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Denumerably many positive radial solutions for the iterative system of Minkowski-Curvature equations

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Abstract

This paper deals with the existence of denumerably many positive radial solutions to the iterative system of Dirichlet problems

$$\mathtt{div}\left(\frac{\nabla \mathtt{z_j}}{\sqrt{1-|\nabla \mathtt{z_j}|^2}}\right)+\mathtt{g_j}\left(\mathtt{z_{j+1}}\right)=0 \;\text{ in }\; \Omega,$$
$$\mathtt{z_j}=0 \;\text{ on }\; \partial\Omega,$$

where $\mathtt{j}\in\{1,2,\cdots,n\}$, $\mathtt{z}_1=\mathtt{z}_{n+1}$, Ω is a unit ball in $\mathbb{R}^{\mathrm{N}}$ involving the mean curvature operator in Minkowski space by applying Krasnoselskii's fixed point theorem, Avery-Henderson fixed point theorem and a new (Ren-Ge-Ren) fixed point theorem in cones.

Keywords: Positive radial solution, Minkowski-curvature equation, fixed point theorem, cone
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1. Introduction

The Dirichlet problems involving the mean curvature operator in Minkowski space

$$\mathcal{M}_{\mathcal{C}}(\mathtt{z})=\mathtt{div}\left(\frac{\nabla \mathtt{z}}{\sqrt{1-|\nabla \mathtt{z}|^2}}\right)$$

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arise from the study of spacelike submanifolds of codimension in the flat Minkowski space with prescribed mean extrinsic curvature [3, 13]. Due to its important background, the existence of radial solutions for such problems has been studied by many researchers, see [2, 4, 5, 6, 7, 10, 11, 17] and the references therein. However, most of the results in the above mentioned references are concerned with nonsingular problems, there are only a few works on singular problems, see [14] for the weakly singular cases and [15] for the strongly singular cases. Recently, [16] Pei and Wang established the existence and uniqueness of positive radial solutions are obtained for a mean curvature equation in Lorentz–Minkowski space of the form

$$\begin{gathered}\mathcal{M}_{\mathcal{C}}(\mathtt{z})+\mathtt{f}\big(|x|,\mathtt{z}\big)=0 \ \text{ in } \ \Omega,\\ \mathtt{z}=0 \ \text{ on } \ \partial\Omega,\end{gathered}$$

where Ω is a unit ball in $\mathbb{R}^{\mathrm{N}}$, $\mathtt{f}(r,u)$ may be singular at $r=0$ and 1, and strongly singular at $\mathtt{z}=0$, by applying the perturbation technique and Schauder fixed point theorem.

Motivated by the results mentioned above, in this paper, we aim to establish the existence denumerably many positive radial solutions of the following iterative system of Dirichlet problems associated with the Minkowski-curvature equations

$$\left.\begin{gathered}\mathtt{div}\left(\frac{\nabla \mathtt{z}_{\mathtt{j}}}{\sqrt{1-|\nabla \mathtt{z}_{\mathtt{j}}|^2}}\right)+\mathtt{g}_{\mathtt{j}}\left(\mathtt{z}_{\mathtt{j}+1}\right)=0 \ \text{ in } \ \Omega,\\ \mathtt{z}_{\mathtt{j}}=0 \ \text{ on } \ \partial\Omega,\end{gathered}\right\} \tag{1.1}$$

where $\mathtt{j}\in\{1,2,\cdots,n\}$, $\mathtt{z}_1=\mathtt{z}_{n+1}$, Ω is a unit ball in $\mathbb{R}^{\mathrm{N}}$ by applying the Krasnoselskii's fixed point theorem, Avery-Hender fixed point theorem and a new(Ren-Ge-Ren) fixed point theorem in cones.

The study of positive radial solutions to (1.1) reduces to the study of positive solutions to the following boundary value problems:

$$\left.\begin{gathered}\left[r^{\mathtt{N}-1}\phi\big(\mathtt{z}_{\mathtt{j}}'\big)\right]'+r^{\mathtt{N}-1}\mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1})=0, \ \mathtt{N}\geq 1,\\ \mathtt{z}_{\mathtt{j}}'(0)=\mathtt{z}_{\mathtt{j}}(1)=0,\end{gathered}\right\} \tag{1.2}$$

where $\mathtt{j}\in\{1,2,\cdots,n\}$, $\phi(\tau)=\tau/\sqrt{1-\tau^2}$, $\tau\in(-1,1)$, and $\mathtt{z}_1=\mathtt{z}_{n+1}$, by the change of variable $\mathtt{z}_{\mathtt{j}}(x)=\mathtt{z}_{\mathtt{j}}(r)$ with $r=|x|$.

Throughout this paper, we make the following assumptions:

$(\mathcal{H}_1)$ $\mathtt{g}_{\mathtt{j}}:(0,+\infty)\to[0,+\infty)$ is continuous.

$(\mathcal{H}_2)$ there exists a sequence $\{r_k\}_{k=1}^{\infty}$ such that $0<r_{k+1}<r_k<\frac{1}{2}$, $k\in\mathbb{N}$,

$$\lim_{k\to\infty} r_k=r^*<\frac{1}{2}.$$

The rest of the paper is organized in the following fashion. In Section 2, we convert the boundary value problem (1.2) into equivalent integral equation. In Section 3, we establish a criteria for the existence of denumerably many positive solutions for the boundary value problem (1.2) by applying Krasnoselskii's fixed point theorem, Avery-Henderson fixed point theorem and new(Ren-Ge-Ren) fixed point theorem in cones.

2. Preliminaries

In this section we provide some lemmas which are useful in the main results of the paper.

Lemma 2.1 ([14]). *Let* $\phi(\tau) = \dfrac{\tau}{\sqrt{1-\tau^2}}$. *Then* $\phi^{-1}(\tau) = \dfrac{\tau}{\sqrt{1+\tau^2}}$ *and*

$$\phi^{-1}(\tau_1)\phi^{-1}(\tau_2) \le \phi^{-1}(\tau_1\tau_2) \le \tau_1\tau_2, \quad \textit{for all} \;\; \tau_1, \tau_2 \in (0, +\infty).$$

Lemma 2.2 ([12]). *Let* $h \in C[0,1]$. *Then the boundary value problem*

$$\left[r^{\mathtt{N}-1}\phi\big(\mathtt{z}_1'\big)\right]' + h(t) = 0, \;\; r \in (0,1), \tag{2.1}$$

$$\mathtt{z}_1'(0) = \mathtt{z}_1(1) = 0, \tag{2.2}$$

has a unique solution

$$\mathtt{z}_1(r) = \int_r^1 \phi^{-1}\left(\frac{1}{t^{\mathtt{N}-1}}\int_0^t \tau^{\mathtt{N}-1}h(\tau)d\tau\right)dt. \tag{2.3}$$

From Lemma 2.2, we note that an n-tuple $\big(\mathtt{z}_1, \mathtt{z}_2, \cdots, \mathtt{z}_n\big)$ is a solution of the boundary value problem (1.2) if and only if

$$\begin{aligned}\mathtt{z}_1(r) = \int_r^1 \phi^{-1}\Bigg[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\Bigg(\int_{\tau_1}^1 \phi^{-1}\Bigg[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\Bigg(\int_{\tau_2}^1 \phi^{-1}\Bigg[\cdots \\ \mathtt{g}_{n-1}\Bigg(\int_{\tau_{n-1}}^1 \phi^{-1}\Bigg[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\Bigg]dt_n\Bigg)\cdots d\tau_1\Bigg]dt_1.\end{aligned}$$

and

$$\begin{aligned}\mathtt{z}_\mathtt{j}(r) &= \int_r^1 \phi^{-1}\left(\frac{1}{t^{\mathtt{N}-1}}\int_0^t \tau^{\mathtt{N}-1}\mathtt{g}_\mathtt{j}(\mathtt{z}_{\mathtt{j}+1}(\tau))d\tau\right)dt, \; \mathtt{j} = 2, 3, \cdots, n, \\ \mathtt{z}_{n+1}(r) &= \mathtt{z}_1(r), \; r \in (0,1).\end{aligned}$$

We denote the Banach space $\mathcal{C}([0,1], \mathbb{R})$ by $\mathscr{B}$ with the norm $\|\mathtt{z}\| = \max\limits_{r\in[0,1]}|\mathtt{z}(r)|$. For $\delta \in (0, 1/2)$, the cone $\mathtt{P}_\delta \subset \mathscr{B}$ is defined by

$$\mathtt{P}_\delta = \left\{\mathtt{z} \in \mathscr{B} : \mathtt{z}(r) \ge 0, \min_{r\in[\delta,\,1-\delta]}\mathtt{z}(r) \ge \delta\|\mathtt{z}\|\right\},$$

For any $\mathtt{z}_1 \in \mathtt{P}_\delta$, define an operator $\aleph : \mathtt{P}_\delta \to \mathscr{B}$ by

$$\begin{aligned}(\aleph\mathtt{z}_1)(r) = \int_r^1 \phi^{-1}\Bigg[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\Bigg(\int_{\tau_1}^1 \phi^{-1}\Bigg[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\Bigg(\int_{\tau_2}^1 \phi^{-1}\Bigg[\cdots \\ \mathtt{g}_{n-1}\Bigg(\int_{\tau_{n-1}}^1 \phi^{-1}\Bigg[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\Bigg]dt_n\Bigg)\cdots d\tau_1\Bigg]dt_1.\end{aligned} \tag{2.4}$$

Lemma 2.3 ([9]). *Let* $\mathtt{z} \in \mathcal{C}\big([0,1], [0, +\infty)\big)$ *be such that* $\mathtt{z}'(r)$ *is decreasing in* $[0,1]$. *Then, we have*

$$\min_{r\in[\delta,\,1-\delta]}\mathtt{z}(r) \ge \delta\|\mathtt{z}\|.$$

Lemma 2.4. *Assume that* $(\mathcal{H}_1)$ *holds. Then, for each* $\delta \in (0, 1/2)$, $\aleph(\mathtt{P}_\delta) \subset \mathtt{P}_\delta$ *and* $\aleph : \mathtt{P}_\delta \to \mathtt{P}_\delta$ *is completely continuous.*

Proof . It is easy to see that $\aleph(\mathtt{z}_1(r)) \in \mathscr{B}$ with $\aleph(\mathtt{z}_1(1)) = 0$. Now, for any $\mathtt{z}_1 \in \mathtt{P}_\delta$, we have

$$\begin{aligned}(\aleph\mathtt{z}_1)'(r) = &-\phi^{-1}\Bigg[\frac{1}{r^{\mathtt{N}-1}}\int_0^r \tau_1^{\mathtt{N}-1}\mathtt{g}_1\Bigg(\int_{\tau_1}^1 \phi^{-1}\Bigg[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\Bigg(\int_{\tau_2}^1\phi^{-1}\Bigg[\cdots\\ &\mathtt{g}_{n-1}\Bigg(\int_{\tau_{n-1}}^1 \phi^{-1}\Bigg[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\Bigg]dt_n\Bigg)\cdots d\tau_1\Bigg]\\ &\le 0.\end{aligned}$$

It follows that $(\aleph\mathtt{z}_1)(r)$ is decreasing on $[0, 1]$, i.e.,

$$(\aleph\mathtt{z}_1)(r) \ge (\aleph\mathtt{z}_1)(1) = 0 \quad \text{for any} \quad r \in [0, 1]. \tag{2.5}$$

Since

$$\begin{aligned}r^{\mathtt{N}-1}\phi\big((\aleph\mathtt{z}_1)'(r)\big) = &-\int_0^r \tau_1^{\mathtt{N}-1}\mathtt{g}_1\Bigg(\int_{\tau_1}^1 \phi^{-1}\Bigg[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\Bigg(\int_{\tau_2}^1\phi^{-1}\Bigg[\cdots\\ &\mathtt{g}_{n-1}\Bigg(\int_{\tau_{n-1}}^1 \phi^{-1}\Bigg[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\Bigg]dt_n\Bigg)\cdots d\tau_2\Bigg)d\tau_1,\end{aligned}$$

is decreasing on $[0, 1]$, it follows from the fact that ϕ is increasing and $\mathtt{N} \ge 1$ that $(\aleph\mathtt{z}_1)'(r)$ is decreasing on $[0, 1]$. Thus, by Lemma 2.3, we have

$$\min_{r\in[\delta,\,1-\delta]} \aleph\mathtt{z}_1(r) \ge \delta\|\aleph\mathtt{z}_1\|. \tag{2.6}$$

From (2.5) and (2.6), we see that $\aleph(\mathtt{P}_\delta) \subset \mathtt{P}_\delta$. Finally by standard methods and the Arzela-Ascoli theorem, the operator $\aleph$ is completely continuous. □

3. Denumerably Many Positive Radial Solutions

For the the existence of denumerably many positive radial solutions for the boundary value problem (1.2), we utilize the following cone fixed point theorems in a Banach space.

Theorem 3.1. *(Krasnoselskii [8]) Let* $\mathscr{B}$ *be Banch space and* $\mathtt{P}$ *be a cone in* $\mathscr{B}$. *Suppose* $\mathtt{E}$ *and* $\mathtt{F}$ *are two open sets with* $0 \in \mathtt{E}, \overline{\mathtt{E}} \subset \mathtt{F}$. *Let* $\aleph : \mathtt{P} \cap (\overline{\mathtt{F}}\backslash\mathtt{E}) \to \mathtt{P}$ *be a completely continuous operator such that*

(a) $\|\aleph\mathtt{z}\| \le \|\mathtt{z}\|$, $\mathtt{z} \in \mathtt{P} \cap \partial\mathtt{E}$, *and* $\|\aleph\mathtt{z}\| \ge \|\mathtt{z}\|$, $\mathtt{z} \in \mathtt{P} \cap \partial\mathtt{F}$, *or*

(b) $\|\aleph\mathtt{z}\| \ge \|\mathtt{z}\|$, $\mathtt{z} \in \mathtt{P} \cap \partial\mathtt{E}$, *and* $\|\aleph\mathtt{z}\| \le \|\mathtt{z}\|$, $\mathtt{z} \in \mathtt{P} \cap \partial\mathtt{F}$.

Then $\aleph$ *has a fixed point in* $\mathtt{P} \cap (\overline{\mathtt{F}}\backslash\mathtt{E})$.

Let ψ be a nonnegative continuous functional on a cone $\mathtt{P}$ of the real Banach space $\mathscr{B}$. Then for any two positive real numbers a' and c', we define the sets

$$\mathtt{P}(\psi, c') = \{\mathtt{z} \in \mathtt{P} : \psi(\mathtt{z}) < c'\}$$

and

$$\mathtt{P}_{a'} = \{\mathtt{z} \in \mathtt{P} : \|\mathtt{z}\| < a'\}.$$

Theorem 3.2. *(Avery-Henderson[1]) Let* $\mathtt{P}$ *be a cone in a real Banach space* $\mathscr{B}$. *Suppose* α *and* γ *are increasing, nonnegative continuous functionals on* $\mathtt{P}$ *and* θ *is nonnegative continuous functional on* $\mathtt{P}$ *with* $\theta(0) = 0$ *such that, for some positive numbers* c' *and* k, $\gamma(\mathtt{z}) \leq \theta(\mathtt{z}) \leq \alpha(\mathtt{z})$ *and* $\|\mathtt{z}\| \leq k\gamma(\mathtt{z})$, *for all* $\mathtt{z} \in \overline{\mathtt{P}(\gamma, c')}$. *Suppose that there exist positive numbers* a' *and* b' *with* $a' < b' < c'$ *such that* $\theta(\mathtt{z}) \leq \lambda\theta(\mathtt{z})$, *for all* $0 \leq \lambda \leq 1$ *and* $\mathtt{z} \in \partial\mathtt{P}(\theta, b')$. *Further, let* $\aleph : \overline{\mathtt{P}(\gamma, c')} \to \mathtt{P}$ *be a completely continuous operator such that*

(a) $\gamma(\aleph\mathtt{z}) > c'$, *for all* $\mathtt{z} \in \partial\mathtt{P}(\gamma, c')$,

(b) $\theta(\aleph\mathtt{z}) < b'$, *for all* $\mathtt{z} \in \partial\mathtt{P}(\theta, b')$,

(c) $\mathtt{P}(\alpha, a') \neq \emptyset$ *and* $\alpha(\aleph\mathtt{z}) > a'$, *for all* $\partial\mathtt{P}(\alpha, a')$.

Then, $\aleph$ *has at least two fixed points* ${}^1\mathtt{z}, {}^2\mathtt{z} \in \mathtt{P}(\gamma, c')$ *such that* $a' < \alpha({}^1\mathtt{z})$ *with* $\theta({}^1\mathtt{z}) < b'$ *and* $b' < \theta({}^2\mathtt{z})$ *with* $\gamma({}^2\mathtt{z}) < c'$.

Theorem 3.3. *(Ren-Ge-Ren[18]) Let* $\mathtt{P}$ *be a cone in a Banach space* $\mathscr{B}$ *Let* α, β *and* γ *be three increasing, nonnegative and continuous functionals on* $\mathtt{P}$, *satisfying for some* $c' > 0$ *and* $M > 0$ *such that* $\gamma(\mathtt{z}) \leq \beta(\mathtt{z}) \leq \alpha(\mathtt{z})$ *and* $\|\mathtt{z}\| \leq M\gamma(\mathtt{z})$, *for all* $\mathtt{z} \in \overline{\mathtt{P}(\gamma, c')}$. *Suppose there exists a completely continuous operator* $\aleph : \overline{\mathtt{P}(\gamma, c')} \to \mathtt{P}$ *and* $0 < a' < b' < c'$ *such that*

(a) $\gamma(\aleph\mathtt{z}) > c'$, *for all* $\mathtt{z} \in \partial\mathtt{P}(\gamma, c')$,

(b) $\beta(\aleph\mathtt{z}) < b'$, *for all* $\mathtt{z} \in \partial\mathtt{P}(\beta, b')$,

(c) $\mathtt{P}(\alpha, a') \neq \emptyset$ *and* $\alpha(\aleph\mathtt{z}) > a'$, *for all* $\partial\mathtt{P}(\alpha, a')$.

Then, $\aleph$ *has at least three fixed points* ${}^1\mathtt{z}, {}^2\mathtt{z}, {}^3\mathtt{z} \in \mathtt{P}(\gamma, c')$ *such that* $\alpha({}^1\mathtt{z}) < a' < \alpha({}^2\mathtt{z})$, $\beta({}^2\mathtt{z}) < b' < \beta({}^3\mathtt{z})$ *and* $\beta({}^3\mathtt{z}) < c'$.

3.1. Existence of (Atleast One) Denumerably Many Positive Radial Solutions

Theorem 3.4. *Assume that* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *hold and let* $\{\delta_k\}_{k=1}^{\infty}$ *be a sequence with* $r_{k+1} < \delta_k < r_k$, $k \in \mathbb{N}$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be any two sequences which satisfies the relation*

$$\mathtt{R}_{k+1} < \delta_k\mathtt{S}_k < \mathtt{S}_k < \mathtt{R}_k, \quad \beta_k\phi(\delta_k^{-1}\mathtt{S}_k) < \phi\big(\mathtt{R}_k\big),\ k \in \mathbb{N},$$

Furthermore for each natural number k, *we assume that* $\mathtt{g}_\mathtt{j}(\mathtt{j} = 1, 2, \cdots, n)$ *satisfies*

$(\mathcal{H}_3)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}(r)) \leq \mathtt{N}\phi\big(\mathtt{R}_k\big)$ *for all* $0 \leq \mathtt{z}(r) \leq \mathtt{R}_k$, $r \in [0, 1]$,

$(\mathcal{H}_4)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}(r)) \geq \beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k)$ *for all* $\delta_k\mathtt{S}_k \leq \mathtt{z}(r) \leq \mathtt{S}_k$, $r \in [\delta_k, 1 - \delta_k]$, *where*

$$\beta_k = \frac{(1 - \delta_k)^{\mathtt{N}} - \delta_k^{\mathtt{N}}}{(1 - \delta_k)^{\mathtt{N}-1}}.$$

The iterative system (1.1) *has denumerably many radial solutions* $\{(\mathtt{z}_1^{[k]}, \mathtt{z}_2^{[k]}, \cdots, \mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{z}_\mathtt{j}^{[k]}(r) \geq 0$ *on* $(0, 1)$, $\mathtt{j} = 1, 2, \cdots, n$ *and* $k \in \mathbb{N}$.

Proof . Consider the sequences $\{\mathtt{E}_k\}_{k=1}^{\infty}$ and $\{\mathtt{F}_k\}_{k=1}^{\infty}$ of open subsets of $\mathscr{B}$ defined by

$$\mathtt{E}_k = \{\mathtt{z} \in \mathscr{B} : \|\mathtt{z}\| < \mathtt{R}_k\},$$

and

$$\mathtt{F}_k = \{\mathtt{z} \in \mathscr{B} : \|\mathtt{z}\| < \mathtt{S}_k\}.$$

Let $\{\delta_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $r^* < r_{k+1} < \delta_k < r_k < 1/2$, for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathtt{P}_{\delta_k}$ by

$$\mathtt{P}_{\delta_k} = \left\{ \mathtt{z} \in \mathscr{B} : \mathtt{z}(r) \geq 0 \ \text{ and } \min_{r \in [\delta_k,\, 1-\delta_k]} \mathtt{z}(r) \geq \delta_k \|\mathtt{z}\| \right\}.$$

Let $\mathtt{z}_1 \in \mathtt{P}_{\delta_k} \cap \partial \mathtt{E}_k$. Then,

$$\mathtt{z}_1(\tau) \leq \mathtt{R}_k = \|\mathtt{z}_1\|$$

for all $\tau \in [0,1]$. Let $0 < \tau_{n-1} < 1$. Then by Lemma 2.1 and $(\mathcal{H}_3)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}} \int_0^{t_n} \tau_n^{\mathtt{N}-1} \mathrm{g}_n\big(\mathtt{z}_1(\tau_n)\big) d\tau_n\right] dt_n &\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}} \int_0^{t_n} \tau_n^{\mathtt{N}-1} \mathrm{g}_n\big(\mathtt{z}_1(\tau_n)\big) d\tau_n\right] dt_n \\
&\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}} \int_0^{t_n} \tau_n^{\mathtt{N}-1} \mathtt{N}\phi\big(\mathtt{R}_k\big) d\tau_n\right] dt_n \\
&\leq \int_0^1 \phi^{-1}\left(\phi\big(\mathtt{R}_k\big) t_n\right) dt_n \\
&\leq \mathtt{R}_k.
\end{aligned}$$

It follows in similar manner for $0 < \tau_{n-2} < 1$, we have

$$\begin{aligned}
&\int_{\tau_{n-2}}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}} \int_0^{t_{n-1}} \tau_{n-1}^{\mathtt{N}-1} \mathrm{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}} \int_0^{t_n} \tau_n^{\mathtt{N}-1} \mathrm{g}_n\big(\mathtt{z}_1(\tau_n)\big) d\tau_n\right] dt_n\right) d\tau_{n-1}\right] dt_{n-1} \\
&\qquad \leq \int_0^1 \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}} \int_0^{t_{n-1}} \tau_{n-1}^{\mathtt{N}-1} \mathrm{g}_{n-1}\big(\mathtt{R}_k\big) d\tau_{n-1}\right] dt_{n-1} \\
&\qquad \leq \int_0^1 \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}} \int_0^{t_{n-1}} \tau_{n-1}^{\mathtt{N}-1} \mathtt{N}\phi\big(\mathtt{R}_k\big) d\tau_{n-1}\right] dt_{n-1} \\
&\qquad \leq \int_0^1 \phi^{-1}\left(\phi\big(\mathtt{R}_k\big) t_{n-1}\right) dt_{n-1} \\
&\qquad \leq \mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
(\aleph \mathtt{z}_1)(r) = &\int_r^1 \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}} \int_0^{t_1} \tau_1^{\mathtt{N}-1} \mathrm{g}_1\left(\int_{\tau_1}^1 \phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}} \int_0^{t_2} \tau_2^{\mathtt{N}-1} \mathrm{g}_2\left(\int_{\tau_2}^1 \phi^{-1}\left[\cdots\right.\right.\right.\right.\right. \\
&\left.\left.\mathrm{g}_{n-1}\left(\int_{\tau_{n-1}}^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}} \int_0^{t_n} \tau_n^{\mathtt{N}-1} \mathrm{g}_n\big(\mathtt{z}_1(\tau_n)\big) d\tau_n\right] dt_n\right) \cdots d\tau_1\right] dt_1 \\
\leq &\, \mathtt{R}_k.
\end{aligned}$$

Since $\mathtt{R}_k = \|\mathtt{z}_1\|$ for $\mathtt{z}_1 \in \mathtt{P}_{\delta_k} \cap \partial\mathtt{E}_k$, we get

$$\|\aleph\mathtt{z}_1\| \leq \|\mathtt{z}_1\|. \tag{3.1}$$

Let $r \in [\delta_k, 1-\delta_k]$. Then,

$$\mathtt{S}_k = \|\mathtt{z}_1\| \geq \mathtt{z}_1(r) \geq \min_{r\in[\delta_k,1-\delta_k]} \mathtt{z}_1(r) \geq \delta_k\|\mathtt{z}_1\| \geq \delta_k\mathtt{S}_k.$$

By $(\mathcal{H}_4)$ and for $\tau_{n-1} \in [\delta_k, 1-\delta_k]$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n &\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\\
&\geq \delta_k\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]\\
&\geq \delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)d\tau_n\right]\\
&\geq \delta_k\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}}-\delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)\right]\\
&\geq \mathtt{S}_k.
\end{aligned}$$

In similar manner (for $\tau_{n-2} \in [\delta_k, 1-\delta_k]$,) that

$$\begin{aligned}
&\int_{\tau_{n-2}}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\big(\mathtt{S}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)d\tau_{n-1}\right]\\
&\quad\geq \delta_k\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}}-\delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)\right]\\
&\quad\geq \mathtt{S}_k.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
(\aleph\mathtt{z}_1)(r) = &\int_r^1 \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1} \tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^1 \phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2} \tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^1 \phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\left.\left.\left.\left.\left.\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1\right.\right.\right.\right.\\
\geq\ &\mathtt{S}_k.
\end{aligned}$$

Thus, if $\mathtt{z}_1 \in \mathtt{P}_{\delta_k} \cap \partial\mathtt{F}_k$, then

$$\|\aleph \mathtt{z}_1\| \geq \|\mathtt{z}_1\|. \tag{3.2}$$

It is evident that $0 \in \mathtt{F}_k \subset \overline{\mathtt{F}}_k \subset \mathtt{E}_k$. From (3.1),(3.2), it follows from Theorem 3.1 that the operator $\aleph$ has a fixed point $\mathtt{z}_1^{[k]} \in \mathtt{P}_{\delta_k} \cap \left(\overline{\mathtt{E}}_k \backslash \mathtt{F}_k\right)$ such that $\mathtt{z}_1^{[k]}(t) \geq 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next setting $\mathtt{z}_{\mathtt{j}+1} = \mathtt{z}_1$, we obtain denumerably many positive solutions $\{(\mathtt{z}_1^{[k]}, \mathtt{z}_2^{[k]}, \cdot\cdot\cdot, \mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ of (1.1) given iteratively by

$$\mathtt{z}_{\mathtt{j}}(r) = \int_r^1 \phi^{-1}\left(\frac{1}{t^{\mathtt{N}-1}}\int_0^t \tau^{\mathtt{N}-1}\mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1}(\tau))d\tau\right)dt,\ \mathtt{j} = n, n-1, \cdots, 2, 1.$$

The proof is completed. □

Example 3.5. *Consider the following iterative system of Dirichlet problems*

$$\left.\begin{array}{ll} \mathcal{M}_{\mathcal{C}}(\mathtt{z}_{\mathtt{j}}) + \mathtt{g}_{\mathtt{j}}(\mathtt{z}_{\mathtt{j}+1}) = 0 & in\ \ \Omega \\ \mathtt{z}_{\mathtt{j}} = 0 & on\ \ \partial\Omega, \end{array}\right\} \tag{3.3}$$

where $\mathtt{j} = 1, 2$, $\mathtt{N} = 1$ and $\mathtt{z}_1 = \mathtt{z}_3$. Let

$$r_k = \frac{31}{64} - \sum_{s=1}^{k} \frac{1}{4(s+1)^4},\ \ \delta_k = \frac{1}{2}(r_k + r_{k+1}),\ \ k = 1, 2, 3, \cdots,$$

then

$$\delta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$r_{k+1} < \delta_k < r_k,\ \delta_k > \frac{1}{5}.$$

It is easy to see

$$r_1 = \frac{15}{32} < \frac{1}{2},\ \ r_k - r_{k+1} = \frac{1}{4(k+2)^4},\ \ k = 1, 2, 3, \cdots.$$

Since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, it follows that

$$\begin{aligned} r^* = \lim_{k\to\infty} r_k &= \frac{31}{64} - \sum_{j=1}^{\infty} \frac{1}{4(j+1)^4} \\ &= \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}. \end{aligned}$$

In addition if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{S}_k = 10^{-(4k+2)},$$

then

$$\begin{aligned} \mathtt{R}_{k+1} = 10^{-(4k+4)} &< \frac{1}{5} \times 10^{-(4k+2)} < \delta_k \mathtt{S}_k \\ &< \mathtt{S}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k}, \end{aligned}$$

$$\phi(\mathtt{R}_k) = \frac{1}{\sqrt{10^{8k} - 1}},\ \ \phi(\mathtt{S}_k) = \frac{1}{\sqrt{10^{8k+4} - 1}},\ \ \text{and}\ \ \phi\left(\frac{\mathtt{S}_k}{\delta_k}\right) = \frac{1}{\sqrt{10^{8k+4} \times \delta_k^2 - 1}}.$$

Since ϕ is increasing and $\delta_k^{-1}\mathtt{S}_k \leq 5\mathtt{S}_k \leq \mathtt{R}_k$, it follows that

$$\beta_k \mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right) \leq \mathtt{N}\phi(\mathtt{R}_k), \quad \beta_k = 1 - 2\delta_k < 1.$$

Let $\mathtt{M}_1$ and $\mathtt{M}_2$ be two positive numbers such that

$$\beta_k \mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right) \leq \mathtt{M}_1 \times 10^{-8k} \leq \mathtt{M}_2 \times 10^{8k} \leq \mathtt{N}\phi(\mathtt{R}_k),$$

and

$$\mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) = \begin{cases} \mathtt{M}_2 \times 10^{-4}, & \mathtt{z} \in (10^{-4}, +\infty), \\ \frac{\mathtt{M}_1 \times 10^{-(4k+2)} - \mathtt{M}_2 \times 10^{-4k}}{10^{-(4k+2)} - 10^{-4k}}(\mathtt{z} - 10^{-4k}) + \mathtt{M}_2 \times 10^{-4k}, & \\ \qquad \mathtt{z} \in \left[10^{-(4k+2)}, 10^{-4k}\right], & \\ \mathtt{M}_1 \times 10^{-(4k+2)}, & \mathtt{z} \in \left(\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right), \\ \frac{\mathtt{M}_1 \times 10^{-(4k+2)} - \mathtt{M}_2 \times 10^{-(4k+4)}}{\frac{1}{5} \times 10^{-(4k+2)} - 10^{-(4k+4)}}(\mathtt{z} - 10^{-(4k+4)}) + \mathtt{M}_2 \times 10^{-(4k+4)}, & \\ \qquad \mathtt{z} \in \left(10^{-(4k+4)}, \frac{1}{5} \times 10^{-(4k+2)}\right], & \\ 0, & \mathtt{z} = 0. \end{cases}$$

Then, $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\begin{aligned} \mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) &\leq \mathtt{N}\phi(\mathtt{R}_k), \quad \mathtt{z} \in \left[0, 10^{-4k}\right], \\ \mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) &\geq \beta_k \mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right), \quad \mathtt{z} \in \left[\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right]. \end{aligned}$$

All the conditions of Theorem 3.4 are satisfied. Therefore, by Theorem 3.4, the boundary value problem (3.3) has denumerably many positive radial solutions $\{(\mathtt{z}_1^{[k]}, \mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|\mathtt{z}_\mathtt{j}^{[k]}\| \leq 10^{-4k}$ for each $k = 1, 2, 3, \cdots$, $\mathtt{j} = 1, 2$.

3.2. Existence of Atleast Two Denumerably Many Families of Positive Radial Solutions

In order to use Theorem 3.2, let $\delta_k < r_k < 1 - \delta_k$ and δ_k of Theorem 3.1, we define the nonnegative, increasing, continuous functional γ_k, β_k, and α_k by

$$\begin{aligned} \gamma(\mathtt{z}) &= \min_{r \in [r_k, 1-\delta_k]} \mathtt{z}(r) = \mathtt{z}(r_k), \\ \theta(\mathtt{z}) &= \max_{r \in [\delta_k, r_k]} \mathtt{z}(r) = \mathtt{z}(r_k), \\ \alpha(\mathtt{z}) &= \max_{r \in [\delta_k, 1-\delta_k]} \mathtt{z}(r) = \mathtt{z}(1 - \delta_k). \end{aligned}$$

It is obvious that for each $\mathtt{z} \in \mathtt{P}$,

$$\gamma_k(\mathtt{z}) = \theta_k(\mathtt{z}) \le \alpha_k(\mathtt{z}).$$

In addition, by Lemma 2.3, for each $\mathtt{z} \in \mathtt{P}$,

$$\gamma(\mathtt{z}) = \mathtt{z}(r_k) \ge \delta_k \|\mathtt{z}\|.$$

Thus

$$\|\mathtt{z}\| \le \delta_k^{-1}\gamma(\mathtt{z}), \quad \text{for all} \;\; \mathtt{z} \in \mathtt{P}.$$

Finally, we also note that

$$\theta(\lambda\mathtt{z}) = \lambda\theta(\mathtt{z}), \quad 0 \le \lambda \le 1 \;\; \text{and} \;\; \mathtt{z} \in \mathtt{P}.$$

Theorem 3.6. *Assume that* $(\mathcal{H}_1)$*–*$(\mathcal{H}_2)$ *hold and let* $\{\delta_k\}_{k=1}^{\infty}$ *be a sequence with* $r_{k+1} < \delta_k < r_k$, $k \in \mathbb{N}$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$, $\{\mathtt{Q}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be three sequences which satisfies the relation*

$$\mathtt{R}_{k+1} < \mathtt{Q}_k < \delta_k\mathtt{S}_k < \mathtt{S}_k < \mathtt{R}_k, \quad \beta_k\phi(\delta_k^{-1}\mathtt{S}_k) < \phi(\mathtt{R}_k),\; k \in \mathbb{N}.$$

Furthermore for each natural number k, *we assume that* $\mathtt{g_j}$ $(\mathtt{j} = 1, 2, \cdots, n)$ *satisfies*

$(\mathcal{H}_5)$ $\mathtt{g_j}(\mathtt{z}) > \beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k)$, *for all* $\mathtt{S}_k \le \mathtt{z}(r) \le \delta_k^{-1}\mathtt{S}_k$,

$(\mathcal{H}_6)$ $\mathtt{g_j}(\mathtt{z}) < \mathtt{N}\phi(\mathtt{R}_k)$, *for all* $0 \le \mathtt{z}(r) \le \delta_k^{-1}\mathtt{R}_k$,

$(\mathcal{H}_7)$ $\mathtt{g_j}(\mathtt{z}) > \beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{Q}_k)$, *for all* $0 \le \mathtt{z}(r) \le \mathtt{Q}_k$.

Then the boundary value problem (1.1) *has two denumerably many families of radial solutions* $\{({}^1\mathtt{z}_1^{[k]}, {}^1\mathtt{z}_2^{[k]}, \cdots, {}^1\mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ *and* $\{({}^2\mathtt{z}_1^{[k]}, {}^2\mathtt{z}_2^{[k]}, \cdots, {}^2\mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ *satisfying*

$$\mathtt{Q}_k < \alpha_k\left({}^1\mathtt{z}_\mathtt{j}^{[k]}\right) \quad \text{with} \quad \theta_k\left({}^1\mathtt{z}_\mathtt{j}^{[k]}\right) < \mathtt{S}_k,\; \mathtt{j} = 1, 2, \cdots, n,\; k \in \mathbb{N}$$

and

$$\mathtt{S}_k < \theta_k\left({}^2\mathtt{z}_\mathtt{j}^{[k]}\right) \quad \text{with} \quad \gamma_k\left({}^2\mathtt{z}_\mathtt{j}^{[k]}\right) < \mathtt{R}_k,\; \mathtt{j} = 1, 2, \cdots, n,\; k \in \mathbb{N}.$$

Proof . We begin by defining the completely continuous operator $\aleph$ by (2.4). So it is easy to check that $\aleph : \overline{\mathtt{P}(\gamma, \mathtt{S}_k)} \to \mathtt{P}$, for $k \in \mathbb{N}$. Firstly, we shall verify that condition (a) of Theorem 3.2 is satisfied. So, let us choose $\mathtt{z}_1 \in \partial\mathtt{P}(\gamma, \mathtt{S}_k)$. Then $\gamma(\mathtt{z}_1) = \min_{r \in [r_k, 1-\delta_k]} \mathtt{z}_1(r) = \mathtt{z}_1(r_k) = \mathtt{S}_k$ this implies that $\mathtt{S}_k \le \mathtt{z}_1(r)$ for $r \in [r_k, 1]$. Since $\|\mathtt{z}_1\| \le \delta_k^{-1}\gamma(\mathtt{z}_1) = \delta_k^{-1}\mathtt{S}_k$. So we have

$$\mathtt{S}_k \le \mathtt{z}_1(r) \le \delta_k^{-1}\mathtt{S}_k,\; r \in [r_k, 1-\delta_k].$$

Let $\tau_{n-1} \in [r_k, 1-\delta_k]$. Then by $(\mathcal{H}_5)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n &\ge \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n \\
&\ge \delta_k\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right] \\
&\ge \delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_n^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)d\tau_n\right] \\
&\ge \delta_k\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}} - \delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{S}_k}{\delta_k}\right)\right] \\
&\ge \mathtt{S}_k.
\end{aligned}$$

In similar manner (for $\tau_{n-2} \in\in [r_k, 1-\delta_k],$) that

$$\begin{aligned}
&\int_{\tau_{n-2}}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{N-1}}\int_0^{t_{n-1}} \tau_{n-1}^{N-1} g_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1} g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{N-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{N-1} g_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1} g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{N-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{N-1} g_{n-1}\big(S_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{N-1}}\int_{\delta_k}^{1-\delta_k}\tau_{n-1}^{N-1}\beta_k N\phi\left(\frac{S_k}{\delta_k}\right)d\tau_{n-1}\right]\\
&\quad\geq \delta_k\phi^{-1}\left[\frac{(1-\delta_k)^{N}-\delta_k^{N}}{(1-\delta_k)^{N-1}}\beta_k\phi\left(\frac{S_k}{\delta_k}\right)\right]\\
&\quad\geq S_k.
\end{aligned}$$

Continuing with bootstrapping argument, we get

$$\begin{aligned}
\gamma(\aleph z_1) &= \min_{r\in[r_k,1-\delta_k]}(\aleph z_1)(r) = (\aleph z_1)(r_k)\\
&= \int_{r_k}^{1}\phi^{-1}\left[\frac{1}{t_1^{N-1}}\int_0^{t_1}\tau_1^{N-1}g_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{N-1}}\int_0^{t_2}\tau_2^{N-1}g_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad g_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\bigg]dt_1\\
&\geq \int_{1-\delta_k}^{1}\phi^{-1}\left[\frac{1}{t_1^{N-1}}\int_0^{t_1}\tau_1^{N-1}g_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{N-1}}\int_0^{t_2}\tau_2^{N-1}g_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad g_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\bigg]dt_1\\
&\geq S_k.
\end{aligned}$$

This proves (i) of Theorem 3.2. We next address (ii) of Theorem 3.2. So, we choose $z_1 \in \partial P(\theta, R_k)$. Then $\theta(z_1) = \max_{r\in[\delta_k,r_k]} z_1(r) = z_1(r_k) = R_k$ this implies that $0 \leq z_1(r) \leq R_k$ for $r \in [0, r_k]$. Since $\|z_1\| \leq \delta_k^{-1}\gamma(z_1) = \delta_k^{-1}\theta(z_1) = \delta_k^{-1}R_k$. So we have

$$0 \leq z_1(r) \leq \delta_k^{-1}R_k,\ r \in [0,1].$$

Let $0 < \tau_{n-1} < 1$. Then by $(\mathcal{H}_6)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n &\leq \int_0^1\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\\
&\leq \int_0^1\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}N\phi\big(R_k\big)d\tau_n\right]dt_n\\
&\leq \int_0^1\phi^{-1}\left(\phi\big(R_k\big)t_n\right)dt_n\\
&\leq R_k.
\end{aligned}$$

It follows in similar manner for $0<\tau_{n-2}<1$, we have

$$\begin{aligned}
&\int_{\tau_{n-2}}^{1}\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\qquad\le\int_0^1\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\big(\mathtt{S}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\qquad\le\int_0^1\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{N}\phi\big(\mathtt{R}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\qquad\le\int_0^1\phi^{-1}\left(\phi\big(\mathtt{R}_k\big)t_{n-1}\right)dt_{n-1}\\
&\qquad\le\mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
\theta_k(\aleph\mathtt{z}_1)&=\max_{r\in[\delta_k,r_k]}(\aleph\mathtt{z}_1)(r)=(\aleph\mathtt{z}_1)(r_k)\\
&=\int_{r_k}^{1}\phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1\\
&\le\int_{0}^{1}\phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad\left.\left.\left.\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1\\
&\le\mathtt{R}_k.
\end{aligned}$$

Hence condition (c) is satisfied. Finally we verify that (c) of Theorem 3.2 is also satisfied. We note that $\mathtt{z}_1(r)=\mathtt{Q}_k/4,\ r\in[0,1]$ is a member of $\mathtt{P}(\alpha_k,\mathtt{Q}_k)$ and $\mathtt{Q}_k/4<\mathtt{Q}_k$. So $\mathtt{P}(\alpha_k,\mathtt{Q}_k)\neq\emptyset$. Now let $\mathtt{z}_1\in\mathtt{P}(\alpha_k,\mathtt{Q}_k)$. Then $\alpha_k(\mathtt{z}_1)=\max_{r\in[\delta_k,1-\delta_k]}\mathtt{z}_1(r)=\mathtt{z}_1(1-\delta_k)=\mathtt{Q}_k$, i.e., $0\le\mathtt{z}_1(r)\le\mathtt{Q}_k$, for $r\in[\delta_k,1-\delta_k]$. Let $0<\tau_{n-1}<1$. Then by $(\mathcal{H}_7)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n&\ge\int_{1-\delta_k}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\\
&\ge\delta_k\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]\\
&\ge\delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{Q}_k}{\delta_k}\right)d\tau_n\right]\\
&\ge\delta_k\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}}-\delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{Q}_k}{\delta_k}\right)\right]\\
&\ge\mathtt{Q}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}\alpha_k(\aleph z_1) &= \max_{r\in[\delta_k,1-\delta_k]}(\aleph z_1)(r) = (\aleph z_1)(1-\delta_k)\\ &= \int_{1-\delta_k}^{1}\phi^{-1}\left[\frac{1}{t_1^{N-1}}\int_0^{t_1}\tau_1^{N-1}g_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{N-1}}\int_0^{t_2}\tau_2^{N-1}g_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\ &\qquad g_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{N-1}}\int_0^{t_n}\tau_n^{N-1}g_n\big(z_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\Bigg]dt_1\\ &\geq Q_k.\end{aligned}$$

Thus condition (c) of Theorem 3.2 is satisfied. Since all hypotheses of Theorem 3.2 are satisfied, the assertion follows. □

Example 3.7. *Consider the following iterative system of Dirichlet problems*

$$\left.\begin{aligned}&\mathcal{M}_{\mathcal{C}}(z_j)+g_j(z_{j+1})=0 \quad in \ \ \Omega\\ &z_j=0 \qquad\qquad\qquad\qquad on \ \ \partial\Omega,\end{aligned}\right\} \tag{3.4}$$

where $j=1,2$, $N=1$ and $z_1=z_3$. Let r_k,δ_k be the same as of Example 3.5. In addition if we take

$$R_k=10^{-4k},\ Q_k=10^{-(4k+3)} \ \text{ and } \ S_k=10^{-(4k+2)},$$

then

$$\begin{aligned}R_{k+1}=10^{-(4k+4)}<Q_k=10^{-(4k+3)}&<\frac{1}{5}\times 10^{-(4k+2)}<\delta_k S_k\\ &<S_k=10^{-(4k+2)}<R_k=10^{-4k},\end{aligned}$$

$$\phi(R_k)=\frac{1}{\sqrt{10^{8k}-1}},\ \phi(Q_k)=\frac{1}{\sqrt{10^{8k+6}-1}},\ \phi(S_k)=\frac{1}{\sqrt{10^{8k+4}-1}},$$

$$\phi(\delta_k^{-1}R_k)=\frac{1}{\sqrt{10^{8k}\times\delta_k^2-1}} \ \text{ and } \ \phi(\delta_k^{-1}Q_k)=\frac{1}{\sqrt{10^{8k+6}\times\delta_k^2-1}}.$$

Since ϕ is increasing and $\delta_k^{-1}S_k\leq 5S_k\leq R_k$, it follows that

$$\beta_k N\phi(\delta_k^{-1}S_k)\leq N\phi(R_k),\quad \beta_k=1-2\delta_k<1.$$

Let N_1 and N_2 be two positive numbers such that

$$N_1\times 10^{-8k}\leq N_2\times 10^{-8k}\leq \beta_k N\phi(\delta_k^{-1}Q_k)\leq \beta_k N\phi(\delta_k^{-1}S_k)\leq N_2\times 10^{8k}\leq N\phi(R_k),$$

and

$$\mathtt{g}_1(\mathtt{z})=\mathtt{g}_2(\mathtt{z})=\begin{cases}\mathtt{N}_2\times 10^{-6}, & \mathtt{z}\in(5\times 10^{-4},+\infty),\\ \mathtt{N}_2\times 10^{-(4k+2)}, & \mathtt{z}\in\left(10^{-(4k+2)},5\times 10^{-4k}\right],\\ \frac{\mathtt{N}_2\times 10^{-(4k+3)}-\mathtt{N}_2\times 10^{-(4k+2)}}{5\times 10^{-(4k+3)}-10^{-(4k+2)}}(\mathtt{z}-10^{-(4k+2)})+\mathtt{N}_2\times 10^{-(4k+2)}, & \mathtt{z}\in\left[5\times 10^{-(4k+3)},10^{-(4k+2)}\right],\\ \mathtt{N}_2\times 10^{-(4k+3)}, & \mathtt{z}\in\left(10^{-(4k+3)},5\times 10^{-(4k+3)}\right),\\ \frac{\mathtt{N}_1\times 10^{-(4k+4)}-\mathtt{N}_2\times 10^{-(4k+3)}}{10^{-(4k+4)}-10^{-(4k+3)}}(\mathtt{z}-10^{-(4k+3)})+\mathtt{N}_2\times 10^{-(4k+3)}, & \mathtt{z}\in\left(10^{-(4k+4)},10^{-(4k+3)}\right],\\ 0, & \mathtt{z}=0.\end{cases}$$

Then, $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\begin{aligned}&\mathtt{g}_1(\mathtt{z})=\mathtt{g}_2(\mathtt{z})\geq\beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k),\quad \mathtt{z}\in\left[10^{-(4k+2)},5\times 10^{-(4k+2)}\right],\\ &\mathtt{g}_1(\mathtt{z})=\mathtt{g}_2(\mathtt{z})\leq\mathtt{N}\phi(\mathtt{R}_k),\quad \mathtt{z}\in\left[0,5\times 10^{-4k}\right],\\ &\mathtt{g}_1(\mathtt{z})=\mathtt{g}_2(\mathtt{z})\geq\beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{Q}_k),\quad \mathtt{z}\in\left[0,10^{-(4k+3)}\right].\end{aligned}$$

All the conditions of Theorem 3.6 are satisfied. Therefore, by Theorem 3.6, the boundary value problem (3.4) has two denumerably many families of radial solutions $\{({}^1\mathtt{z}_1^{[k]},{}^1\mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$ and $\{({}^2\mathtt{z}_1^{[k]},{}^2\mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$ satisfying

$$10^{-(4k+3)}<\max_{r\in[\delta_k,1-\delta_k]}{}^1\mathtt{z}_\mathtt{j}^{[k]}(r)\ \text{ with }\ \max_{r\in[\delta_k,r_k]}{}^1\mathtt{z}_\mathtt{j}^{[k]}(r)<10^{-(4k+2)},\ \mathtt{j}=1,2,\ k\in\mathbb{N}$$

and

$$10^{-(4k+2)}<\max_{r\in[\delta_k,r_k]}{}^2\mathtt{z}_\mathtt{j}^{[k]}(r)\ \text{ with }\ \min_{r\in[r_k,1-\delta_k]}{}^2\mathtt{z}_\mathtt{j}^{[k]}(r)<10^{-4k},\ \mathtt{j}=1,2,\ k\in\mathbb{N}.$$

3.3. Existence of Atleast Three Denumerably Many Families of Positive Radial Solutions

In order to use Theorem 3.3, let $\delta_k<r_k<1-\delta_k$ and δ_k of Theorem 3.1, we define the nonnegative, increasing, continuous functional γ_k, β_k, and α_k by

$$\begin{aligned}\gamma(\mathtt{z})&=\max_{r\in[\delta_k,r_k]}\mathtt{z}(r)=\mathtt{z}(r_k),\\ \beta(\mathtt{z})&=\min_{r\in[r_k,1-\delta_k]}\mathtt{z}(r)=\mathtt{z}(r_k),\\ \alpha(\mathtt{z})&=\max_{r\in[\delta_k,1-\delta_k]}\mathtt{z}(r)=\mathtt{z}(1-\delta_k).\end{aligned}$$

It is obvious that for each $\mathtt{z} \in \mathtt{P}$,

$$\gamma_k(\mathtt{z}) \leq \beta_k(\mathtt{z}) \leq \alpha_k(\mathtt{z}).$$

In addition, by Lemma 2.3, for each $\mathtt{z} \in \mathtt{P}$,

$$\gamma(\mathtt{z}) = \mathtt{z}(r_k) \geq \delta_k \|\mathtt{z}\|.$$

Thus

$$\|\mathtt{z}\| \leq \delta_k^{-1}\gamma(\mathtt{z}), \quad \text{for all } \ \mathtt{z} \in \mathtt{P}.$$

Theorem 3.8. *Assume that* $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ *hold and let* $\{\delta_k\}_{k=1}^{\infty}$ *be a sequence with* $r_{k+1} < \delta_k < r_k$, $k \in \mathbb{N}$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$, $\{\mathtt{Q}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be three sequences which satisfies the relation*

$$\mathtt{R}_{k+1} < \mathtt{Q}_k < \delta_k \mathtt{S}_k < \mathtt{S}_k < \mathtt{R}_k, \quad \beta_k \phi(\delta_k^{-1}\mathtt{S}_k) < \phi(\mathtt{R}_k),\ k \in \mathbb{N}.$$

Furthermore for each natural number k, *we assume that* $\mathtt{g}_\mathtt{j}(\mathtt{j} = 1, 2, \cdots, n)$ *satisfies*

$(\mathcal{H}_8)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) < \mathtt{N}\phi(\mathtt{R}_k)$, *for all* $0 \leq \mathtt{z}(r) \leq \delta_k^{-1}\mathtt{R}_k$,

$(\mathcal{H}_9)$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) > \beta_k \mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k)$, *for all* $\mathtt{S}_k \leq \mathtt{z}(r) \leq \delta_k^{-1}\mathtt{S}_k$,

$(\mathcal{H}_{10})$ $\mathtt{g}_\mathtt{j}(\mathtt{z}) < \mathtt{N}\phi(\mathtt{Q}_k)$, *for all* $0 \leq \mathtt{z}(r) \leq \delta_k^{-1}\mathtt{Q}_k$.

Then the boundary value problem (1.1) *has three denumerably many families of radial solutions* $\{({}^1\mathtt{z}_1^{[k]}, {}^1\mathtt{z}_2^{[k]}, \cdots, {}^1\mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$, $\{({}^2\mathtt{z}_1^{[k]}, {}^2\mathtt{z}_2^{[k]}, \cdots, {}^2\mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ *and* $\{({}^3\mathtt{z}_1^{[k]}, {}^3\mathtt{z}_2^{[k]}, \cdots, {}^3\mathtt{z}_n^{[k]})\}_{k=1}^{\infty}$ *satisfying*

$$0 \leq \alpha_k\left({}^1\mathtt{z}_\mathtt{j}^{[k]}\right) < \mathtt{Q}_k < \alpha_k\left({}^2\mathtt{z}_\mathtt{j}^{[k]}\right),\ \mathtt{j} = 1, 2, \cdots, n,\ k \in \mathbb{N},$$

$$\beta_k\left({}^2\mathtt{z}_\mathtt{j}^{[k]}\right) < \mathtt{S}_k < \beta_k\left({}^3\mathtt{z}_\mathtt{j}^{[k]}\right),\ \mathtt{j} = 1, 2, \cdots, n,\ k \in \mathbb{N},$$

and

$$\gamma_k\left({}^3\mathtt{z}_\mathtt{j}^{[k]}\right) < \mathtt{R}_k,\ \mathtt{j} = 1, 2, \cdots, n,\ k \in \mathbb{N}.$$

Proof . We define the completely continuous operator $\aleph$ by (2.4). So it is easy to check that $\aleph : \overline{\mathtt{P}(\gamma, \mathtt{R}_k)} \to \mathtt{P}$, for $k \in \mathbb{N}$. In order to prove that all the conditions of Theorem 3.3 are satisfied, we choose $\mathtt{z}_1 \in \partial\mathtt{P}(\gamma, \mathtt{R}_k)$. Then $\gamma(\mathtt{z}_1) = \max_{r\in[\delta_k, r_k]} \mathtt{z}_1(r) = \mathtt{z}_1(r_k) = \mathtt{R}_k$ this implies that $0 \leq \mathtt{z}_1(r) \leq \mathtt{R}_k$ for $r \in [0, r_k]$. Since $\|\mathtt{z}_1\| \leq \delta_k^{-1}\gamma(\mathtt{z}_1) = \delta_k^{-1}\mathtt{R}_k$. So we have

$$0 \leq \mathtt{z}_1(r) \leq \delta_k^{-1}\mathtt{R}_k,\ r \in [0, 1].$$

Let $0 < \tau_{n-1} < 1$. Then by Lemma 2.1 and $(\mathcal{H}_8)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n &\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n \\
&\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{N}\phi\big(\mathtt{R}_k\big)d\tau_n\right]dt_n \\
&\leq \int_0^1 \phi^{-1}\left(\phi\big(\mathtt{R}_k\big)t_n\right)dt_n \\
&\leq \mathtt{R}_k.
\end{aligned}$$

It follows in similar manner for $0<\tau_{n-2}<1$, we have

$$\begin{aligned}
\int_{\tau_{n-2}}^{1}&\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\le\int_0^1\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\big(\mathtt{R}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\le\int_0^1\phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}}\tau_{n-1}^{\mathtt{N}-1}\mathtt{N}\phi\big(\mathtt{R}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\le\int_0^1\phi^{-1}\left(\phi\big(\mathtt{R}_k\big)t_{n-1}\right)dt_{n-1}\\
&\le\mathtt{R}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
\gamma_k(\aleph\mathtt{z}_1)&=\max_{r\in[\delta_k,r_k]}(\aleph\mathtt{z}_1)(r)=(\aleph\mathtt{z}_1)(r_k)\\
&=\int_{r_k}^{1}\phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\Bigg]dt_1\\
&\le\int_{0}^{1}\phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1}\tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\Bigg]dt_1\\
&\le\mathtt{R}_k.
\end{aligned}$$

Hence condition (a) is satisfied. Secondly, we show that (b) of Theorem 3.3 is fulled. For this we select $\mathtt{z}_1\in\partial\mathtt{P}(\beta,\mathtt{S}_k)$. Then $\beta(\mathtt{z}_1)=\min_{r\in[r_k,1-\delta_k]}\mathtt{z}_1(r)=\mathtt{z}_1(r_k)=\mathtt{S}_k$, i.e. $\mathtt{z}_1(r)\ge\mathtt{S}_k$, for $r\in[r_k,1-\delta_k]$. So we have $\|\mathtt{z}_1\|\ge\mathtt{S}_k$, for $r\in[r_k,1-\delta_k]$. Noticing that $\|\mathtt{z}_1\|\le\delta_k^{-1}\gamma_k(\mathtt{z}_1)\le\delta_k^{-1}\beta_k(\mathtt{z}_1)=\delta_k^{-1}\mathtt{S}_k$. we have

$$\mathtt{S}_k\le\mathtt{z}_1(r)\le\delta_k^{-1}\mathtt{S}_k,\ \text{ for }\ r\in[r_k,1-\delta_k].$$

Let $0<\tau_{n-1}<1$. The by $(\mathcal{H}_9)$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n&\ge\int_{1-\delta_k}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\\
&\ge\delta_k\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]\\
&\ge\delta_1\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k}\tau_n^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_1}\right)d\tau_n\right]\\
&\ge\delta_1\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}}-\delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{S}_k}{\delta_1}\right)\right]\\
&\ge\mathtt{S}_k.
\end{aligned}$$

In similar manner (for $\tau_{n-2} \in [\delta_k, 1-\delta_k],$) that

$$\begin{aligned}
&\int_{\tau_{n-2}}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_0^{t_{n-1}} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_{n-1}^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\mathtt{g}_{n-1}\big(\mathtt{S}_k\big)d\tau_{n-1}\right]dt_{n-1}\\
&\quad\geq \delta_k\phi^{-1}\left[\frac{1}{(1-\delta_k)^{\mathtt{N}-1}}\int_{\delta_k}^{1-\delta_k} \tau_{n-1}^{\mathtt{N}-1}\beta_k\mathtt{N}\phi\left(\frac{\mathtt{S}_k}{\delta_1}\right)d\tau_{n-1}\right]\\
&\quad\geq \delta_1\phi^{-1}\left[\frac{(1-\delta_k)^{\mathtt{N}}-\delta_k^{\mathtt{N}}}{(1-\delta_k)^{\mathtt{N}-1}}\beta_k\phi\left(\frac{\mathtt{S}_k}{\delta_1}\right)\right]\\
&\quad\geq \mathtt{S}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
\beta_k(\aleph\mathtt{z}_1) &= \min_{r\in[r_k,1-\delta_k]}(\aleph\mathtt{z}_1)(r) = (\aleph\mathtt{z}_1)(r_k)\\
&= \int_{r_k}^{1} \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1} \tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1} \phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2} \tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1} \phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad \left.\left.\left.\left.\left.\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1\right.\right.\right.\right.\\
&\geq \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1} \tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1} \phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2} \tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1} \phi^{-1}\left[\cdots\right.\right.\right.\right.\right.\\
&\qquad \left.\left.\left.\left.\left.\mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1\right.\right.\right.\right.\\
&\geq \mathtt{S}_k.
\end{aligned}$$

Hence condition (b) is satisfied. Finally we verify that (c) of Theorem 3.3 is also satisfied. Since $0 \in \mathtt{P}$ and $\mathtt{Q}_k > 0$, it follows that $\mathtt{P}(\alpha_k, \mathtt{Q}_k) \neq \emptyset$. Now let $\mathtt{z}_1 \in \mathtt{P}(\alpha_k, \mathtt{Q}_k)$. Then $\alpha_k(\mathtt{z}_1) = \max_{r\in[\delta_k,1-\delta_k]} \mathtt{z}_1(r) = \mathtt{z}_1(1-\delta_k) = \mathtt{Q}_k$, i.e., $0 \leq \mathtt{z}_1(r) \leq \mathtt{Q}_k$, for $r \in [\delta_k, 1-\delta_k]$. Also, $\|\mathtt{z}_1\| \leq \delta_k^{-1}\gamma_k(\mathtt{z}_1) \leq \delta_k^{-1}\alpha_k(\mathtt{z}_1) = \delta_k^{-1}\mathtt{Q}_k$. Then we get

$$0 \leq \mathtt{z}_1(r) \leq \delta_k^{-1}\mathtt{Q}_k, \quad \text{for} \quad r \in [0,1].$$

Let $0 < \tau_{n-1} < 1$. Then by $(\mathcal{H}_{10})$, we have

$$\begin{aligned}
\int_{\tau_{n-1}}^{1} \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n &\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\\
&\leq \int_0^1 \phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n} \tau_n^{\mathtt{N}-1}\mathtt{N}\phi\big(\mathtt{Q}_k\big)d\tau_n\right]dt_n\\
&\leq \int_0^1 \phi^{-1}\left(\phi\big(\mathtt{Q}_k\big)t_n\right)dt_n\\
&\leq \mathtt{Q}_k.
\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}
\alpha_k(\aleph \mathtt{z}_1) &= \max_{r\in[\delta_k,1-\delta_k]} (\aleph \mathtt{z}_1)(r) \\
&= (\aleph \mathtt{z}_1)(1-\delta_k) \\
&= \int_{1-\delta_k}^{1} \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1} \tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\right[\cdots\right.\right.\right. \\
&\qquad \left.\left.\left. \mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1 \\
&\le \int_{0}^{1} \phi^{-1}\left[\frac{1}{t_1^{\mathtt{N}-1}}\int_0^{t_1} \tau_1^{\mathtt{N}-1}\mathtt{g}_1\left(\int_{\tau_1}^{1}\phi^{-1}\left[\frac{1}{t_2^{\mathtt{N}-1}}\int_0^{t_2}\tau_2^{\mathtt{N}-1}\mathtt{g}_2\left(\int_{\tau_2}^{1}\phi^{-1}\right[\cdots\right.\right.\right. \\
&\qquad \left.\left.\left. \mathtt{g}_{n-1}\left(\int_{\tau_{n-1}}^{1}\phi^{-1}\left[\frac{1}{t_n^{\mathtt{N}-1}}\int_0^{t_n}\tau_n^{\mathtt{N}-1}\mathtt{g}_n\big(\mathtt{z}_1(\tau_n)\big)d\tau_n\right]dt_n\right)\cdots d\tau_1\right]dt_1 \\
&\le \mathtt{Q}_k.
\end{aligned}$$

Thus condition (c) of Theorem 3.3 is satisfied. Since all hypotheses of Theorem 3.3 are satisfied, the assertion follows. $\square$

Example 3.9. *Consider the following iterative system of Dirichlet problems*

$$\left.\begin{array}{ll} \mathcal{M}_{\mathcal{C}}(\mathtt{z}_\mathtt{j}) + \mathtt{g}_\mathtt{j}(\mathtt{z}_{\mathtt{j}+1}) = 0 & \textit{in}\ \ \Omega \\ \mathtt{z}_\mathtt{j} = 0 & \textit{on}\ \ \partial\Omega, \end{array}\right\} \tag{3.5}$$

where $\mathtt{j} = 1, 2$, $\mathtt{N} = 1$ and $\mathtt{z}_1 = \mathtt{z}_3$. Let Let r_k, δ_k be the same as of Example 3.5. In addition if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{Q}_k = 10^{-(4k+3)} \ \text{ and }\ \mathtt{S}_k = 10^{-(4k+2)},$$

then

$$\begin{aligned}
\mathtt{R}_{k+1} = 10^{-(4k+4)} < \mathtt{Q}_k = 10^{-(4k+3)} &< \frac{1}{5}\times 10^{-(4k+2)} < \delta_k \mathtt{S}_k \\
&< \mathtt{S}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k},
\end{aligned}$$

$$\phi(\mathtt{R}_k) = \frac{1}{\sqrt{10^{8k}-1}},\ \phi(\mathtt{Q}_k) = \frac{1}{\sqrt{10^{8k+6}-1}},\ \phi(\mathtt{S}_k) = \frac{1}{\sqrt{10^{8k+4}-1}},$$

$$\phi(\delta_k^{-1}\mathtt{R}_k) = \frac{1}{\sqrt{10^{8k}\times\delta_k^2-1}},\ \phi(\delta_k^{-1}\mathtt{Q}_k) = \frac{1}{\sqrt{10^{8k+6}\times\delta_k^2-1}}\ \text{ and }\ \phi(\delta_k^{-1}\mathtt{S}_k) = \frac{1}{\sqrt{10^{8k+4}\times\delta_k^2-1}}.$$

Since ϕ is increasing and $\delta_k^{-1}\mathtt{S}_k \le 5\mathtt{S}_k \le \mathtt{R}_k$, it follows that

$$\beta_k \mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k) \le \mathtt{N}\phi(\mathtt{R}_k),\quad \beta_k = 1-2\delta_k < 1.$$

Let $\mathtt{N}_2$ be a positive number such that

$$\mathtt{N}_2\times 10^{-8k} \le \beta_k\mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k) \le \mathtt{N}_2\times 10^{8k} \le \mathtt{N}\phi(\mathtt{Q}_k) \le \mathtt{N}\phi(\mathtt{R}_k),$$

and

$$\mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) = \begin{cases} \mathtt{N}_2 \times 10^{-6}, & \mathtt{z} \in (5 \times 10^{-4}, +\infty), \\ \mathtt{N}_2 \times 10^{-(4k+2)}, & \mathtt{z} \in \left(10^{-(4k+2)}, 5 \times 10^{-4k}\right], \\ \frac{\mathtt{N}_2 \times 10^{-(4k+3)} - \mathtt{N}_2 \times 10^{-(4k+2)}}{5 \times 10^{-(4k+3)} - 10^{-(4k+2)}}(\mathtt{z} - 10^{-(4k+2)}) + \mathtt{N}_2 \times 10^{-(4k+2)}, & \\ \qquad\qquad \mathtt{z} \in \left[5 \times 10^{-(4k+3)}, 10^{-(4k+2)}\right], & \\ \mathtt{N}_2 \times 10^{-(4k+3)}, & \mathtt{z} \in \left(10^{-(4k+3)}, 5 \times 10^{-(4k+3)}\right), \\ 0, & \mathtt{z} = 0. \end{cases}$$

Then, $\mathtt{g}_1$ and $\mathtt{g}_2$ satisfies the following growth conditions:

$$\mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) \leq \mathtt{N}\phi(\mathtt{R}_k), \quad \mathtt{z} \in \left[0, 5 \times 10^{-4k}\right],$$

$$\mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) \geq \beta_k \mathtt{N}\phi(\delta_k^{-1}\mathtt{S}_k), \quad \mathtt{z} \in \left[10^{-(4k+2)}, 5 \times 10^{-(4k+2)}\right],$$

$$\mathtt{g}_1(\mathtt{z}) = \mathtt{g}_2(\mathtt{z}) \leq \mathtt{N}\phi(\mathtt{Q}_k), \quad \mathtt{z} \in \left[0, 5 \times 10^{-(4k+3)}\right].$$

All the conditions of Theorem 3.8 are satisfied. Therefore, by Theorem 3.8, the boundary value problem (3.5) has three denumerably many families of radial solutions $\{({}^1\mathtt{z}_1^{[k]}, {}^1\mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$, $\{({}^2\mathtt{z}_1^{[k]}, {}^2\mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$ and $\{({}^3\mathtt{z}_1^{[k]}, {}^3\mathtt{z}_2^{[k]})\}_{k=1}^{\infty}$ satisfying

$$0 \leq \max_{r \in [\delta_k, 1-\delta_k]} {}^1\mathtt{z}_\mathtt{j}^{[k]} < 10^{-(4k+3)} < \max_{r \in [\delta_k, 1-\delta_k]} {}^2\mathtt{z}_\mathtt{j}^{[k]}, \ \mathtt{j} = 1, 2, \ k \in \mathbb{N},$$

$$\min_{r \in [r_k, 1-\delta_k]} {}^2\mathtt{z}_\mathtt{j}^{[k]} < 10^{-(4k+2)} < \min_{r \in [r_k, 1-\delta_k]} {}^3\mathtt{z}_\mathtt{j}^{[k]}, \ \mathtt{j} = 1, 2, \ k \in \mathbb{N},$$

and

$$\max_{r \in [\delta_k, r_k]} {}^3\mathtt{z}_\mathtt{j}^{[k]} < 10^{-4k}, \ \mathtt{j} = 1, 2, \ k \in \mathbb{N}.$$

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