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# Measuring the efficiency of three methods for estimating the parameters and reliability of a two-parameter weighted exponential distribution

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**Abstract**

The purpose of this research is to investigate a weighted exponential distribution with two parameters [shape & scale], including calculating the cumulative distribution and reliability functions, as well as estimating the two parameters and reliability function using three methods (MLE, MOM, Jackknife), and comparing the results using MSE with simulation. The results show the jackknife estimator of parameters and reliability function is the best.

*Keywords:* estimate, Jackknife, MLE, MOM, reliability function, weighted exponential distribution

## 1. Introduction

Rao (1965) proposed and formulated the weighted probability distribution in general form in relation with modeling statistical data. This distribution is employed in many domains of real life such as medicine, ecology, and reliability. Jing and Priyadarshani introduced Patel and Rao, Gupta & Kirmani, Gupta & Keating as application examples for weighted distribution. see [1, 5, 6, 9]. Xingye Qiao and Ling Song Zhang [10] used a distance weighted distribution to detect the direction by minimizing distance weighted discrimination in 2015. The objectives of this study are to determine some of the statistical properties of the weighted Exponential distribution with two-parameters, like cumulative, Reliability, and Hazard functions, as well as their moments and the moment generating function, and then estimate shape , scale parameters and Reliability function using various methods to determine the best estimator.

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## 2. Weighted Exponential Distribution Properties

This distribution's probability density function is provided by [4]

$$f\left(x;\delta,\gamma\right)=\frac{1+\delta}{\delta}\ \gamma\ e^{-\gamma x}\left(1-e^{-\gamma\delta x}\right)\qquad \gamma>0,\ \delta>0\ x>0, \tag{2.1}$$

where $\delta,\gamma$ are the shape and scale parameters respectively. So the cumulative distribution function driving as:

$$\begin{aligned}F\left(x\right)&=\int_0^x f\left(t\right)dt\\&=\frac{1+\delta}{\delta}\ \gamma\ e^{-\gamma t}\left(1-e^{-\gamma\delta t}\right)dt\\&=\frac{1+\delta}{\delta}\ \left\{\int_0^x\gamma\ e^{-\gamma t}\ dt-\int_0^x\gamma\ e^{-\gamma t(1+\delta)}\ dt\right\}\\&=\frac{1+\delta}{\delta}\ \left\{\left(1-\ e^{-\gamma x}\right)\ -\frac{1}{1+\delta}\left(1-\ e^{-\gamma x(1+\delta)}\right)\right\}.\end{aligned}$$

So

$$F\left(x\right)=\frac{1+\delta}{\delta}\ \frac{1}{1+\delta}\left\{\left(1+\delta\right)\left(1-\ e^{-\gamma x}\right)\ -\left(1-\ e^{-\gamma x(1+\delta)}\right)\right\}$$

$$F\left(x\right)=\frac{1}{\delta}\left\{e^{-\gamma x(1+\delta)}+\delta-e^{-\gamma x}(1+\delta)\right\}. \tag{2.2}$$

The reliability function define as:

$$\begin{aligned}R\left(x\right)&=1-F\left(x\right)\\&=1-\frac{1}{\delta}\left\{e^{-\gamma x(1+\delta)}+\delta-e^{-\gamma x}(1+\delta)\right\},\end{aligned} \tag{2.3}$$

and the m.g.f is given by:

$$\begin{aligned}M_X\left(t\right)=E\left(e^{tx}\right)&=\int_0^\infty e^{tx}f(x;\delta,\gamma)dx\\&=\left(\frac{1+\delta}{\delta}\right)\gamma\int_0^\infty e^{tx}e^{-\gamma x}\left(1-e^{-\gamma\delta x)}\right)dx\\&=\left(\frac{1+\delta}{\delta}\right)\gamma\left[\int_0^\infty e^{-x(\gamma-t)}dx-\int_0^\infty e^{-x(\gamma-t+\gamma\delta)}dx\right]\\&=\left(\frac{1+\delta}{\delta}\right)\gamma\left\{\frac{1}{(\gamma-t)}-\frac{1}{(\gamma-t+\gamma\delta)}\right\}\\&=\left(\frac{1+\delta}{\delta}\right)\gamma\left\{\frac{\gamma\delta}{(\gamma-t)(\gamma-t+\gamma\delta)}\right\}\end{aligned}$$

$$\rightarrow\ M_X\left(t\right)=(1+\delta)\gamma^2(\gamma-t)^{-1}(\gamma-t+\gamma\delta)^{-2}. \tag{2.4}$$

Now the mean and variance:

$$M_X'(t)=(1+\delta)\gamma^2(\gamma-t)^{-1}(-2)(\gamma-t+\gamma\delta)^{-3}(-1)$$

$$+(\gamma - t + \gamma\delta)^{-2}(-1)(\gamma - t)^{-2}(-1)\}$$

$$M'_X(t=0) = (1+\delta)\gamma^2\{2(\gamma - t)^{-1}(\gamma - t + \gamma\delta)^{-3} + (\gamma - t)^{-2}(\gamma - t + \gamma\delta)^{-2}\}.$$

Therefore the mean is

$$E(x) = \frac{2 + (1+\delta)}{\gamma^2(1+\delta)^2} = \frac{3+\delta}{\gamma^2(1+\delta)^2} \tag{2.5}$$

$$M''_X(t) = (1+\delta)\gamma^2\{2(\gamma - t)^{-1}(-3)(\gamma - t + \gamma\delta)^{-4}(-1) + (\gamma - t + \gamma\delta)^{-3}$$
$$- 2(\gamma - t)^{-2}(-1) + (\gamma - t)^{-2}(-2)(\gamma - t + \gamma\delta)^{-3}(-1)$$
$$+ (\gamma - t + \gamma\delta)^{-2}(-2)(\gamma - t)^{-3}(-1)\}$$

Then

$$M''_X(0) = (1+\delta)\gamma^2\left\{\frac{6}{\gamma^5(1+\delta)^4} + \frac{2}{\gamma^5(1+\delta)^3} + \frac{2}{\gamma^5(1+\delta)^3} + \frac{2}{\gamma^5(1+\delta)^2}\right\}.$$

Thus,

$$E\left(x^2\right) = \frac{1}{(1+\delta)\gamma^3}\left\{\frac{2\delta^2 + 8\delta + 12}{(1+\delta)^2}\right\} = \frac{2\delta^2 + 8\delta + 12}{\gamma^3(1+\delta)^3}.$$

Therefore the variance is

$$v(x) = \frac{2\delta^2 + 8\delta + 12}{\gamma^3(1+\delta)^3} - (\frac{3+\delta}{\gamma^2(1+\delta)^2})^2 \tag{2.6}$$

## 3. Methods of Estimation

### *3.1. Method of moments*

the moment estimators for ($\delta$ & $\gamma$), is by solve the eq.'s [3];

$$\mu_r = E(x^r)$$

and $m_r = \frac{\sum x_i^r}{n}$

$$\mu_1 = E(x) = \frac{\sum x_i}{n} = \frac{3+\delta}{\gamma^2(1+\delta)^2}$$

$$\mu_2 = E\left(x^2\right) \to \frac{\sum x_i^2}{n} = \frac{2\delta^2 + 8\delta + 12}{\gamma^3(1+\delta)^3}$$

$$\gamma^3(1+\delta)^3\frac{\sum x_i^2}{n} = 2\delta^2 + 8\delta + 1$$

$$\frac{\gamma(1+\delta)}{\overline{x}} \qquad \frac{\sum x_i^2}{n} = \frac{2(\delta^2 + 4\delta + 6)}{3+\delta}$$

$$\overline{x} = \frac{3+\delta}{\gamma^2(1+\delta)^2} \tag{3.1}$$

Now: $\gamma^2\,(1+\delta)^2 = \frac{3+\delta}{\overline{x}}$

$$\frac{\sum x_i^2}{n} = \frac{2(\delta^2+4\delta+6)}{\gamma\,(1+\delta)\,\frac{3+\delta}{\overline{x}}}$$

$$\gamma\frac{\sum x_i^2}{n} = \frac{2\;\overline{x}\;(\delta^2+4\delta+6)}{(1+\delta)\,(3+\delta)}$$

$$\to\quad \widehat{\gamma}_{mm} = \frac{2n\overline{x}}{\sum x_i^2}\left[1+\frac{3}{(\delta^2+4\delta+3)}\right] \tag{3.2}$$

can be achieved by iteratively solving eq. (3.2) until $|\widehat{\gamma}_{i+1}-\widehat{\gamma}_i| < \epsilon$ that used to find $(\widehat{\delta}_{mm})$ by eq.(3.1) as:

$$\widehat{\delta}_{mm} = \gamma^2\left(1+\widehat{\delta}_0\right)^2\,\overline{x}-3 \tag{3.3}$$

which is numerically solvable. So the estimation of Reliability function by eq. (2.3) as:

$$\hat{R}_{mm} = 1-\frac{1}{\widehat{\delta}_{mm}}\left\{e^{-\widehat{\gamma}_{mm}x(1+\widehat{\delta}_{mm})}+\widehat{\delta}_{mm}-e^{-\widehat{\gamma}_{mm}x}(1+\widehat{\delta}_{mm})\right\}. \tag{3.4}$$

*3.2. Method of Maximum likelihood*

By eq. (2.1), the pdf is $\to f\,(x;\delta,\gamma) = \frac{1+\delta}{\delta}\;\gamma\;e^{-\gamma x}\left(1-e^{-\gamma\delta x}\right)$ the ML estimator as [2]:

$$L = \prod_{i=1}^{n} f\,(x_i,\delta,\gamma) = (1+\delta)^n\delta^{-n}\gamma^n e^{-\gamma\sum x_i}\prod_{i=1}^{n}\left(1-e^{-\gamma\delta x_i}\right)$$

$$\log L = n\log(1+\delta)\;-n\log\delta\;+n\log\gamma\;-\gamma\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log\left(1-e^{-\gamma\delta x_i}\right)$$

$$\frac{\partial\log L}{\partial\delta} = \frac{n}{1+\delta}-\frac{n}{\delta}+\sum_{i=1}^{n}\frac{\delta x_i e^{-\gamma\delta x_i}}{(1-e^{-\gamma\delta x_i})}$$

$$\frac{\partial\log L}{\partial\delta} = 0\quad\to\frac{n}{1+\widehat{\delta}}+\sum_{i=1}^{n}\frac{\gamma x_i e^{-\gamma\delta x_i}}{(1-e^{-\gamma\delta x_i})} = \frac{n}{\widehat{\delta}}.$$

So

$$\widehat{\delta}_{ML} = \frac{n}{\frac{n}{1+\widehat{\delta}}\;+\sum_{i=1}^{n}\frac{\gamma x_i e^{-\gamma\delta x_i}}{\left(1-e^{-\gamma\delta x_i}\right)}}. \tag{3.5}$$

Now to ML of $\widehat{\gamma}$:

$$\frac{\partial\log L}{\partial\gamma} = \frac{n}{\gamma}-\sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{-\delta x_i e^{-\gamma\delta x_i}}{(1-e^{-\gamma\delta x_i})}$$

$$\frac{\partial\log L}{\partial\gamma} = 0\quad\to\quad\frac{n}{\widehat{\gamma}} = \sum_{i=1}^{n}x_i-\sum_{i=1}^{n}\frac{\widehat{\delta}x_i e^{-\gamma\delta x_i}}{(1-e^{-\gamma\delta x_i})}.$$

So

$$\widehat{\gamma}_{ML} = \frac{n}{\sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{\widehat{\delta} x_i e^{-\gamma_0 \widehat{\delta} x_i}}{\left(1-e^{-\gamma_0 \widehat{\delta} x_i}\right)}} \tag{3.6}$$

and by eq. (2.3) the estimation of reliability function

$$\hat{R}_{ML} = 1 - \frac{1}{\widehat{\delta}_{ML}} \left\{ e^{-\widehat{\gamma}_{ML} x (1+\widehat{\delta}_{ML})} + \widehat{\delta}_{ML} - e^{-\widehat{\gamma}_{ML} x} (1 + \widehat{\delta}_{ML}) \right\}. \tag{3.7}$$

*3.3. Jackknife method*

This method used to improve the values of estimators [7] rely on one value is removed from the observation values ,then re estimate the (n-1) estimator values for all samples respectively, let $\widehat{\delta}_{ML}\,(J)$ *and* $\widehat{\gamma}_{ML}(J)$ represent the ML estimator after excluding one value, so the Jackknife estimator for $\delta, \gamma$ as:

$$\widehat{\delta}_{J(ML)} = n\widehat{\delta}_{ML} - (n-1) \frac{\sum_{j=1}^{n} \widehat{\delta}_{ML(j)}}{n} \tag{3.8}$$

$$\widehat{\gamma}_{J(ML)} = n\widehat{\gamma}_{ML} - (n-1)\, \frac{\sum_{j=1}^{n} \widehat{\gamma}_{ML(j)}}{n} \tag{3.9}$$

as well as the Reliability function, which is described as follows:

$$\hat{R}_{J(ML)} = 1 - \frac{1}{\widehat{\delta}_{J(ML)}} \left\{ e^{-\widehat{\gamma}_{J(ML)} x (1+\widehat{\delta}_{J(ML)})} + \widehat{\delta}_{J(ML)} - e^{-\widehat{\gamma}_{J(ML)} x} (1 + \widehat{\delta}_{J(ML)}) \right\} \tag{3.10}$$

## 4. Simulation

In this section, generating of random variables of this distribution will be done , with three sample sizes[ small=25 middle=75 and large=100], and parameter's default values:($\delta$=0.5,1.2,1.5),($\gamma$=0.3,0.6,1.8),in time ($t_i$ = 1.0, 1.2, 1.4, 1.6) four experiment with repeat sample (1000), equation (2.2) generated the following data as: let $G = e^{-\gamma x} \quad \rightarrow \ \ln G \ = -\gamma x \rightarrow x = -\frac{\ln G}{\gamma}$

$$\rightarrow \quad U = 1 - G \left( \frac{1+\delta}{\delta} \right) \left(1 - G^{\delta}\right), \tag{4.1}$$

where $U$ is an interval (0,1) random variable.

Equation (4.1) is used to generate different values of $x$, and equations (3.2,3.3,3.4,3.5,3.6,3.7,3.8,3.9, and 3.10) are used to calculate estimators for this distribution. The three methods are then compared using MSE to select the best in four experiments. The following tables show the outcomes.

Table 1: the values of MSE [ $\delta$=0.5 & $\gamma$=0.3]

| n | | | ML | MM | Jakknife | Best |
|---|---|---|---|---|---|---|
| 25 | $\widehat{\delta}$ | | 0.23716 | 0.76126 | 0.19855 | J |
| | $\widehat{\gamma}$ | | 0.02172 | 0.07465 | 0.02086 | J |
| | $\hat{R}$ | $t_1$ | 0.31432 | 0.31548 | 0.33677 | ML |
| | | $t_2$ | 0.27443 | 0.28432 | 0.30777 | ML |
| | | $t_3$ | 0.28654 | 0.25876 | 0.25655 | J |
| | | $t_4$ | 0.25876 | 0.26655 | 0.24766 | J |
| 75 | $\widehat{\delta}$ | | 0.23760 | 0.75824 | 0.31761 | ML |
| | $\widehat{\gamma}$ | | 0.01372 | 0.06123 | 0.01195 | J |
| | $\hat{R}$ | $t_1$ | 0.31632 | 0.32748 | 0.30966 | J |
| | | $t_2$ | 0.28643 | 0.26632 | 0.23977 | J |
| | | $t_3$ | 0.27854 | 0.28076 | 0.25855 | J |
| | | $t_4$ | 0.25076 | 0.25855 | 0.25566 | ML |
| 100 | $\widehat{\delta}$ | | 0.23757 | 0.75631 | 0.22072 | J |
| | $\widehat{\gamma}$ | | 0.01284 | 0.06785 | 0.00890 | J |
| | $\hat{R}$ | $t_1$ | 0.32432 | 0.29654 | 0.26766 | J |
| | | $t_2$ | 0.24654 | 0.24443 | 0.21564 | J |
| | | $t_3$ | 0.22566 | 0.27654 | 0.22877 | ML |
| | | $t_4$ | 0.21767 | 0.20691 | 0.20667 | J |

Table 2: the values of MSE [ $\delta$=0.5 & $\gamma$=0.6 ]

| n | | | ML | MM | Jakknife | Best |
|---|---|---|---|---|---|---|
| 25 | $\widehat{\delta}$ | | 0.00114 | 0.01040 | 0.00038 | J |
| | $\widehat{\gamma}$ | | 0.00384 | 0.20245 | 0.00381 | J |
| | $\hat{R}$ | $t_1$ | 0.00170 | 0.00172 | 0.00137 | J |
| | | $t_2$ | 0.00122 | 0.00123 | 0.00107 | J |
| | | $t_3$ | 0.00107 | 0.00107 | 0.00031 | J |
| | | $t_4$ | 0.00021 | 0.00013 | 0.00011 | J |
| 75 | $\widehat{\delta}$ | | 0.00113 | 0.0040 | 0.00109 | J |
| | $\widehat{\gamma}$ | | 0.00290 | 0.15390 | 0.00038 | J |
| | $\hat{R}$ | $t_1$ | 0.00139 | 0.00142 | 0.00128 | J |
| | | $t_2$ | 0.00112 | 0.00113 | 0.00106 | J |
| | | $t_3$ | 0.00092 | 0.00104 | 0.00032 | J |
| | | $t_4$ | 0.00013 | 0.00011 | 0.00013 | J, ML |
| 100 | $\widehat{\delta}$ | | 0.00121 | 0.00040 | 0.00030 | J |
| | $\widehat{\gamma}$ | | 0.00042 | 0.00848 | 0.00033 | J |
| | $\hat{R}$ | $t_1$ | 0.00132 | 0.00127 | 0.00122 | J |
| | | $t_2$ | 0.00109 | 0.00107 | 0.00105 | J |
| | | $t_3$ | 0.00031 | 0.00032 | 0.00031 | J, ML |
| | | $t_4$ | 0.00011 | 0.00013 | 0.00012 | ML |

Table 3: the values of MSE [ $\delta$=1.2 & $\gamma$=1.8 ]

| n | | | ML | MM | Jakknife | Best |
|---|---|---|---|---|---|---|
| 25 | $\widehat{\delta}$ | | 0.03120 | 9.70149 | 0.02915 | J |
| | $\widehat{\gamma}$ | | 0.14217 | 0.18581 | 0.01809 | J |
| | $\hat{R}$ | $t_1$ | 0.24154 | 0.23316 | 0.25338 | MM |
| | | $t_2$ | 0.09811 | 0.09866 | 0.09277 | J |
| | | $t_3$ | 0.08938 | 0.07935 | 0.07639 | J |
| | | $t_4$ | 0.10776 | 0.06671 | 0.03669 | J |
| 75 | $\widehat{\delta}$ | | 0.03060 | 8.12616 | 0.04551 | ML |
| | $\widehat{\gamma}$ | | 0.04943 | 0.14509 | 0.02776 | J |
| | $\hat{R}$ | $t_1$ | 0.24243 | 0.25140 | 0.23427 | J |
| | | $t_2$ | 0.09801 | 0.09805 | 0.09966 | ML |
| | | $t_3$ | 0.07287 | 0.08874 | 0.06028 | J |
| | | $t_4$ | 0.08265 | 0.06623 | 0.03758 | J |
| 100 | $\widehat{\delta}$ | | 0.03148 | 7.76765 | 0.03098 | J |
| | $\widehat{\gamma}$ | | 0.02308 | 0.09603 | 0.03966 | ML |
| | $\hat{R}$ | $t_1$ | 0.24233 | 0.25099 | 0.23437 | J |
| | | $t_2$ | 0.0991 | 0.09956 | 0.09276 | J |
| | | $t_3$ | 0.07037 | 0.06843 | 0.04038 | J |
| | | $t_4$ | 0.02875 | 0.05698 | 0.03687 | ML |

Table 4: the values of MSE [$\delta$=1.5 & $\gamma$=1.8 ]

| n | | | ML | MM | Jakknife | Best |
|---|---|---|---|---|---|---|
| 25 | $\widehat{\delta}$ | | 0.78800 | 4.73200 | 0.70051 | J |
| | $\widehat{\gamma}$ | | 1.61231 | 1.83286 | 0.97644 | J |
| | $\hat{R}$ | $t_1$ | 0.01046 | 0.01048 | 0.01013 | J |
| | | $t_2$ | 0.00998 | 0.00999 | 0.00983 | J |
| | | $t_3$ | 0.00983 | 0.00983 | 0.00097 | J |
| | | $t_4$ | 0.00097 | 0.00098 | 0.00097 | J,ML |
| 75 | $\widehat{\delta}$ | | 0.75760 | 4.13006 | 0.78967 | ML |
| | $\widehat{\gamma}$ | | 1.30520 | 1.23014 | 0.93521 | J |
| | $\hat{R}$ | $t_1$ | 0.01015 | 0.01018 | 0.01004 | J |
| | | $t_2$ | 0.00988 | 0.00989 | 0.00982 | J |
| | | $t_3$ | 0.00098 | 0.00980 | 0.00097 | J |
| | | $t_4$ | 0.00097 | 0.00097 | 0.00096 | J |
| 100 | $\widehat{\delta}$ | | 0.75458 | 4.12708 | 0.73708 | J |
| | $\widehat{\gamma}$ | | 1.11255 | 1.17858 | 0.85740 | J |
| | $\hat{R}$ | $t_1$ | 0.01008 | 0.01003 | 0.00998 | J |
| | | $t_2$ | 0.00985 | 0.00983 | 0.00981 | J |
| | | $t_3$ | 0.00097 | 0.00098 | 0.00098 | ML |
| | | $t_4$ | 0.00097 | 0.00099 | 0.00096 | J |

## 5. Conclusion

Based on the simulation findings in the tables above, the jackknife approach is the best and most efficient way for estimating the parameters and reliability function for this distribution, as evidenced by the MSE values for all experiments and sample sizes.

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