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Some fractional weighted trapezoid type inequalities for preinvex functions

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Abstract

In this paper, we establish a weighted integral identity for preinvex functions. Some new trapezoidal type inequalities are derived for functions whose modulus of the first derivatives are preinvex via Riemann-Liouville fractional operators.

Keywords: Weighted functions, preinvex functions, Hölder inequality, Condition C.
2010 MSC: 26D10, 26D15, 26A51.

1. Introduction

Let I be an interval of real numbers. A function $f : I \to \mathbb{R}$ is said to be convex, if for all $x, y \in I$ and all $t \in [0, 1]$ (see [43]), we have

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y).$$

Hermite-Hadamard inequality is one of the well-known inequalities for the class of convex functions (see [21, 24]). It can be stated as follows: For every convex function f on the interval $[a, b]$ with $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\left(x\right)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}. \tag{1.1}$$

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The aformentioned inequality has attracted different researchers' attention. Accordingly, several variants, extentions, generalizations and improvements have appeared in the literature. We recommend the references [1, 2, 14, 17, 18, 22, 28, 29, 33, 34, 48] for further details.

In [18], Dragomir and Agarwal provided the following results related to inequality (1.1).

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Pearce and Pečarić [42], established another variant of the second results given in [18] as follows

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Kavurmaci et al. [29], showed the following results which represents a refinement of those established in [18, 42] as follows

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{12}\left(|f'(a)|+\left|f'\left(\frac{a+b}{2}\right)\right|+|f'(b)|\right)$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{4}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}}$$
$$\times\left(\left(|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right),$$

and

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\,du\right| \leq \frac{b-a}{8}\left(\frac{1}{3}\right)^{\frac{1}{q}}$$
$$\times\left(\left(2|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}+\left(2|f'(b)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right),$$

Various mathematicians have recently moved to great lengths in order to generalize classical convexity. Hanson [23] introduced one of the vital generalizations called preinvex functions. It is worth noting that different authors have studied their fundamental properties and their role in optimization, variational inequalities, and equilibrium problems. We suggest [39, 40, 44, 50, 51] to interested readers.

We recall that a function $f : K \to \mathbb{R}$ defined on the invex set K (i.e. $x + t\eta(y,x) \in K$ for all $x, y \in K$ and $t \in [0,1]$) is said to be preinvex with respect to η, if

$$f(x + t\eta(y,x)) \le (1-t) f(x) + tf(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$ (see [50]).

Noor [41], provided the analogue preinvex of inequality (1.1)

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1.2}$$

According to Barani et al. [6], the analogue preinvex of the results provided in [18] is as follows:

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,du \right| \le \frac{|\eta(b,a)|}{8} \left(|f'(a)| + |f'(b)|\right)$$

and

$$\left| \frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)\,du \right| \le \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}} \left(|f'(a)|^q + |f'(b)|^q\right)^{\frac{1}{q}}.$$

Fractional calculus is a field of mathematics tackling arbitrary derivation and integration. It has been extended to other disciplines such as biology (see [19]), economics (see [20]), physics (see [25]), and engineering sciences (see [13]).

Numerous mathematicians have presented a great number of fractional integral and derivative operators as well as their generalizations and extensions. Further essential details can be found in [3, 9, 26, 27, 30, 31, 38, 45, 47].

The above mentioned fractional operators has been widely used by many researchers mostly in the fields of integral inequalities see [5, 7, 10, 11, 12, 32, 35, 46].

In [46], Sarikaya et al. proved the following inequalities of Hermite-Hadamard including Riemann-Liouville fractional integrals.

Theorem 1.1. *Let $g : [a,b] \to \mathbb{R}$ be a convex function with $0 \le a < b$ and $g \in L([a,b])$. If $g' \in L([a,b])$, then the following inequality for fractional integrals holds:*

$$g\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a^+}^\alpha g(b) + J_{b^-}^\alpha g(a)\right] \le \frac{g(a)+g(b)}{2} \tag{1.3}$$

where $J_{a^+}^\alpha g$ and $J_{b^-}^\alpha g$ are the Riemann-Liouville integrals of order $\alpha > 0$ defined by

$$J_{a^+}^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t)dt, \quad x > a,$$

$$J_{b^-}^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t)dt, \quad b > x,$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$, is the Gamma function and $J_{a^+}^0 g(x) = J_{b^-}^0 g(x) = g(x)$.

The trapezoid type inequality related to (1.3) have been obtained in the same paper as follow.

Theorem 1.2. *Let $g : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|g|'$ is a convex function on $[a, b]$, then the following inequality for Riemann-Liouville fractional integrals holds:*

$$\left|\frac{g(a)+g(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} g(b) + J_{b^-}^{\alpha} g(a)\right]\right| \leq \frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^{\alpha}}\right)\left[|g'(a)|+|g'(b)|\right] \tag{1.4}$$

M.R. Delavar [16], in his recent research work, presented some refinements for inequality (1.4), where the convexity condition for the absolute value of the derivative of considered function is replaced by boundless and a Lipschitzian condition for the derivative.
Inspired and motivated by the obtained results in [4, 6, 15, 18, 29], we, first, establish, in the present paper, a new identity for the product of two functions. Then, we derive a new weighted inequalities, of the Hermite-Hadamard type, for preinvex functions differentiable via Reimann-Liouville fractional operators.

2. Preliminaries

Condition C. ([37]) Let K be an invex set with respect to the bi-function $\eta(.,.)$ then, for any $a, b \in K$ and $t \in [0.1]$, we have

$$\eta(a, a+t\eta(b,a)) = -t\eta(b,a) \text{ and } \eta(b, a+t\eta(b,a)) = (1-t)\,\eta(b,a)\,.$$

From Condition C, it follows that

$$\eta(a+t_2\eta(b,a), a+t_1\eta(b,a)) = (t_2-t_1)\,\eta(b,a)$$

holds for any $a, b \in K$ and $t_1, t_2 \in [0.1]$.

Lemma 2.1. *([49]) For any $0 \leq a < b$ in $\mathbb{R}$ and $0 < \alpha \leq 1$, we have*

$$|a^{\alpha} - b^{\alpha}| \leq (b-a)^{\alpha}\,.$$

Lemma 2.2. *([8]) For any $a, b > 0$ and $\alpha \leq 1$, we have*

$$a^{\alpha} + b^{\alpha} \leq 2^{1-\alpha}(a+b)^{\alpha}\,.$$

3. Main results

In order to prove our results, we need the following lemmas

Lemma 3.1. *Let $w : [a, a+\eta(b,a)] \to \mathbb{R}$ be symmetric with respect to $\frac{2a+\eta(b,a)}{2}$ and $w \in L([a, a+\eta(b,a)])$, then we have*

$$J_{a^+}^{\alpha} w\left(\frac{2a+\eta(b,a)}{2}\right) = J_{(a+\eta(b,a))^-}^{\alpha} w\left(\frac{2a+\eta(b,a)}{2}\right) \tag{3.1}$$

and

$$J_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}^{\alpha} w(a) = J_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}^{\alpha} w(a+\eta(b,a)). \tag{3.2}$$

If we take $w(u) = \frac{1}{\eta(b,a)}$ for all $u \in [a, a+\eta(b,a)]$, then we have

$$\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta^{\alpha-1}(b,a)}\left(J_{a^+}^{\alpha} w\left(\frac{2a+\eta(b,a)}{2}\right) + J_{\left(\frac{2a+\eta(b,a)}{2}\right)^+}^{\alpha} w(a+\eta(b,a))\right) = \frac{1}{\alpha}. \tag{3.3}$$

Proof . From definition of the Riemann-Liouville integrals and symmetry of $w(u)$, we have

$$\begin{aligned}
J_{a^+}^{\alpha} w(\frac{2a+\eta(b,a)}{2}) &= \frac{1}{\Gamma(\alpha)} \int_a^{\frac{2a+\eta(b,a)}{2}} \left(\frac{2a+\eta(b,a)}{2} - t\right)^{\alpha-1} w(t)dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(\frac{2a+\eta(b,a)}{2} - (2a+\eta(b,a)-t)\right)^{\alpha-1} \\
&\quad \times w(2a+\eta(b,a)-t)dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(t - \frac{2a+\eta(b,a)}{2}\right)^{\alpha-1} w(t)dt \\
&= J_{(a+\eta(b,a))^-}^{\alpha} w(\frac{2a+\eta(b,a)}{2}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
J_{(\frac{2a+\eta(b,a)}{2})^-}^{\alpha} w(a) &= \frac{1}{\Gamma(\alpha)} \int_a^{\frac{2a+\eta(b,a)}{2}} (t-a)^{\alpha-1} w(t)dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} (a+\eta(b,a)-t)^{\alpha-1} w(2a+\eta(b,a)-t)dt \\
&= \frac{1}{\Gamma(\alpha)} \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} (a+\eta(b,a)-t)^{\alpha-1} w(t)dt \\
&= J_{(\frac{2a+\eta(b,a)}{2})^+}^{\alpha} f(a+\eta(b,a)).
\end{aligned}$$

Now, assume that $w(u) = \frac{1}{\eta(b,a)}$ for all $u \in [a, a+\eta(b,a)]$, we have

$$\begin{aligned}
&\frac{2^{\alpha-1}\Gamma(\alpha)}{\eta^{\alpha-1}(b,a)} \left(J_{a^+}^{\alpha} w\left(\frac{2a+\eta(b,a)}{2}\right) + J_{(\frac{2a+\eta(b,a)}{2})^+}^{\alpha} w(a+\eta(b,a)) \right) \\
&= \frac{2^{\alpha-1}}{\eta^{\alpha}(b,a)} \left(\int_a^{\frac{2a+\eta(b,a)}{2}} \left(\frac{2a+\eta(b,a)}{2} - t\right)^{\alpha-1} dt + \int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} (a+\eta(b,a)-t)^{\alpha-1} dt \right) \\
&= \frac{2^{\alpha-1}}{\eta^{\alpha}(b,a)} \left(\frac{1}{\alpha}\left(\frac{\eta(b,a)}{2}\right)^{\alpha} + \frac{1}{\alpha}\left(\frac{\eta(b,a)}{2}\right)^{\alpha} \right) = \frac{2^{\alpha-1}}{\eta^{\alpha}(b,a)} \times \frac{2}{\alpha}\left(\frac{\eta(b,a)}{2}\right)^{\alpha} = \frac{1}{\alpha}.
\end{aligned}$$

□

Lemma 3.2. *Let $f : I = [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable function on I°, with $a < a+\eta(b,a)$, and let $w : [a, a+\eta(b,a)] \to \mathbb{R}$ be symmetric with respect to $\frac{2a+\eta(b,a)}{2}$. If $f', w \in L([a, a+\eta(b,a)])$,* *then*

$$\Xi\left(a,b,\eta,\alpha,w,f\right)=\frac{\left(\eta\left(b,a\right)\right)^{2}}{8}$$
$$\times\left(\int_{0}^{1}p_{2}\left(t\right)f'\left(a+\left(1-\frac{1}{2}t\right)\eta\left(b,a\right)\right)dt-\int_{0}^{1}p_{1}\left(t\right)f'\left(a+\frac{1}{2}t\eta\left(b,a\right)\right)dt\right),$$

where

$$\Xi\left(a,b,\eta,\alpha,w,f\right)=\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{\eta^{\alpha-1}\left(b,a\right)}$$
$$\times\left(J_{a^{+}}^{\alpha}w\left(\frac{2a+\eta\left(b,a\right)}{2}\right)+J_{\left(\frac{2a+\eta(b,a)}{2}\right)^{+}}^{\alpha}w(a+\eta\left(b,a\right))\right)\frac{f\left(a\right)+f\left(a+\eta\left(b,a\right)\right)}{2}$$
$$-\frac{2^{\alpha-1}\Gamma\left(\alpha\right)}{2\eta^{\alpha-1}\left(b,a\right)}\left(J_{a^{+}}^{\alpha}wf\left(\frac{2a+\eta\left(b,a\right)}{2}\right)+J_{\left(\frac{2a+\eta(b,a)}{2}\right)^{-}}^{\alpha}wf(a)\right.$$
$$\left.+J_{(a+\eta(b,a))^{-}}^{\alpha}wf\left(\frac{2a+\eta\left(b,a\right)}{2}\right)+J_{\left(\frac{2a+\eta(b,a)}{2}\right)^{+}}^{\alpha}wf(a+\eta\left(b,a\right))\right). \tag{3.4}$$

and

$$p_{1}\left(t\right)=\int_{t}^{1}\left(\left(1-s\right)^{\alpha-1}+s^{\alpha-1}\right)w\left(a+\frac{1}{2}s\eta\left(b,a\right)\right)ds \tag{3.5}$$

and

$$p_{2}\left(t\right)=\int_{t}^{1}\left(\left(1-s\right)^{\alpha-1}+s^{\alpha-1}\right)w\left(a+\left(1-\frac{1}{2}s\right)\eta\left(b,a\right)\right)ds. \tag{3.6}$$

Proof . Integrating by parts and changing the variables, we obtain

$$\begin{aligned}
&\int_0^1 p_1(t)\, f'\left(a+\frac{1}{2}t\eta(b,a)\right) dt \\
=\;& \int_0^1 \left(\int_t^1 \left((1-s)^{\alpha-1}+s^{\alpha-1}\right) w\left(a+\frac{1}{2}s\eta(b,a)\right) ds\right) f'\left(a+\frac{1}{2}t\eta(b,a)\right) dt \\
=\;& \frac{2}{\eta(b,a)}\left(\int_t^1 \left((1-s)^{\alpha-1}+s^{\alpha-1}\right) w\left(a+\frac{1}{2}s\eta(b,a)\right) ds\right) f\left(a+\frac{1}{2}t\eta(b,a)\right)\Bigg|_{t=0}^{t=1} \\
&+\frac{2}{\eta(b,a)}\int_0^1 \left(\left((1-t)^{\alpha-1}+t^{\alpha-1}\right) w\left(a+\frac{1}{2}t\eta(b,a)\right)\right) f\left(a+\frac{1}{2}t\eta(b,a)\right) dt \\
=\;& -\frac{2}{\eta(b,a)}\left(\int_0^1 \left((1-s)^{\alpha-1}+s^{\alpha-1}\right) w\left(a+\frac{1}{2}s\eta(b,a)\right) ds\right) f(a) \\
&+\frac{2}{\eta(b,a)}\int_0^1 \left(\left((1-t)^{\alpha-1}+t^{\alpha-1}\right) w\left(a+\frac{1}{2}t\eta(b,a)\right)\right) f\left(a+\frac{1}{2}t\eta(b,a)\right) dt \\
=\;& -\frac{2^{\alpha+1}}{\eta^{\alpha+1}(b,a)}\left(\left(\int_a^{\frac{2a+\eta(b,a)}{2}} \left(\frac{2a+\eta(b,a)}{2}-u\right)^{\alpha-1} w(u)\,du + \int_a^{\frac{2a+\eta(b,a)}{2}} (u-a)^{\alpha-1} w(u)\,du\right) f(a)\right. \\
&\left.+\int_a^{\frac{2a+\eta(b,a)}{2}} (u-a)^{\alpha-1} w(u) f(u)\,du + \int_a^{\frac{2a+\eta(b,a)}{2}} \left(\frac{2a+\eta(b,a)}{2}-u\right)^{\alpha-1} w(u) f(u)\,du\right) \\
=\;& -\frac{2^{\alpha+1}\Gamma(\alpha)}{\eta^{\alpha+1}(b,a)}\left(\left(J_{a^+}^{\alpha} w\left(\frac{2a+\eta(b,a)}{2}\right) + J_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}^{\alpha} w(a)\right) f(a)\right. \\
&\left.+J_{\left(\frac{2a+\eta(b,a)}{2}\right)^-}^{\alpha} wf(a) + J_{a^+}^{\alpha} wf\left(\frac{2a+\eta(b,a)}{2}\right)\right).
\end{aligned} \tag{3.7}$$

Similarly, we have

$$
\begin{aligned}
&\int_0^1 p_2(t) f'\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right)dt \\
=\ & \int_0^1 \left(\int_t^1 \left((1-s)^{\alpha-1}+s^{\alpha-1}\right) w\left(a+\left(1-\frac{1}{2}s\right)\eta(b,a)\right)\right) f'\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right)dt \\
=\ & \frac{2}{\eta(b,a)}\left(\int_0^1 \left((1-s)^{\alpha-1}+s^{\alpha-1}\right) w\left(a+\left(1-\frac{1}{2}s\right)\eta(b,a)\right)ds\right) f\left(a+\eta(b,a)\right) \\
& -\frac{2}{\eta(b,a)}\int_0^1 \left((1-t)^{\alpha-1}+t^{\alpha-1}\right) w\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right) f\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right)dt \\
=\ & \frac{2^{\alpha+1}}{\eta^{\alpha+1}(b,a)}\left(\left(\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(u-\frac{2a+\eta(b,a)}{2}\right)^{\alpha-1} w(u)\,du\right.\right. \\
& \left.+\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(a+\eta(b,a)-u\right)^{\alpha-1} w(u)\,du\right) f\left(a+\eta(b,a)\right) \\
& -\left(\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(u-\frac{2a+\eta(b,a)}{2}\right)^{\alpha-1} w(u) f(u)\,du\right. \\
& \left.\left.+\int_{\frac{2a+\eta(b,a)}{2}}^{a+\eta(b,a)} \left(a+\eta(b,a)-u\right)^{\alpha-1} w(u) f(u)\,du\right)\right) \\
=\ & \frac{2^{\alpha+1}\Gamma(\alpha)}{\eta^{\alpha+1}(b,a)} \\
& \times\left(\left(J^{\alpha}_{(a+\eta(b,a))^-} w\left(\frac{2a+\eta(b,a)}{2}\right)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^+} w(a+\eta(b,a))\right) f\left(a+\eta(b,a)\right)\right. \\
& \left.-\left(J^{\alpha}_{(a+\eta(b,a))^-} wf\left(\frac{2a+\eta(b,a)}{2}\right)+J^{\alpha}_{\left(\frac{2a+\eta(b,a)}{2}\right)^+} wf(a+\eta(b,a))\right)\right).
\end{aligned}
\tag{3.8}
$$

Subtracting (3.7) from (3.8), using (3.1) and (3.2), and then multiplying the resulting equality by $\frac{(\eta(b,a))^2}{8}$, we obtain the desired result. □

Theorem 3.3. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function on $[a,a+\eta(b,a)]$ and $|f'|\in L([a,a+\eta(b,a)])$ with $\eta(b,a)>0$ and assume that Condition C holds, and let $w:[a,a+\eta(b,a)]\to$*

$\mathbb{R}$ *be continuous and symmetric function with respect to* $\frac{2a+\eta(b,a)}{2}$. *If* $|f'|$ *is preinvex function, we have*

$$\begin{aligned}&|\Xi(a,b,\eta,\alpha,w,f)|\\ \leq\ & \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left[\left(1-\frac{2\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\right.\\ &\left.+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)(|f'(a)|+|f'(a+\eta(b,a))|)\right].\end{aligned}$$

Proof . From Lemma 3.2, properties of modulus, Condition C, and preinvexity of $|f'|$, we have

$$\begin{aligned}&|\Xi(a,b,\eta,\alpha,w,f)|\\ \leq\ & \frac{(\eta(b,a))^2}{8}\left[\int_0^1|p_2(t)|\left|f'\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right)\right|dt+\int_0^1|p_1(t)|\left|f'\left(a+\frac{1}{2}t\eta(b,a)\right)\right|dt\right]\\ \leq\ & \frac{(\eta(b,a))^2}{8\alpha}\|\omega\|_{[a,a+\eta(b,a)],\infty}\\ &\times\left[\int_0^1[1-t^\alpha+(1-t)^\alpha]\left|f'\left(a+\left(1-\frac{1}{2}t\right)\eta(b,a)\right)\right|dt\right.\\ &\left.+\int_0^1[1-t^\alpha+(1-t)^\alpha]\left|f'\left(a+\frac{1}{2}t\eta(b,a)\right)\right|dt\right].\end{aligned}\tag{3.9}$$

Note that from Condition C, we have

$$\frac{1}{2}\eta(b,a)=\eta\left(\frac{2a+\eta(b,a)}{2},a\right)$$

which implies

$$a+t\frac{\eta(b,a)}{2}=a+t\eta\left(\frac{2a+\eta(b,a)}{2},a\right),\tag{3.10}$$

on the other hand, we have

$$\frac{1}{2}\eta(b,a)=\left(1-\frac{1}{2}\right)\eta(b,a)=\eta\left(a+\eta(b,a),\frac{2a+\eta(b,a)}{2}\right),$$

which gives

$$\frac{2a+\eta(b,a)}{2}+t\frac{\eta(b,a)}{2}=\frac{2a+\eta(b,a)}{2}+t\eta\left(a+\eta(b,a),\frac{2a+\eta(b,a)}{2}\right).\tag{3.11}$$

Since $|f'|$ is preinvex function, and from (3.10) and (3.11), we have

$$\begin{aligned}&\left|f'\left(a+t\frac{\eta(b,a)}{2}\right)\right|=\left|f'\left(a+t\eta\left(\frac{2a+\eta(b,a)}{2},a\right)\right)\right|\\ \leq\ & (1-t)|f'(a)|+t\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|,\end{aligned}\tag{3.12}$$

and

$$\begin{aligned}
&\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right| = \left|f'\left(a+\left(\frac{1}{2}+\frac{1-t}{2}\right)\eta(b,a)\right)\right| \\
= &\left|f'\left(\frac{2a+\eta(b,a)}{2}+(1-t)\left(\frac{1}{2}\eta(b,a)\right)\right)\right| \\
= &\left|f'\left(\frac{2a+\eta(b,a)}{2}+(1-t)\eta\left(a+\eta(b,a),\frac{2a+\eta(b,a)}{2}\right)\right)\right| \\
\leq &\ t\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+(1-t)\left|f'\left(a+\eta(b,a)\right)\right|. \qquad (3.13)
\end{aligned}$$

Using (3.12) and (3.13) in (3.9), we get

$$\begin{aligned}
&\left|\Xi(a,b,\eta,\alpha,w,f)\right| \\
\leq &\ \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty} \\
&\left[\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right]\left(t\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|+(1-t)\left|f'(a+\eta(b,a))\right|\right)dt\right. \\
&\left.+\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right]\left((1-t)\left|f'(a)\right|+t\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\right)dt\right] \\
= &\ \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left[2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right]tdt\right. \\
&\left.+\left(\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right)\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right](1-t)\,dt\right] \\
= &\ \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left[\left(1-\frac{2\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|\right. \\
&\left.+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right)\right],
\end{aligned}$$

where we have used the fact that

$$\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right]tdt=\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}, \qquad (3.14)$$

and

$$\int_0^1\left[1-t^\alpha+(1-t)^\alpha\right](1-t)\,dt=\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}. \qquad (3.15)$$

□

Corollary 3.4. *In Theorem 3.3, if we use the preinvexity of $|f'|$, we get*

$$\begin{aligned}
&\left|\Xi(a,b,\eta,\alpha,w,f)\right| \\
\leq &\ \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\left|f'(a)\right|+\left|f'(a+\eta(b,a))\right|\right). \qquad (3.16)
\end{aligned}$$

Remark 3.5. *Corollary 3.4 will be reduced to Theorem 2.1 from [6], if we take* $w(u) = \frac{1}{\eta(b,a)}$ *and* $\alpha = 1$. *Moreover, if we choose* $\eta(b,a) = b - a$ *we get Theorem 2.2 from [18].*

Corollary 3.6. *In Theorem 3.3, if we take* $\eta(b,a) = b - a$ *we obtain*

$$\begin{aligned} & \left|\Xi\left(a, b, b-a, \alpha, w, f\right)\right| \\ \leq\ & \frac{(b-a)^2}{8\alpha}\|w\|_{[a,b],\infty}\left[\left(1 - \frac{2\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2}\right)\right| \right. \\ & \left. + \left(\frac{1}{2} + \frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\right]. \end{aligned} \tag{3.17}$$

Moreover, if we use the convexity of $|f'|$, *we get*

$$\left|\Xi\left(a, b, b-a, \alpha, w, f\right)\right| \leq \frac{(b-a)^2}{8\alpha}\|w\|_{[a,b],\infty}\left(\left|f'(a)\right| + \left|f'(b)\right|\right). \tag{3.18}$$

Corollary 3.7. *In Theorem 3.3, if we take* $\alpha = 1$, *we obtain*

$$\begin{aligned} & \left|\frac{f(a) + f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du - \int_a^{a+\eta(b,a)} w(u) f(u)\,du\right| \\ \leq\ & \frac{(\eta(b,a))^2}{12}\|w\|_{[a,a+\eta(b,a)],\infty} \\ & \times\left(\left|f'(a)\right| + \left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right| + \left|f'(a+\eta(b,a))\right|\right). \end{aligned} \tag{3.19}$$

Moreover, if we use the preinvexity of $|f'|$, *we get*

$$\begin{aligned} & \left|\frac{f(a) + f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du - \int_a^{a+\eta(b,a)} w(u) f(u)\,du\right| \\ \leq\ & \frac{(\eta(b,a))^2}{8}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\left|f'(a)\right| + \left|f'(a+\eta(b,a))\right|\right). \end{aligned} \tag{3.20}$$

Remark 3.8. *Theorem 3.3 will be reduced to Corollary 2 from [29], if we take* $w(u) = \frac{1}{\eta(b,a)}$, $\eta(b,a) = b - a$ *and* $\alpha = 1$.

Theorem 3.9. *Let* $f : [a, a+\eta(b,a)] \to \mathbb{R}$ *be a differentiable function on* $[a, a+\eta(b,a)]$ *and* $|f'| \in L([a, a+\eta(b,a)])$ *with* $\eta(b,a) > 0$ *and assume that Condition C holds, and let* $w : [a, a+\eta(b,a)] \to \mathbb{R}$ *be continuous and symmetric function with respect to* $\frac{2a+\eta(b,a)}{2}$. *If* $|f'|^q$ *is preinvex function and* $q > 1$ *with* $\frac{1}{q} + \frac{1}{p} = 1$, *we have*

$$\begin{aligned} & \left|\Xi\left(a, b, \eta, \alpha, w, f\right)\right| \leq \frac{(\eta(b,a))^2}{4\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \\ & \times\left(\left(\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q + \left|f'(a+\eta(b,a))\right|^q\right)^{\frac{1}{q}} + \left(\left|f'(a)\right|^q + \left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.21}$$

Proof . From Lemma 3.2, properties of modulus, Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\Xi\left(a,b,\eta,\alpha,w,f\right)\right| \\
\leq\;& \frac{\left(\eta\left(b,a\right)\right)^{2}}{8}\left[\left(\int_{0}^{1}\left|p_{2}(t)\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_{0}^{1}\left|p_{1}(t)\right|^{p}dt\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right] \\
\leq\;& \frac{\left(\eta\left(b,a\right)\right)^{2}}{8\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left(\int_{0}^{1}\left[1-t^{\alpha}+(1-t)^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}} \\
&\times\left[\left(\int_{0}^{1}\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{0}^{1}\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].
\end{aligned} \tag{3.22}$$

Now, from Lemma 2.1 we have

$$1-t^{\alpha}\leq\left(1-t\right)^{\alpha}. \tag{3.23}$$

So, from (3.23) we can easily see that

$$\begin{aligned}
\left(\int_{0}^{1}\left[1-t^{\alpha}+(1-t)^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}} &\leq \left(\int_{0}^{1}\left[(1-t)^{\alpha}+(1-t)^{\alpha}\right]^{p}dt\right)^{\frac{1}{p}} \\
&= \left(\int_{0}^{1}2^{p}\left(1-t\right)^{\alpha p}dt\right)^{\frac{1}{p}} \\
&= 2\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.24}$$

Substituting (3.12), (3.13), (3.24) in (3.22), then using the preinvexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\Xi\left(a,b,\eta,\alpha,w,f\right)\right| \\
\leq\ & \frac{\left(\eta\left(b,a\right)\right)^2}{4\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q\int\limits_0^1 tdt+\left|f'\left(a+\eta(b,a)\right)\right|^q\int\limits_0^1(1-t)dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\left|f'\left(a\right)\right|^q\int\limits_0^1(1-t)dt+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q\int\limits_0^1 tdt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{\left(\eta\left(b,a\right)\right)^2}{4\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{q}} \\
&\times\left[\left(\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+\left|f'\left(a+\eta(b,a)\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(a\right)\right|^q+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Corollary 3.10. *In Theorem 3.9, if we use the preinvexity of $|f'|^q$, we get*

$$\begin{aligned}
&\left|\Xi\left(a,b,\eta,\alpha,w,f\right)\right|\leq\frac{\left(\eta\left(b,a\right)\right)^2}{4\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\times\left(\left(\frac{\left|f'\left(a\right)\right|^q+3\left|f'\left(a+\eta(b,a)\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(a\right)\right|^q+\left|f'\left(a+\eta\left(b,a\right)\right)\right|^q}{4}\right)^{\frac{1}{q}}\right).
\end{aligned}\tag{3.25}$$

Corollary 3.11. *In Theorem 3.9, if we take $\eta\left(b,a\right)=b-a$ we obtain*

$$\begin{aligned}
&\times\left|\Xi\left(a,b,b-a,\alpha,w,f\right)\right|\leq\frac{\left(b-a\right)^2}{4\alpha}\left\|w\right\|_{[a,b)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\left[\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(b\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(a\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{3.26}$$

In addition, if we use the convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\Xi\left(a,b,b-a,\alpha,w,f\right)\right|\leq\frac{\left(b-a\right)^2}{4\alpha}\left\|w\right\|_{[a,b)],\infty}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{\left|f'\left(a\right)\right|^q+3\left|f'\left(b\right)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'\left(a\right)\right|^q+\left|f'\left(b\right)\right|^q}{4}\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{3.27}$$

Corollary 3.12. *In Theorem 3.9, if we take $\alpha = 1$, we obtain*

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right|\\ \leq\ & \frac{(\eta(b,a))^2}{4}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left[\left(\frac{\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+|f'(a+\eta(b,a))|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{3.28}$$

Furthermore, if we utilize the preinvexity of $|f'|^q$, we obtain

$$\begin{aligned}&\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right|\\ \leq\ & \frac{(\eta(b,a))^2}{4}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left(\left(\frac{|f'(a)|^q+3|f'(a+\eta(b,a))|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(a+\eta(b,a))|^q}{4}\right)^{\frac{1}{q}}\right).\end{aligned} \tag{3.29}$$

Corollary 3.13. *In Corollary 3.12 taking $\eta(b,a) = b - a$, we obtain*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b w(u)\,du-\int_a^b w(u)f(u)\,du\right|\\ \leq\ & \frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\times\left(\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right).\end{aligned} \tag{3.30}$$

Moreover, if we use the convexity of $|f'|^q$, we get

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b w(u)\,du-\int_a^b w(u)f(u)\,du\right|\leq\frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\\ &\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned} \tag{3.31}$$

Remark 3.14. *Theorem 3.9 will be reduced to Corollary 3 from [29], if we take $w(u) = \frac{1}{\eta(b,a)}$, $\eta(b,a) = b - a$ and $\alpha = 1$.*

Corollary 3.15. *In Corollary 3.12, using the discrete power mean inequality given in Lemma 2.2, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)} w(u)\,du-\int_{a}^{a+\eta(b,a)} w(u)\,f(u)\,du\right|$$
$$\leq \frac{(\eta(b,a))^2}{2}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}$$
$$\left[\frac{|f'(a)|^q+2\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+|f'(a+\eta(b,a))|^q}{4}\right]^{\frac{1}{q}}. \tag{3.32}$$

Additionally, if we use the preinvexity of $|f'|^q$ we get

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_{a}^{a+\eta(b,a)} w(u)\,du-\int_{a}^{a+\eta(b,a)} w(u)\,f(u)\,du\right| \tag{3.33}$$
$$\leq \frac{(\eta(b,a))^2}{2}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(a+\eta(b,a))|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.16. *In inequality* (3.33)*, taking $w(u)=\frac{1}{\eta(b,a)}$, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)} f(u)\,du\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+|f'(a+\eta(b,a))|^q}{2}\right)^{\frac{1}{q}}.$$

Moreover, if we choose $\eta(b,a)=b-a$, we get Theorem 2.3 from [18].

Theorem 3.17. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a differentiable function on $[a,a+\eta(b,a)]$ and $|f'|\in L([a,a+\eta(b,a)])$ with $\eta(b,a)>0$ and assume that Condition C holds, and let $w:[a,a+\eta(b,a)]\to\mathbb{R}$ be continuous and symmetric function with respect to $\frac{2a+\eta(b,a)}{2}$. If $|f'|^q$ is preinvex function and $q\geq 1$, we have*

$$|\Xi(a,b,\eta,\alpha,w,f)|\leq\frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty}\left[\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\right.\right.$$
$$\left.\times\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)|f'(a+\eta(b,a))|^q\right)^{\frac{1}{q}}$$
$$\left.+\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)|f'(a)|^q\right)^{\frac{1}{q}}\right].$$

Proof . From lemma 3.2, properties of modulus and power mean inequality, we get

$$\begin{aligned}
&\left|\Xi\left(a,b,\eta,\alpha,w,f\right)\right| \\
\leq\ & \frac{\left(\eta\left(b,a\right)\right)^{2}}{8}\left[\left(\int\limits_{0}^{1}\left|p_{2}(t)\right|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{0}^{1}\left|p_{2}(t)\right|\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int\limits_{0}^{1}\left|p_{1}(t)\right|dt\right)^{1-\frac{1}{q}}\left(\int\limits_{0}^{1}\left|p_{1}(t)\right|\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\left(\eta\left(b,a\right)\right)^{2}}{8\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left(\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)dt\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)\left|f'\left(a+\left(1-\frac{t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)\left|f'\left(a+\frac{t}{2}\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right].
\end{aligned}\tag{3.34}$$

Substituting (3.12) and (3.13) in (3.34) and using (3.14) and (3.15), we get

$$\begin{aligned}
&\left|\Xi\left(a,b,\eta,\alpha,w,f\right)\right| \\
\leq\ & \frac{\left(\eta\left(b,a\right)\right)^{2}}{8\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left[\left(\left|f'\left(\frac{2a+\eta\left(b,a\right)}{2}\right)\right|^{q}\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)tdt\right.\right. \\
&\left.+\left|f'\left(a+\eta\left(b,a\right)\right)\right|^{q}\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)(1-t)dt\right)^{\frac{1}{q}} \\
&+\left(\left|f'\left(\frac{2a+\eta\left(b,a\right)}{2}\right)\right|^{q}\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)tdt\right. \\
&\left.\left.+\left|f'\left(a\right)\right|^{q}\int\limits_{0}^{1}\left(1-t^{\alpha}+(1-t)^{\alpha}\right)(1-t)dt\right)^{\frac{1}{q}}\right] \\
=\ & \frac{\left(\eta\left(b,a\right)\right)^{2}}{8\alpha}\left\|w\right\|_{[a,a+\eta(b,a)],\infty}\left[\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{2a+\eta\left(b,a\right)}{2}\right)\right|^{q}\right.\right. \\
&\left.+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(a+\eta\left(b,a\right)\right)\right|^{q}\right)^{\frac{1}{q}} \\
&\left.+\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{2a+\eta\left(b,a\right)}{2}\right)\right|^{q}+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(a\right)\right|^{q}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

□

Corollary 3.18. *In Theorem 3.17, if we use the preinvexity of $|f'|^q$, we get*

$$\begin{aligned} |\Xi(a,b,\eta,\alpha,w,f)| &\leq \frac{(\eta(b,a))^2}{8\alpha}\|w\|_{[a,a+\eta(b,a)],\infty} \\ &\times\left(\left(\left(\frac{1}{4}-\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a)|^q+\left(\frac{3}{4}+\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a+\eta(b,a))|^q\right)^{\frac{1}{q}}\right. \\ &+\left.\left(\left(\frac{3}{4}+\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a)|^q+\left(\frac{1}{4}-\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a+\eta(b,a))|^q\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.35}$$

Corollary 3.19. *In Theorem 3.17, if we take $\eta(b,a)=b-a$ we obtain*

$$\begin{aligned} &|\Xi(a,b,b-a,\alpha,w,f)| \\ \leq\ & \frac{(b-a)^2}{8\alpha}\|w\|_{[a,b],\infty} \\ &\times\left(\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\ &+\left.\left(\left(\frac{1}{2}-\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left(\frac{1}{2}+\frac{\alpha}{(\alpha+1)(\alpha+2)}\right)|f'(a)|^q\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.36}$$

Moreover, if we use the convexity of $|f'|^q$, we obtain

$$\begin{aligned} &|\Xi(a,b,b-a,\alpha,w,f)| \\ \leq\ & \frac{(b-a)^2}{8\alpha}\|w\|_{[a,b],\infty} \\ &\left(\left(\left(\frac{1}{4}-\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a)|^q+\left(\frac{3}{4}+\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right. \\ &+\left.\left(\left(\frac{3}{4}+\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(a)|^q+\left(\frac{1}{4}-\frac{\alpha}{2(\alpha+1)(\alpha+2)}\right)|f'(b)|^q\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.37}$$

Corollary 3.20. *In Theorem 3.17, if we take $\alpha=1$, we obtain*

$$\begin{aligned} &\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right| \\ \leq\ & \frac{(\eta(b,a))^2}{8}\|w\|_{[a,a+\eta(b,a)],\infty} \\ &\times\left(\left(\frac{\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+2|f'(a+\eta(b,a))|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+2|f'(a)|^q}{3}\right)^{\frac{1}{q}}\right). \end{aligned} \tag{3.38}$$

Moreover, if we use the preinvexity of $|f'|^q$*, we get*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right| \tag{3.39}$$

$$\le \frac{(\eta(b,a))^2}{8}\|w\|_{[a,a+\eta(b,a)],\infty}$$

$$\times\left(\left(\frac{|f'(a)|^q+5|f'(a+\eta(b,a))|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|f'(a+\eta(b,a))|^q+5|f'(a)|^q}{6}\right)^{\frac{1}{q}}\right).$$

Remark 3.21. *Theorem 3.17 will be reduced to Corollary 4 from [18], if we take* $w(u)=\frac{1}{\eta(b,a)}$, $\eta(b,a)=b-a$ *and* $\alpha=1$.

Corollary 3.22. *In Corollary 3.20, using the discrete power mean inequality given in Lemma 2.2, we obtain*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right| \tag{3.40}$$

$$\le \frac{(\eta(b,a))^2}{4}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+\eta(b,a)}{2}\right)\right|^q+|f'(a+\eta(b,a))|^q}{3}\right)^{\frac{1}{q}}.$$

Moreover, if we use the preinvexity of $|f'|^q$*, we get*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}\int_a^{a+\eta(b,a)} w(u)\,du-\int_a^{a+\eta(b,a)} w(u)f(u)\,du\right| \tag{3.41}$$

$$\le \frac{(\eta(b,a))^2}{4}\|w\|_{[a,a+\eta(b,a)],\infty}\left(\frac{|f'(a)|^q+|f'(a+\eta(b,a))|^q}{2}\right)^{\frac{1}{q}}.$$

Corollary 3.23. *In Corollary 3.22, choosing* $\eta(b,a)=b-a$*, we obtain*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(u)\,du-\int_a^b w(u)f(u)\,du\right| \tag{3.42}$$

$$\le \frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}.$$

Moreover, if we use the preinvexity of $|f'|^q$*, we get*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b w(u)\,du-\int_a^b w(u)f(u)\,du\right|\le \frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}. \tag{3.43}$$

Remark 3.24. *Taking* $w(u)=\frac{1}{b-a}$ *in inequality* (3.43) *of Corollary 3.23, we obtain Theorem 1 from [42].*

4. Applications involving the arithmetic and logarithmic means

We shall consider the means for arbitrary real numbers a, b.
The Arithmetic mean: $A(a,b) = \frac{a+b}{2}$.
The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R} \setminus \{0, -1\}$.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^3, b^3\right) - L_3^3(a,b)\right| \leq \frac{(b-a)}{4}\left(a^2 + \left(\frac{a+b}{2}\right)^2 + b^2\right).$$

Proof . The assertion follows from inequality (3.19) of Corollary 3.7, applied to the function $f(x) = x^3$ which $f'(x) = 3x^2$ with $w(u) = \frac{1}{\eta(b,a)}$ and $\eta(b,a) = b - a$ □

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b \leq 1$, then we have*

$$\begin{aligned} &\left|A\left(a^{\frac{3}{2}}, (a + A(a,b))^{\frac{3}{2}}\right) - L_{\frac{3}{2}}^{\frac{3}{2}}(a, a + A(a,b))\right| \\ \leq\ & \frac{3(A(a,b))}{8}\left(\frac{a^{\frac{q}{2}}|f'(a)|^q + (a + A(a,b))^{\frac{q}{2}}}{2}\right)^{\frac{1}{q}}. \end{aligned}$$

Proof . The assertion follows from inequality (3.41) of Corollary 3.22, applied to the function $f(x) = \frac{2}{3}x^{\frac{3}{2}}$ which $f'(x) = x^{\frac{1}{2}}$ with $q \geq 4, w(u) = \frac{1}{\eta(b,a)}$ and $\eta(b,a) = A(a,b)$. □

5. Open problems

Further research works can tackle the following open problems:

1. Can we refine the estimation of Theorem 4 by another alternative aside from Lemma 1?
2. Is it possible to prove Theorems 3, 4 and 5 for the preinvex functions whose domains are invex sets in $\mathbb{R}^2$ by taking the symetric function $w(x,y)$ is not a function with separate variable $w(x,y) \neq w(x)w(y)$?
3. Can we establish the q-analogue of Theorems 3, 4 and 5?

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