 2.28, 3.74, 1.30, 1.59, 1.83, 3.85, 6.30, 4.83, 0.50, 3.40, 2.33, 4.25, 3.49, 2.12, 0.83, 0.54, 3.23, 4.50, 0.71, 0.48, 2.30, 7.73".

Data -2: The second data set represents the failure times in hours from an accelerated life test with a sample size of 59 conductors. The observations are as follows [13][14].

"6.545, 9.289, 7.543, 6.956, 6.492, 5.459, 8.120, 4.706, 8.687, 2.997, 8.591, 6.129, 11.038, 5.381, 6.958, 4.288, 6.522, 4.137, 7.459, 7.495, 6.573, 6.538, 5.589, 6.087, 5.807, 6.725, 8.532, 9.663, 6.369, 7.024, 8.336, 9.218, 7.945, 6.869, 6.352, 4.700, 6.948, 9.254, 5.009, 7.489, 7.398, 6.033, 10.092, 7.496, 4.531, 7.974, 8.799, 7.683, 7.224, 7.365, 6.923, 5.640, 5.434, 7.937, 6.515, 6.476, 6.071, 10.941, 5.923".

Data -3: The third data set represents the single fibers that were tensioned at 10mm gauge lengths with a sample size of 63. The observations are as follows [15][16].

"1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977,2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020".

For comparison with the new distribution, six additional distributions also fitted including Rayleigh Gompertz (RGo), Gamma Gompertz ($GaGo$), Marshall Olkin Gompertz ($MOGo$), Kumaraswamy Gompertz ($KuGo$), Gompertz (Go), Beta Gompertz ($BeGo$), and Exponentiated Generalized Gompertz ($EGGo$) distributions (for more details about these distributions, also see references [9][17]-[21]. The R software used to compute MLEs of their parameters and the well-known information criteria: ($-\ell$) Negative Log-Likelihood, (AIC) Akaike Information Criteria, ($CAIC$) Consistent Akaike Information Criteria, (BIC) Bayesian Information Criteria, ($HQIC$) Hanan and Quinn Information Criteria. Minimum values of these measures indicate the distribution has a better fitting. The MLEs of the parameters for each model and the information criteria fitted to the different real-data sets are shown in Tables 1-3.

From the results, RGo has the lowest values of information criteria, making it the best fit to describe three considered real data sets. Furthermore, the plots of empirical cdfs, the histogram, and the estimated densities in Figures 3-8 demonstrate this best-fitting.

Table 1: The MLEs of the parameters for the models and the information criteria fitted to data-1.

Model	Estimates				$-\ell$	AIC	$CAIC$	BIC	$HQIC$
RGo	1.0713	0.1038	0.1496	—	118.4245	242.8489	243.2424	249.3721	245.4227
$GaGo$	0.1847	0.1924	1.3696	—	121.9779	249.9558	250.3492	256.4789	252.5296
$MOGo$	13.8406	-0.1374	1.3422	—	121.7713	249.5427	249.9361	256.0658	252.1165
$KuGo$	6.3026	0.1466	0.0982	2.5538	118.1797	244.3594	245.0261	253.0569	247.7911
Go	0.2262	0.2168	—	—	124.2617	252.5234	252.7170	256.8722	254.2393
$BeGo$	4.4803	0.1361	0.1070	2.6861	118.3171	244.6342	245.3008	253.3317	248.0659
$EGGo$	0.8114	2.7460	-0.0097	0.8114	119.0462	246.0924	246.7591	254.7900	249.5242

Table 2: The MLEs of the parameters for the models and the information criteria fitted to data-2.

Model	Estimates				$-\ell$	AIC	$CAIC$	BIC	$HQIC$
RGo	3.0508	0.3013	0.0304	—	111.3743	228.7486	229.1849	234.9812	231.1815
$GaGo$	0.0927	0.3208	0.3449	—	125.7948	257.5896	258.0260	263.8222	260.0226
$MOGo$	0.6639	0.6290	0.0037	—	116.4862	238.9725	239.4088	245.2051	241.4054
$KuGo$	7.7732	0.9610	0.2178	0.1543	111.2992	230.5983	231.3390	238.9085	233.8423
Go	0.4755	0.0131	—	—	119.0604	242.2296	242.4439	246.3847	243.8516
$BeGo$	7.1155	0.6206	0.2618	0.1554	111.2809	230.5617	231.3025	238.8719	233.8057
$EGGo$	0.1337	2.4942	0.2662	0.5407	116.0791	240.1582	240.8989	248.4683	243.4021

Table 3: The MLEs of the parameters for the models and the information criteria fitted to data-3.

Model	Estimates				$-\ell$	AIC	$CAIC$	BIC	$HQIC$
RGo	51.8522	0.2440	0.4308	—	56.1989	118.3979	118.8047	124.8273	120.9266
$GaGo$	0.1478	0.8856	0.3825	—	73.4993	152.9988	153.4055	159.4282	155.5275
$MOGo$	96.5475	0.3163	0.9067	—	62.9824	131.9650	132.3718	138.3944	134.4937
$KuGo$	43.8886	0.2509	0.6835	0.6550	56.1471	120.2944	120.9840	128.8669	123.6660
Go	1.1165	0.0286	—	—	71.6282	147.2727	147.4727	151.5590	148.9585
$BeGo$	35.1650	0.3847	0.5340	0.7535	56.2292	120.4586	121.1482	129.0311	123.8302
$EGGo$	1.1970	70.2495	0.1587	1.0134	56.2571	120.5143	121.2040	129.0869	123.8860

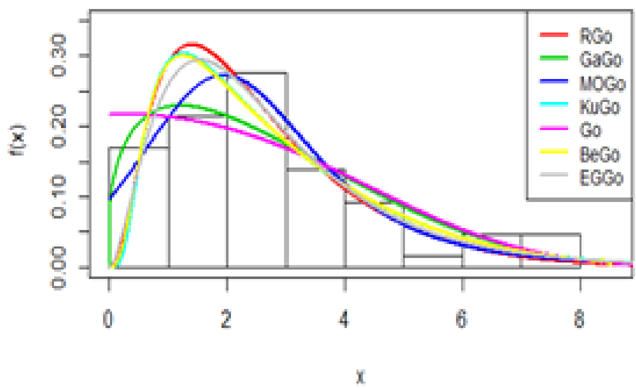

Figure 3: Data-1 histogram plot of RGo with other compared distributions

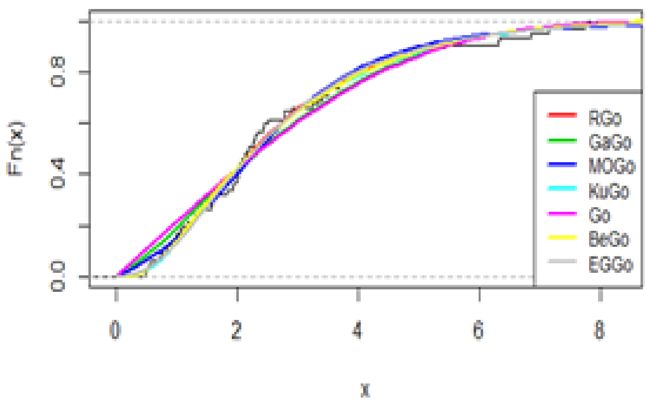

Figure 4: Data-1 empirical cdf plot of RGo with other compared distributions

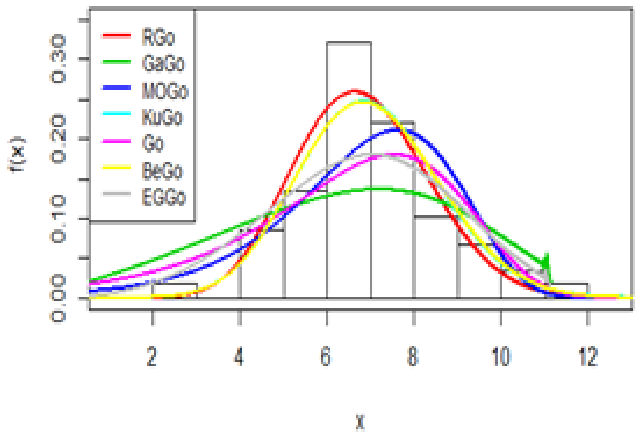

Figure 5: Data-2 histogram plot of *RGo* with other compared distributions

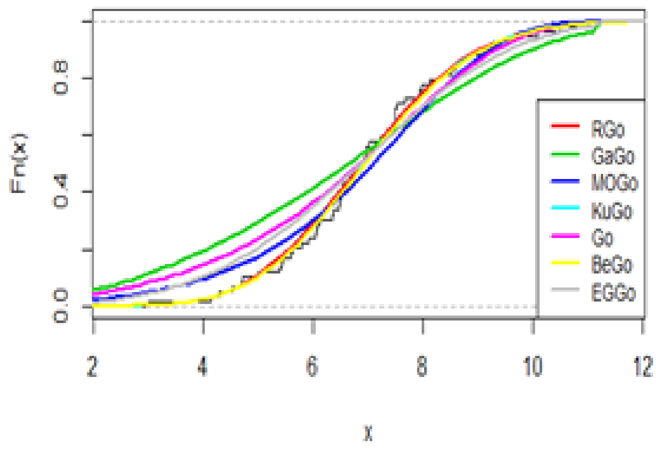

Figure 6: Data-2 empirical cdf plot of *RGo* with other compared distributions

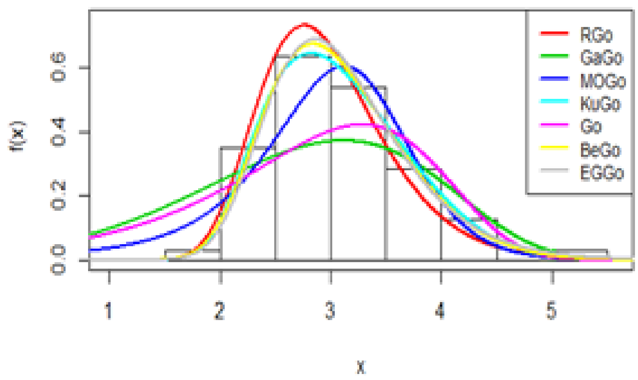

Figure 7: Data-3 histogram plot of *RGo* with other compared distributions

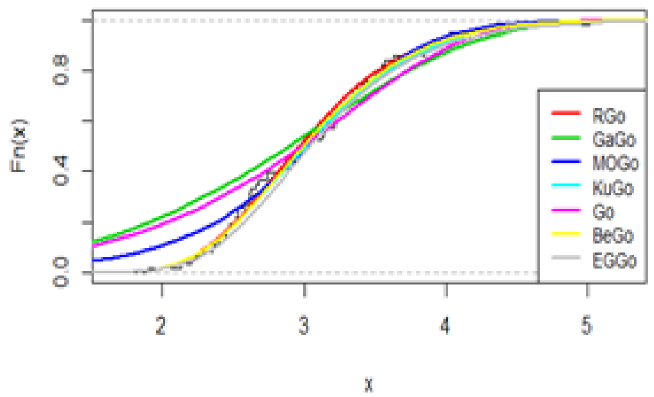

Figure 8: Data-3 empirical cdf plot of *RGo* with other compared distributions

8. Concluding Remarks

A new compound three-parameter model named the Rayleigh Gompertz distribution is introduced. The non-central moments, characteristic function, quantile function, median, simulated data, reliability and hazard functions, Shannon and relative entropies, and the stress-strength model of reliability are provided. The estimation of the parameters by maximum likelihood is discussed. Three applications of the new distribution are given to prove its flexibility to fit different real-life data compared with six other known distributions.

References

[1] N. Eugene, C. Lee and F. Famoye, *Beta-normal distribution and its applications*, Commun. Statist.- Theory Meth. 31(4) (2002) 497–512.

[2] A.W. Marshall and I. Olkin , *A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families*, Biometrika, 84(5) (1997) 641–652.

[3] K. Zografos and N. Balakrishnan, *On families of beta and generalized gamma-generated distributions and associated inference*, Statist. Meth. 6(4) (2009) 344–362.

[4] G.M. Cordeiro and M. de Castro, *A new family of generalized distributions*, J. Stat. Comput. Simul. 81(7) (2011) 883–898.

[5] M. Bourguignon, R.B. Silva and G.M. Cordeiro, *The Weibull-G family of probability distributions*, J. Data Sci. 12(1)(2014) 53–68.

[6] N.H. Al-Noor and N. K. Assi, *Rayleigh-Rayleigh distribution: properties and applications*, J. Phys. Conf. Ser. 1591 (2020) 012038.

[7] T.I. Missov and A. Lenart, *Linking period and cohort life-expectancy linear increases in Gompertz proportional hazards models*, Demographic Res. 24 (2011) 455–468.

[8] J.H. Pollard and E.J. Valkovics, *The Gompertz distribution and its applications*, Genus. 48(3)(1992) 15–28.

[9] A.A. Jafari, S. Tahmasebi and M. Alizadeh, *The beta-Gompertz distribution*, Rev. Colomb. de Estadíst. 37 (1)(2014) 139–156.

[10] N.H. Al-Noor and N.K. Assi, *Rayleigh Gamma Gompertz distribution: properties and applications*, AIP Conf. Proc. 2334(1)(2021) 090003.

[11] A. El-Gohary, A. Alshamrani and A.N. Al-Otaibi, *The generalized Gompertz distribution*, Appl. Math. Modell. 37(1-2) (2013) 13-24.

[12] B.M. Mathers, L. Degenhardt, C. Bucello, J. Lemon, L. Wiessing and M. Hickman, *Mortality among people who inject drugs: a systematic review and meta-analysis*, Bull. World Health Organ. 91(2) (2013) 102–123.

[13] P. Nasiri and H. Pazira, *Bayesian approach on the generalized exponential distribution in the presence of outliers*, J. Statist. Theory Pract. 4(3) (2010) 453–475.

[14] H.A. Schafft, T.C. Staton, J. Mandel and J.D. Shott, *Reproducibility of electromigration measurements*, IEEE Trans. Electron. Devices 34(3) (1987) 673–681.

[15] D. Kundu and M.Z. Raqab, *Estimation of $R = P(X < Y)$ for three parameter Weibull distribution*, Stat. Probab. Lett. 79(17) (2009) 1839–1846.

[16] M.E. Ghitany, R.A. Al-Jarallah and N. Balakrishnan, *On the existence and uniqueness of the MLEs of the parameters of a general class of exponentiated distributions*, Statist. 47(3) (2013) 605–612.

[17] T.I. Missov, *Gamma-Gompertz life expectancy at birth*, Demographic Res. 28 (2013) 259–270.

[18] S. Yaghoobzadeh, *A new generalization of the Marshall–Olkin Gompertz distribution*, Int. J. Syst. Assur. Eng. Manage. 8 (2017) 1580–1587.

[19] R.C. da Silva, J.J.D. Sanchez, F.P. Lima and G.M. Cordeiro, *The Kumaraswamy Gompertz distribution*, J. Data Sci. 13(2015) 241–260.

[20] M.M.M. El-Din, Y. Abdel-Aty and M.H. Abu-Moussa, *Statistical inference for the Gompertz distribution based on type-ii progressively hybrid censored data*, Commun. Stat.- Simul. Comput. 46(8) (2017) 6242–6260.

[21] O.A. Ade, *Performance rating of the exponentiated generalized Gompertz Makeham distribution: An analytical approach*, Am. J. Theor. Appl. Statist. 6(5) (2017) 228–235.