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Wavelet estimation of fractional cointegration vector for multivariate time series

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Abstract

Cointegration analysis is one of the most active areas in the econometrics and time series where different methods are introduced for identifying and estimating cointegration vectors in fractionally integrated time series. In this paper, we estimate the fractional cointegration vector by fully modified narrow band least squares method (FMNBLS) and a proposed method, depending on a linear regression model and wavelet theory, and assuming the errors of the model following ARFIMA model, also estimate the fractional parameter (long memory parameter) for each variable depending on the Wavelet Whittle method. These methods were applied on simulated multivariate data for functional magnetic resonance imaging (fMRI) using the R program and programming the proposed method by MATLAB program.

Keywords: fractional integration, fractional cointegration vector, VARFIMA, wavelet transformation, linear regression model, fMRI Data.

1. Introduction

Tim series with long memory can be observed in many areas of application which has attracted lots of interest in statistics and many applications.

Practically the multiple time series can be said to be cointegrated (one or more linear combination) if there a linear combination between variables. For multivariate time series which have fractional integration (d) where d is the parameter of long memory, then get fractional cointegration that become an important and relevant topic in empirical analysis.

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In 2003 Willa W. Chen and Clifford M. Hurvich introduced semiparametric estimation of multivariate fractional cointegration process of cointegration rank ($r > 0$). They estimate the cointegration relationship by the eigenvectors corresponding to the r smallest eigenvalue of an averaged periodogram matrix of tapered, differenced observations.

In 2009 Bent Jesper Christensen and Morten Ørregaard Nielsen consider semiparametric frequency domain analysis of cointegration between long memory processes, i.e. fractional cointegration, allowing derivation of useful long-run relations even among stationary processes. They form a narrow-band frequency domain least squares (FDLS) estimator of the cointegrating relation.

The aim of our research is to study the estimate cointegration vector in fractionally integrated time series. We proposed new method to estimate cointegration vector and compare with method Fully Modified Narrow Band Least Squares method (FMNBLS), based on criteria Mse.

2. Fractional integration (long memory)

Sowell and Mellon write a general differencing operator as $(1-L)^d$, for $d = \left[-\frac{1}{2}, \frac{1}{2}\right]$ the fractional differencing operator $(1-L)^d$ is defined by its Maclaurin series (binomial theorem) to be

$$(1-L)^d = \sum_{j=0}^{\infty} \binom{d}{j} (-1)^j L^j \tag{2.1}$$

where $\binom{d}{j}(-1)^j = \frac{\Gamma(d+1)(-1)^j}{\Gamma(d-j+1)\Gamma(j+1)} = \frac{\Gamma(-d+1)}{\Gamma(-d)\Gamma(j+1)}$

Because $\frac{1}{\Gamma(a)}$ is bounded and has roots at the nonpositive integers the sum defining $(1-L)^d$ has finite number of nonzero terms for $d = \left[-\frac{1}{2}, \frac{1}{2}\right]$ and $d \neq 0$. [15]

3. Multivariate long memory test

Philipp Sibbertsen and others, 2018, suggested a consistent multivariate test for the null hypothesis of true long memory against the alternative of spurious long memory. [18]

$$H_0 : f(\lambda_j) \sim \Lambda_j(d)\, G\Lambda_j^*(d) \qquad (True\ long\ memory) \qquad vs.$$

$$H_1 : f(\lambda_j) \nsim \Lambda_j(d)\, G\Lambda_j^*(d) \qquad (Spurious\ long\ memory)$$

The test statistic is based on the weighted sum of the partial derivatives of the multivariate local Whittle likelihood function which is given by:

$$\begin{aligned} MLWS = \frac{1}{2} \sup_{r\in[\varepsilon,1]} \Bigg\| & \frac{2}{\sqrt{\sum_{j=1}^{m} v_j^2}} \sum_{a=1}^{q} \eta_a \sum_{j=1}^{[\mathrm{mr}]} v_j \left(aG^{-1}\left(\widehat{d}\right) \mathrm{Re}\left[\Lambda_j\left(\widehat{d}\right)^{-1} I(\lambda_j)\, \Lambda_j^*\left(\widehat{d}\right)^{-1}\right]_a - 1\right) + \\ & \frac{1}{\sqrt{\sum_{j=1}^{m} v_j^2}} \sum_{a=1}^{q} \eta_a \left(aG^{-1}\left(\widehat{d}\right)\right) \sum_{j=1}^{[\mathrm{mr}]} \frac{\lambda_j - \pi}{2} \mathrm{Im}\left[\Lambda_j\left(\widehat{d}\right)^{-1} I(\lambda_j)\, \Lambda_j^*\left(\widehat{d}\right)^{-1}\right]_a \Bigg\| \end{aligned} \tag{3.1}$$

4. Fractional cointegration

The concept of fractional cointegration is attracting increasing attention from both theoretical and empirical researchers in economics and finance.

Using the definition for fractional cointegration integration (FCI) introduced by Robinson and Marinucci (1987) (FCI) for $(k \times 1)$ vector z_t , $t = 1, \ldots, n$ whose i$^{\text{th}}$ element $z_t \equiv I(d_i)$, $d_i > 0$, $i = 1, \ldots, k$ and $z_t \equiv FCI(d_1, \ldots, d_k, d_e)$ if there exists a $(k \times 1)$ vector $\alpha \neq 0$ (usually unknown) such that $\alpha' z_t = e_t \equiv I(d_e)$ where $0 \leq d_e \leq \min_{1 \leq i \leq k} d_i$.

Robinson and Marinucci (2001) partition z_t as $z_t = \left(x_t', y_t\right)'$ where y_t is a scalar and $x_t = (x_{1,t}, \ldots, x_{k-1,t})'$ then it can say that z_t fractionally cointegrated of order $(d_1, \ldots, d_{k-1}, d_y; d_e)$ and written $z_t \in FCI(d_1, \ldots, d_k; d_e)$ if $x_{i,t}$ is $I(d_i)$, $i = 1, \ldots, k-1$, and y_t is $I(d_y)$ and if there exists a certain linear combination of z_t $(k-1) \times 1$ vector α such that $e_t = y_t - \alpha' x_t$ is $I(d_e)$ with $d_e < \min(d_i)$ or $d_e < d_y$ and this definition entails $d_i = d_y > d_e$ for at least one i. [9]

5. Fractional cointegration test

Morten Ørregaard Nielsen proposed a nonparametric variance ratio testing approach for testing fractional cointegration without prior knowledge of the integration order of the data, the strength of the cointegrating relations, or the cointegration vector. [11]

The vector time series (Z_t) with components are integrated of different orders, then $(Z_t \in I(d))$ where (d) is the highest order of integration of the components of (Z_t), then the time series may have cointegration. The test statistic does not depend on the integration order of the observed variables, (d) which is given as: [11]

$$\Lambda_{n,r}(d_1) = T^{2d_1} \sum_{j=1}^{n-r} \lambda_j \qquad , \qquad r = 0, 1, \ldots, n-1 \tag{5.1}$$

where λ_j are eigenvalues.
The large values of $\Lambda_{n,r_0}(d_1)$ are associated with rejection of hypothesis:

$$\begin{aligned} &H_0 : r = r_0 \quad \text{(no fractional cointegration)} \qquad\qquad \text{vs.} \\ &H_1 : r > r_0 \quad \text{(fractional cointegration)} \end{aligned}$$

When $\Lambda_{n,r_0}(d_1) > CV_{\zeta,n-r_0}(d, d_1)$ the test rejects the null hypothesis H_0 and has asymptotic size ζ and it is consistent against H_1, where $CV_{\zeta,n-r_0}(d, d_1)$ critical value is calculated from:

$$P\left(U_{n-r_0}(d, d_1) > CV_{\zeta,n-r_0}(d, d_1)\right) = \zeta \tag{5.2}$$

6. ARFIMA model

The best model for representing data having long memory properties is ARFIMA model (Auto Regressive Fractionally Integrated Moving Average) so it can be modeling a multivariate time series z_t having fractional integration by VARFIMA (Vector of Auto Regressive Fractionally Integrated Moving Average).
In 1980, Granger and Joyeux introduced ARFIMA model which is written as: [8]

$$\phi(L)(1-L)^d Y_t = \theta(L) e_t \begin{cases} |d| < \frac{1}{2} \\ e_t i.i.d \ \sim \ N(0, \sigma_e^2) \end{cases} \tag{6.1}$$

where L : Backshift operator.

$\phi(L)$: Auto regressive polynomial of order (p).

$\theta(L)$: Moving average polynomial of order (q).

e_t : White noise.

The multivariate generalization VARFIMA model would be z_t $(a\ k \times 1)$ vector time series such that:

$$\varphi(L)\, D(L)\, V z_t = \vartheta(L)\, \varepsilon_t \tag{6.2}$$

where $\varphi(L) = (\varphi_0 - \varphi_1 L - \varphi_2 L^2 - \ldots - \varphi_p L^p)$ and $\vartheta(L) = (\vartheta_0 + \vartheta_1 L + \vartheta_2 L^2 + \ldots + \vartheta_q L^q)$.
$\varphi(L)$ and $\theta(L)$ are $k \times k$ matrix polynomials in the lag operator L. It will be assumed that $D(L) = diag\left[(1-L)^{d_1}, (1-L)^{d_2}, \ldots, (1-L)^{d_k}\right]$, $\varphi(L)$ is of order p, $\theta(L)$ is of order q, $\varphi(0) = \theta(0) = I_k$, the roots of $|\varphi(a)|$ and $|\theta(a)|$ are outside the unit circle and $\varepsilon_t \sim IIDN_k(0, \Sigma)$. [3, 7, 13].
The constant $k \times k$ matrix V is nonsingular. The simple form of the differencing matrix $D(L)$ means that the characteristics of the fractional z_t vector series stated below can be obtained by the univariate proofs applied by element, in particular: [5], [6]

1) z_t is stationary if $d_i < \frac{1}{2}$ for $i = 1, 2, \ldots, k$.

2) z_t possess an invertible moving average representation if $d_i > -\frac{1}{2}$.

3) If the spectral density of z_t is denoted $f_z(\lambda)$ then as $\lambda \to 0$, $f_z(\lambda) \sim \left[\kappa_{\mathrm{ij}} \lambda^{-(d_i + d_j)}\right]$ where each κ_{ij} is constant and is independent of d_i and d_j.

4) If the autocovariances of z_t are denoted $\gamma_z(s) = E\left[x_t x'_{t-s}\right]$ then as $s \to \infty$, $\gamma_z(s) \sim \left[h_{\mathrm{ij}} s^{d_i + d_j - 1}\right]$ where each h_{ij} is constant and is depend on d_i and d_j.

7. Wavelet theory

Wavelet analysis is a new development in the area of applied mathematics. The use of wavelets as a tool for time series analysis and signal processing has increased in recent years due to their potential for solving a number of practical problems; Daubechies (1992) provides an extensive look at the mathematical properties of wavelets. Chui (1992), and Strang and Nguyen (1996) are good introductions to wavelets.

The text by Gen , Cay et al. (2002) gives a good discussion on how wavelets can be applied in economics and finance. Ramsey (1999, 2002) and Schleicher (2002) also give some additional insights on how wavelet analysis can be adopted in economics and finance.

Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale.

From many papers the wavelet can be defined as (They are small waves with a value (amplitude) starting from zero and a specific time period with zero mean).

The wavelet transform is defined by many authors as a mathematical technique in which a particular signal is analysed in both the time domain and frequency domain by applying dilations and translations (or shifted) of the mother function (detailed or wavelet function) $\psi(.)$ and father function (Approximation or scaling function) $\varphi(.)$.

For a given resolution $j \geq 0$ and $p \in \mathbb{Z}$, the dilated and translated mother and father functions (function for the next stage of transformation) can be defined as: [1]

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \tag{7.1}$$

$$\varphi_{a,b}(t) = \frac{1}{\sqrt{a}} \varphi\left(\frac{t-b}{a}\right) \tag{7.2}$$

where $a \in \mathbb{R}_+$: scaling parameter and $b \in \mathbb{R}$: Translation parameter.

The wavelet coefficients can be got by multi resolution analysis process (MRA) then these coefficients arranged as linear combination to represent the original signal as:

$$f_w(t) = \sum_{i=0}^{2^j-1} a_i \varphi(t-i) + \sum_{i=0}^{2^j-1} \sum_{j=0}^{J-1} d_{\text{ji}} \psi\left(2^j t - i\right) \tag{7.3}$$

where a_i : Approximate coefficient and $d_{j,i}$: Detail coefficients

The wavelet transformation for data can be processed by multiplying the data by wavelet transformation matrix (W). There are different types of wavelet function such as Daubechies order 1 (Haar) which is the simplest one, Daubechies order 4, 8, Symmlet order 3, 7, Coiflet order 1, 5 etc. where the basic function used in the wavelet transformation must fit as closely as possible to the signal to be transformed.

8. Wavelet Whittle estimation of (d)

[2] introduced a semiparametric estimation of multivariate long-range dependent processes based on wavelet theory and Whittle estimation which estimate the vector of fractional parameter (d).

The wavelet Whittle likelihood function is defined as:

$$\mathcal{L}(G(d), d) = \frac{1}{n} \sum_{j=j_0}^{j_1} \left[n_j \log\det\left(\Lambda_j(d) G(d) \Lambda_j(d)\right) + \sum_{k=0}^{n_j} W_{j,k}^T \left(\Lambda_j(d) G(d) \Lambda_j(d)\right)^{-1} W_{j,k} \right] \tag{8.1}$$

where $j_1 \geq j_0 \geq 1$ The maximal and minimal resolution level.

$W_{j,k}$: Wavelet coefficients.

$\Lambda_j(d) = diag\left(2^{\text{jd}}\right)$

$G(d)$: Matrix of $(l,m)^{\text{th}}$ element is $G_{l,m}(d) = \Omega_{l,m} K(d_l + d_m) \cos\left(\pi \frac{(d_l + d_m)}{2}\right)$

Then they estimate vector of fractional integration (d) as:

$$\widehat{d} = \text{argmin}_d R(d) \tag{8.2}$$

where $R(d) = \log\det\left(\frac{1}{n}\sum_{j=j_0}^{j_1}\left(\Lambda_j(d)^{-1}\left(\sum_{k=0}^{n_j} W_{j,k} W_{j,k}^T\right)\Lambda_j(d)^{-1}\right)\right) + 2\log_2\left(\frac{1}{n}\sum_{j=j_0}^{j_1} j n_j\right)\left(\sum_{l=1}^{p} d_l\right)$.

9. Estimating fractional cointegration vector

[12] introduced a fully modified narrow-band least squares (FMNBLS) estimation of the fractional cointegrating vector as given:

$$\widehat{\beta}_m(\gamma) = \widehat{F}_{\text{xx}}^{-1}(\gamma, 1, m)\, \widehat{F}_{\text{xy}}(\gamma,\ 1,\ m) \tag{9.1}$$

where $\widehat{F}_{\text{qr}}(\gamma,\ k,\ l) = \frac{2\pi}{T} \sum_{j=k}^{l} \text{Re}\left[\frac{1}{2\pi T} \sum_{t=1}^{T} \sum_{s=1}^{T} \left(\Delta^\gamma q_t\right)\left(\Delta^\gamma r_s\right)' e^{-i(t-s)\lambda_j}\right] \quad , \quad 0 \leq k \leq l \leq T-1.$

$\Delta^\gamma q_t$, $\Delta^\gamma r_s$ cross-periodogram matrix between the observed vectors.

$\lambda_j = \frac{2\pi j}{T}$, $\frac{1}{m} + \frac{m}{T} \to 0 \ as \ T \to \infty$.

10. Proposed method for estimating fractional cointegration vector

After estimating the fractional parameter for each variable (d_i), and according to [9] partition z_t as mentioned in (section 3), it can be written the multivariate variable as linear regression model by taking (Y) is the variable has the smallest fractional parameter and X is a matrix for the remaining variables of order (k-1×1) as:

$$Y = X\beta + u \tag{10.1}$$

where β is the regression coefficient estimation (parameters vector) of the model that needed to estimate which is represent the fractional cointegration vector of the multivariate time series and u is the residual vector.

The process of written the variable as linear regression model caused auto correlation between errors u which mean they have short or long memory. Therefore, this situation must be taken into account in the estimation process, because if they are ignored, they will lead to inaccurate results.

The autocorrelation of errors will be eliminated by using the wavelet transform on the variables to convert the linear regression model to wavelet domain and applying the maximum likelihood function on the new model.

By assuming the multivariate data z_t following VARFIMA model and by applying partition to z_t where Y is a univariate ARFIMA then directly the errors u of the linear regression model in the equation (9.1) take the form of a stable process with long-memory property and follow the ARFIMA model and have a spectral density function, written as:

$$f(\lambda) = \left|1 - e^{i\lambda}\right|^{-2d} f_0(\lambda) \tag{10.2}$$

where $f_0(\lambda)$ piecewise positive symmetric continues function, $\lambda \in (-\pi, \pi]$ and $\left(0 < d < \frac{1}{2}\right)$.

Rewrite model in equation (9.1) in wavelet domain as:

$$Y_W = X_W\beta + u_W \tag{10.3}$$

where $Y_w =$ W.Y, $X_w = W.X$ and $u_w = W.u$ ($u_w \sim N(0, \Sigma_{u_w})$, $\Sigma_{u_w} = \sigma^2\Sigma_w$).

According to Fadili and Bullmore (2002), the variance and covariance matrix is diagonal and its elements represent the variances of the scaling and approximation coefficients as: [4]

$$\Sigma_{u_w} = \begin{bmatrix} S_{aj} & & & & \\ & S_{dj} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} S_{d2} & & \\ & \ddots & \\ & & S_{d2} \end{pmatrix} & \\ & & & & \begin{pmatrix} S_{d1} & & \\ & \ddots & \\ & & S_{d1} \end{pmatrix} \end{bmatrix} \tag{10.4}$$

where S_{a_J} Scaling coefficient variance given by:

$$S_{a_J} = \frac{2^j}{2\pi} \int_{-\infty}^{\infty} \frac{\sigma^2 C_\gamma}{w^\gamma} \left|\varphi(2^j w)\right|^2 \mathrm{dw} \tag{10.5}$$

and S_{d_j} Wavelet coefficients variance given by:

$$S_{\mathrm{dj}} = \frac{2^j}{2\pi}\int_{-\infty}^{\infty}\frac{\sigma^2 C_\gamma}{w^\gamma}\left|\psi(2^j w)\right|^2 \mathrm{dw} \tag{10.6}$$

φ : Fourier transformation for scaling function.
ψ : Fourier transformation for wavelet function.
C_γ : Constant in term of (γ).

Equations (10.5) and (10.6) can be simplified when the wavelet function converges to an ideal band-pass filter and the scaling function converges to an ideal low-pass filter given by:

$$S_{a_J} \approx \frac{2^{J+1}}{(2\pi)^\gamma}\int_0^{2^{-(J+1)}}\frac{\sigma^2 C_\gamma}{f^\gamma}\mathrm{df} \tag{10.7}$$

$$S_{d_j} \approx \frac{2^{j+1}}{(2\pi)^\gamma}\int_{2^{-(j+1)}}^{2^{-j}}\frac{\sigma^2 C_\gamma}{f^\gamma}df \qquad , j \in \{1, \ \ldots, \ J\} \tag{10.8}$$

By integration got:

$$S_{a_J} \approx \frac{\sigma^2 C_\gamma \left(2^{j+1}\right)^\gamma}{(2\pi)^\gamma\,[1-\gamma]} = \sigma^2 S_{a_J}(\gamma) \tag{10.9}$$

$$S_{d_j} \approx \frac{\sigma^2 C_\gamma \left(2^{j}\right)^\gamma}{(2\pi)^\gamma\,[1-\gamma]}\,[2-2^\gamma] = \sigma^2 S_{d_j}(\gamma) \tag{10.10}$$

It is noted from the previous equations that they depends on the correlation parameter (γ) that defines on the domain $(-1 < \gamma < 1)$ that must be estimated for the purpose of writing the variance-covariance matrix in eq. (16), but there is a mechanism depending on the field for (γ) that can Write $(\gamma = 2d)$ where (d) is a parameter of fractional differences, then got:

$$S_{a_J} \approx \frac{\sigma^2 C_d}{[1-2d]}\pi^{-2d}2^{2jd} \tag{10.11}$$

$$S_{d_j} \approx \frac{\sigma^2 C_d}{[1-2d]}\pi^{-2d}2^{2jd}\left[2^{1-2d}-1\right] \qquad , \qquad j = 1, 2, \ldots, J \tag{10.12}$$

where C_d : Constant in term of (d).
The maximum likelihood function for model in eq. (10.3) given by:

$$L(\theta) = \frac{\left|\Sigma_{u_w}^{-1}\right|}{(2\pi)^{\frac{n}{2}}}e^{-\frac{1}{2}(Y_w - X_w\beta)'\,\Sigma_{u_w}^{-1}(Y_w - X_w\beta)} \tag{10.13}$$

where $\Sigma_{u_w}^{-1}$ variance covariance inverse matrix in wavelet domain and $(\theta^t = \beta^t, d, \sigma^2)$.

From formula (10.13) $(Y_w - X_w\beta)'\,\Sigma_{u_w}^{-1}(Y_w - X_w\beta)$ can be simplified by performing some algebraic operations, so got:

$$(Y_w - X_w\beta)'\,\Sigma_{u_w}^{-1}(Y_w - X_w\beta) = \left(Y_w - X_w\widehat{\beta}\right)'\,\Sigma_{u_w}^{-1}\left(Y_w - X_w\widehat{\beta}\right) + \left(\beta - \widehat{\beta}\right)' X_w'\Sigma_{u_w}^{-1}X_w\left(\beta - \widehat{\beta}\right) \tag{10.14}$$

By taking the logarithm for formula (10.13) got:

$$\log L(\theta) = -\frac{1}{2}\left[n\log\left(2\pi\sigma^2\right) + \log\left(S_{a_J}\right) + \sum_j n(j)\log\left(S_{d_j}\right)\right] - \frac{1}{2\sigma^2}\left[\frac{\left(Y_{J,0} - \sum_{s=1}^{P} X^s_{J,0}\beta_s\right)^2}{S_{a_J}} + \sum_{j,l}\frac{\left(Y_{j,l} - \sum_{s=1}^{P} X^s_{j,0}\beta_s\right)^2}{S_{d_j}}\right] \tag{10.15}$$

where $n(j) = \frac{n}{2^j}$.
Then the estimation of coefficients vector β given as:

$$\widehat{\beta} = \left(X'_w \widehat{\Sigma}^{-1}_{u_w} X_w\right)^{-1} X'_w \widehat{\Sigma}^{-1}_{u_w} Y_w \tag{10.16}$$

and variance-covariance matrix for $\widehat{\beta}$ given as:

$$v - cov(b) = \left(X'_w \Sigma^{-1}_{u_w} X_{\ w}\right)^{-1} = \sigma^2_u\left(X'_w \Sigma^{-1}_w X_{\ w}\right)^{-1} \tag{10.17}$$

and the estimation of (σ^2) given by:

$$\widehat{\sigma}^2_{\mathrm{ML}} = \frac{1}{n}\left[\frac{a^2_{J,0}}{S_{a_J}} + \sum_{j,l}\frac{d^2_{j,l}}{S_{d_j}}\right] \tag{10.18}$$

In this paper, researcher has derived and simplified the equation (16) for Σ_{u_w} as given:

$$\Sigma_{u_w} = \sigma^2 \begin{bmatrix} S_{aj}(\gamma) & & & & \\ & S_{dj}(\gamma) & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} S_{d2}(\gamma) & & \\ & \ddots & \\ & & S_{d2}(\gamma) \end{pmatrix} & \\ & & & & \begin{pmatrix} S_{d1}(\gamma) & & \\ & \ddots & \\ & & S_{d1}(\gamma) \end{pmatrix} \end{bmatrix} \tag{10.19}$$

which can be written as:

$$\Sigma_{u_w} = \sigma^2 \Sigma_w \tag{10.20}$$

The main diameter elements can be written in terms of the parameter (d) (long memory parameter) as given:

$$S_{a_J}(\gamma) = \frac{C_d}{[1-2d]}\pi^{-2d}2^{2Jd} = S_{a_J}(d) \tag{10.21}$$

$$S_{d_j}(\gamma) = \frac{C_d}{[1-2d]}\pi^{-2d}2^{2jd}\left[2^{1-2d} - 1\right] = S_{\mathrm{dj}}(d) \tag{10.22}$$

where $\frac{C_d}{[1-2d]}\pi^{-2d}2^{-2d}2^{2dj}\left(2-2^{2d}\right)=\frac{C_d}{[1-2d]}\pi^{-2d}2^{2dj}\left(2^{1-2d}-1\right)$. then

$$|\Sigma_{u_w}|=\sigma^2\left[S_{a_J}(d)\prod_{j=1}^{\frac{n}{2^j}}S_{\mathrm{dj}}(d)\right] \tag{10.23}$$

$$\Sigma_{u_w}=\begin{bmatrix} \frac{1}{S_{aj}(d)} & & & & \\ & \frac{1}{S_{dj}(d)} & & & \\ & & \ddots & & \\ & & & \begin{pmatrix} \frac{1}{S_{d2}(d)} & & \\ & \ddots & \\ & & \frac{1}{S_{d2}(d)} \end{pmatrix} & \\ & & & & \begin{pmatrix} \frac{1}{S_{d1}(d)} & & \\ & \ddots & \\ & & \frac{1}{S_{d1}(d)} \end{pmatrix} \end{bmatrix} \tag{10.24}$$

From equation (10.23) the likelihood function can be written as:

$$L(\theta)=(2\pi)^{-\frac{n}{2}}\left(\sigma^2\right)^{-\frac{n}{2}}|\Sigma_w|^{-\frac{1}{2}}\,e^{\left[-\frac{1}{2\sigma^2}(Y_w-X_w\beta)'\Sigma_w^{-1}(Y_w-X_w\beta)\right]} \tag{10.25}$$

by taking the logarithm got:

$$\log L(\theta)=-\frac{n}{2}\log 2\pi-\frac{n}{2}\log\sigma^2-\frac{1}{2}\left[\log S_{a_J}(d)+\sum_j\left(\frac{n}{2^j}\right)\log S_{\mathrm{dj}}(d)\right]- \frac{1}{2\sigma^2}\left[(Y_w-X_w\beta)'\Sigma_w^{-1}(Y_w-X_w\beta)\right] \tag{10.26}$$

where $(Y_w-X_w\beta)'\Sigma_w^{-1}(Y-X_w\beta)=\left[{y_w}'\Sigma_w^{-1}y_w-Y\Sigma_w^{-1}X_w\beta-\beta'{X_w}'\Sigma_w^{-1}Y_w+\beta'{X_w}'\Sigma_w^{-1}X_w\beta\right]=\left[{y_w}'\Sigma_w^{-1}y_w-2\beta'{X_w}'\Sigma_w^{-1}y_w+\beta'{X_w}'\Sigma_w^{-1}X_w\beta\right]$.
By derived equation (10.25) according to (β):

$$\frac{\partial\log L(\theta)}{\partial\beta'}=-\frac{1}{2\sigma^2}\left[0-2{X_w}'\Sigma_w^{-1}Y_w+2{X_w}'\Sigma_w^{-1}X_w\beta\right] \tag{10.27}$$

and let $\frac{\partial\log L(\theta)}{\partial\beta'}=0$, then

$$\widehat{\beta}=\left({X_w}'\Sigma_w^{-1}X_w\right)^{-1}{X_w}'\Sigma_w^{-1}y_w \tag{10.28}$$

For estimating (σ^2) derived equation (10.25) according to (σ^2):

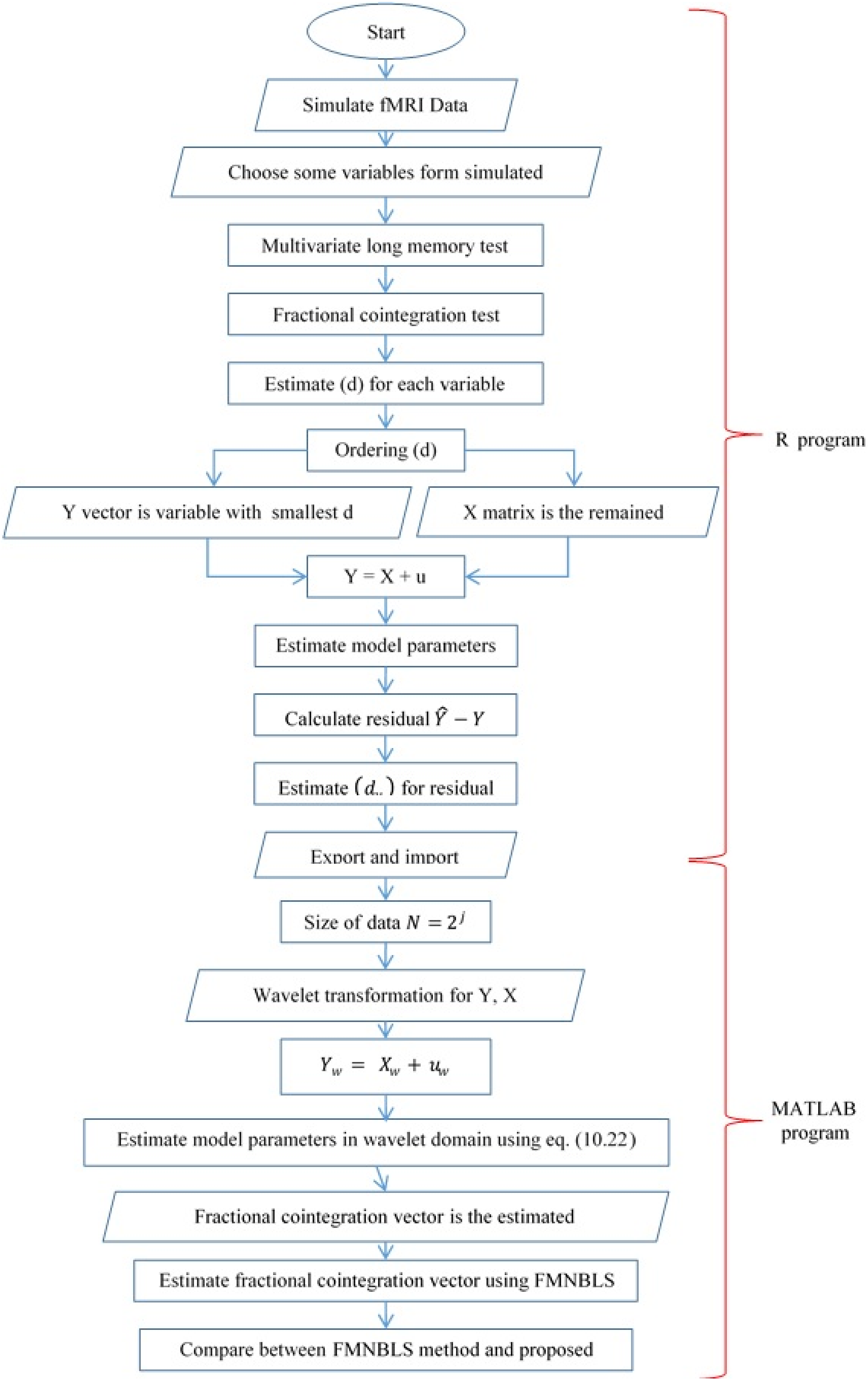
$$\frac{\partial\log L(\theta)}{\partial\sigma^2}=-0-\frac{1}{2\sigma^2}-0+\frac{1}{2\sigma^4}\left[\frac{\left(Y_{J,0}-\sum_{s=1}^P X_{J,0}^s\beta_s\right)^2}{S_{a_J}}+\sum_{j,l}\frac{\left(Y_{j,l}-\sum_{s=1}^P X_{j,0}^s\beta_s\right)^2}{S_{d_j}}\right] \tag{10.29}$$

where $\left(Y_{J,0} - \sum_{s=1}^{P} X_{J,0}^{s}\beta_s\right)^2 = a_{J,0}^2$ and $\left(Y_{j,l} - \sum_{s=1}^{P} X_{j,0}^{s}\beta_s\right)^2 = d_{j,l}^2$.
and let

$$\begin{aligned} \frac{\partial \log L}{\partial \sigma^2} = 0 \Rightarrow -\frac{n}{2\widehat{\sigma}^2} + \frac{\left[\frac{a_{J,0}^2}{S_{a_J}} + \sum_j \frac{d_{j,l}^2}{S_{d_j}}\right]}{2\widehat{\sigma}^4} = 0 \\ \widehat{\sigma}^2 = \frac{1}{n}\left[\frac{a_{J,0}^2}{S_{a_J}} + \sum_{j,l} \frac{d_{j,l}^2}{S_{d_j}}\right] \end{aligned} \tag{10.30}$$

11. Simulation

The fMRI is one of the most advanced neuroimaging techniques which uses the standard magnetic resonance imaging (MRI) to examine the brain functions. It measures the changes in the blood oxygen level-dependent (BOLD) signal which is related to the neuronal activity[16].
R program and MATLAB program will be used for programming the method. With R program a time series for each region of interest in the brain will be simulated as a real fMRI. A data frame with 1200 observations on the following 89 variables will be simulated [19].
The fractional parameter will be estimated for some simulated variables of fMRI using multivariate wavelet Whittle estimation for the long-memory parameter vector (d), by arranged the d's values, the variable of smallest (d) represent the Y vector and the other variables represent the X matrix of linear regression model. The parameter of model will be estimated by (OLS), then calculate the residual of model (u), and then estimate (d_u) for residual by the same method above.
The fractional cointegration vector will be estimated by Fully Modified Narrow Band Least Squares method (FMNBLS). Exporting data to excel file to use in MATLAB and determine the size of data to (1024) because the size of data in wavelet transformation using multi resolution analysis process must be $N = 2^j$ where j is the maximum number of transformation stage (levels) then wavelet transformation will be applied on Y and X using simplest type Haar wavelet (db1).
A comparison between the two methods of estimating fractional cointegration vector will be made using. The flowchart below summaries the proposed method.


Start
Simulate fMRI Data
Choose some variables form simulated
Multivariate long memory test
Fractional cointegration test
Estimate (d) for each variable
Ordering (d)
Y vector is variable with smallest d
X matrix is the remained
Y = X + u
Estimate model parameters
Calculate residual $\hat{Y} - Y$
Estimate ($d_{..}$) for residual
Export and import
Size of data $N = 2^j$
Wavelet transformation for Y, X
$Y_w = X_w + u_w$
Estimate model parameters in wavelet domain using eq. (10.22)
Fractional cointegration vector is the estimated
Estimate fractional cointegration vector using FMNBLS
Compare between FMNBLS method and proposed
R program
MATLAB program

The fMRI simulated data, the six variables that chosen is given below:

F1OG	F1OD	F2G	F2D	F2OG	F2OD

The result of testing long memory and fractional cointegration shown in Table (1):

Table (1)				
Test	**Default (α)**	**Statistic value**	**Critical value**	**Decision**
Multivariate long memory	0.05	1.376663	1.426	Accept H_0
Fractional cointegration	0.01	18.40958	13.85072	Reject H_0

The result of ordered estimating fractional parameter (d_i) for variables and the fractional parameter for residual (d_u) with multivariate wavelet Whittle estimation method shown in table (2) where used Daubechies wavelet db(1) or Haar wavelet:

Table (2)	
Fractional parameter (d) estimation for variables using db(1) J=10	
d_2	0.2274028
d_1	0.2992724
d_5	0.3857055
d_6	0.3337001
d_3	0.5279570
d_4	0.5332106
d_u	0.1661185

From table (2) after ordering (d_i), chosen variable (2) as (Y) because it has the smallest (d) and remaining variable represent (X), also the value of estimated (d_u) is always small than (d_i).
The estimation of fractional cointegration vector is shown in table (3):

Table (3)	
Proposed method using db(1) (Haar)	(FMNBLS) methoed
0.10891711486	0.38888477
0.20208895979	0.79257251
0.19961494736	0.99241467
-0.04516462367	-0.19205951
0.3286213136	0.09973917

From table (4) notice that the MSe for proposed method is smaller than the MSe for FMNBL which mean the proposed method is better for estimate the fractional cointegration vector.

Table (4)	
MSe FMNBL	MSe Proposed method
$1.3500363*10^5$	$2.982702*10^7$

12. Conclusion

1. From multivariate long memory test, Accept H_0 it is means true long memory.

2. From fractional cointegration test, reject H_0 it is means there is a cointegration between variables.

3. Estimating fractional parameter (d_i) by Daubechies wavelet db(1) and chosen variable (2) as (Y) because it has the smallest (d) and remaining variable represent (X).

4. The value of estimated (d_u) is always small than (d_i).

5. Estimation of fractional cointegration vector using (FMNBLS) method and the proposed method.

6. The Mse for proposed method is smaller than the MSe for FMNBL which mean the proposed method is better.

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