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Reparameterization and the conditional inverse of a balanced factorial experiment with three factors

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Abstract

In this research, a factorial experiment 2^3 was studied through a balanced mathematical model applied in a complete random design (CRD) to show the effect of the main factors and the interactions between the factors through the use of the general linear model in which the design matrix $(X'X)$ has less than full rank and thus the parameters vector (β) is neither estimable nor testable. Therefore, the re-parameter method and conditional inverse were used to transform the design matrix $(X'X)$ to a full-rank matrix, so that the parameters vector (β) is capable of estimable and testable, after analyzing the experiment data and testing hypotheses it was found that the interactions $(\alpha\beta\gamma)^*_{ijk}$ and $(\alpha\beta)^*_{ij}$ are not significant, while the factors $(\alpha)^*_i$, $(\beta)^*_j$, $(\gamma)^*_k$ and the interactions $(\alpha\gamma)^*_{ik}$ and $(\beta\gamma)^*_{jk}$ have significant effects.

Keywords: Three factors, balanced, estimable, treatment, general linear model, testable, less than full rank, conditional inverse, ANOVA, full rank, test, statistic, generalization, reparameterization.

1. Introduction

Analysis of variance (ANOVA) is a statistical tool used extensively in the biological, psychological, medical, ecological, and environmental sciences, Design of experiments begins with determining the objectives of an experiment and selecting the process factors for the study. An experimental design is the laying out of a detailed experimental plan in advance of doing the experiment, Well-chosen experimental design maximizes the amount of "information" that can be obtained for a given amount of experimental effort.

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There are two types of experiments in the design of experiments: first, simple experiments, and secondly, factorial experiments. The researcher will be interested in a factorial experiment that contains three factors through the use of a factorial experiment (2^3) applied in a complete random design (CRD) to show the effect of the main factors and the interactions between factors through the use of the general linear model or general multivariate regression model is a compact way of simultaneously writing several multiple linear regression models. In that sense it is not a separate statistical linear model.

The general linear model incorporates a number of different statistical models: ANOVA, ANCOVA, MANOVA, MANCOVA, ordinary linear regression, t-test and F-test. The general linear model is a generalization of multiple linear regression to the case of more than one dependent variable [8]. The aim of the research is to apply the reparameterization method and the conditional inverse to a factorial experiment (2^3) using the general linear model, as the problem was the impossibility of analysis when using the general linear model because the design matrix with less than full rank, where the less than full rank of the matrix was treated using two methods of reparameterization and the conditional inverse, analyze the experiment data and get the results.

2. Factorial Experiments

These are experiments in which the interest is to study the effect of two or more factors in one experiment, using all possible combinations between many different levels of the factors to be studied. These experiments are used to study the main effects of each factor individually, as well as study the interaction at the same time. [1]

The reason for studying the Factorial experiments:

1. The presence of more than one factor affecting the experiment in order to shorten the time instead of designing an independent experiment for each factor
2. Finding the interplay between the factors affecting the experimental material
3. It is not possible to perform several experiments. [2]

3. Three-factor Interaction Model

In factorial experiments in which each factor has only two levels, the number of treatments will be equal to 2^n. If factor A has two levels, Factor B has two levels, and Factor C has two levels. So, the number of processors used in the experiment ($2^3 = 8$). Which ($a_0b_0c_0 = 1$. $a_0b_0c_1 = c$. $a_0b_1c_0 = b$. $a_0b_1c_1 = bc$. $a_1b_0c_0 = a$. $a_1b_0c_1 = ac$. $a_1b_1c_0 = ab$. $a_1b_1c_1 = abc$)

The mathematical model of the factorial experiment is:

$$\begin{aligned} y_{ijkl} = \ & \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijkl} \\ & i = 1,\ 2, \qquad j = 1,\ 2 \qquad k = 1,\ 2 \end{aligned} \tag{3.1}$$

Where, α_i denotes the effect of the i^{th} level of factor A, β_j is the effect of the j^{th} level of factor B, γ_k is the effect of the k^{th} level of factor C, $(\alpha\beta)_{ij}$ denotes the interaction between the i^{th} level of factor A and the j^{th} level of factor B, $(\alpha\gamma)_{ik}$ denotes the interaction between the i^{th} level of factor A and the k^{th} level of factor C, $(\beta\gamma)_{jk}$ denotes the interaction between the j^{th} level of factor B and the k^{th} level of factor C, $(\alpha\beta\gamma)_{ijk}$ denotes the interaction between the i^{th} level of factor A and the j^{th} level of factor B and the k^{th} level of factor C, and ε_{ijkl} the error term associated with y_{ijkl}[4].

4. Determining Interactions for a Completely Randomized Factorial Design

The interactions in a design can be determined by writing down all combinations of treatment letters while preserving the alphabetical order of the letters. For example, the letters for treatments A, B, and C can be combined as follows: AB, AC, BC, and ABC. Interactions that involve two letters are called two-treatment interactions, first-order interactions, or double interactions. If three letters are involved, the interaction is a three-treatment interaction, second order or triple interaction, and so on [5]. Three treatment letters (A, B, C) can be combined as follows:
Tow treatment interaction: AB AC BC
Three treatment interaction: ABC
In general, the number of two-, three-, and so on treatment interactions in completely randomized factorial designs is given by the combination of (t) treatments taken (l) at a time:

$$C_l^t = \frac{t!}{l!(t-1)!} \tag{4.1}$$

where (t) is the number of treatments in the design, and (l) is the number of letters in the interaction. For example, a three-treatment design has two-treatment interactions:

$$C_2^3 = \frac{3!}{2!(3-2)!} = 3 \tag{4.2}$$

for three-treatment interactions:

$$C_2^3 = \frac{3!}{3!(3-3)!} = 1 \tag{4.3}$$

The Figure 1 shows the levels of the three factors with interaction [9].

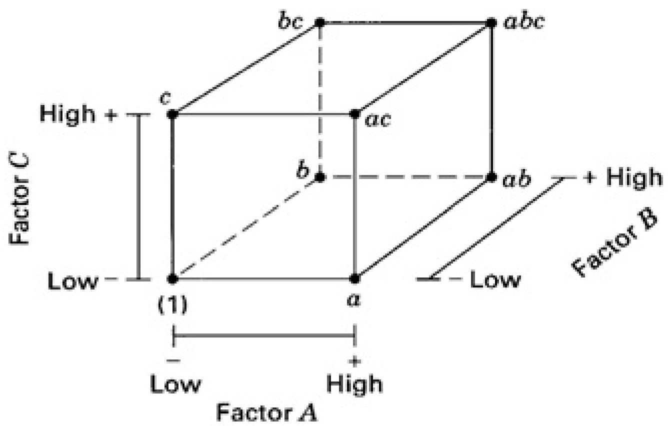

Figure 1: factorial experiment (2^3)

This design enables a researcher to test hypotheses concerning treatments A, B, and C and the AB, AC, BC, and ABC. interactions [5].

5. Tests of Hypotheses

To test hypotheses for a factorial experiment 2^3, where $i = 1,2, \quad j = 1,2, \quad k = 1,2$ and $i \neq i^{'}$, $j \neq j^{'}$, $k \neq k'$,there are three cases of tests which are [6]:

1. Main Effects Hypotheses For each of the factors, we consider the comparison of the mean over all levels of the factor. Thus the three main effects hypotheses are.

$$H_{01} : \mu_{i...} = \mu_{i'...} \quad \text{(Equal levels of factor } \alpha)$$
$$H_{02} : \mu_{.j.} = \mu_{.j'.} \quad \text{(Equal levels of factor } \beta)$$
$$H_{03} : \mu_{..k} = \mu_{..k'} \quad \text{(Equal levels of factor } \gamma)$$

2. Two-Factor Interactions

   For each pair of main effects (Binary interactions), we consider the two-factor interaction constraints, when averaged over all levels of the other factors. Thus the three Binary interactions hypotheses are.

$$H_{12} : \mu_{ij.} - \mu_{i'j.} - \mu_{ij'.} + \mu_{i'j'.} = 0 \quad \text{(No interaction for all } i \text{ and } j)$$
$$H_{13} : \mu_{ik.} - \mu_{i'k.} - \mu_{ik'.} + \mu_{i'k'.} = 0 \quad \text{(No interaction for all } i \text{ and } k)$$
$$H_{23} : \mu_{jk.} - \mu_{j'k.} - \mu_{jk'.} + \mu_{j'k'.} = 0 \quad \text{(No interaction for all } j \text{ and } k)$$

3. Three-Factor Interaction Hypothesis For each of the three-factor interaction, we consider the three-factor interaction constraints, when averaged over all levels of the other factors. Thus the three interactions hypotheses are.

$$H_{123} : \mu_{ijk} + \mu_{ij'k'} + \mu_{i'jk'} + \mu_{i'j'k} - \mu_{ijk'} - \mu_{ij'k} - \mu_{i'jk} - \mu_{i'j'k'} = 0 \text{ (No interaction for all } i \text{ and } j)$$

For the purpose of testing the model in equation (3.1), we take into account a factorial experiment (three factors and two levels for each factor) with a frequency (4.1) for each factorial treatment as shown in the Table 1.

Table 1: Data layout for a three-factor design with 2^3 and $r = 2$

			A			
			1		2	
			B			
		Replicate	1	2	1	2
C	1	1	y_{1111}	y_{1211}	y_{2111}	y_{2211}
		2	y_{1112}	y_{1212}	y_{2112}	y_{2212}
	2	1	y_{1121}	y_{1221}	y_{2121}	y_{2221}
		2	y_{1122}	y_{1222}	y_{2122}	y_{2222}

6. General Linear Model

The General Linear Model (GLM) underlies most of the statistical analyses that are used in applied and social research. It is the foundation for the t-test, Analysis of Variance (ANOVA), Analysis of Covariance (ANCOVA), regression analysis, and many of the multivariate methods including factor analysis, cluster analysis, multidimensional scaling, discriminant function analysis, canonical correlation, and others [8].

1. Design Model.

   Consider the general linear model $Y = XB + \epsilon$, where Y is an observable (n x 1) random vector, X is an (n x p) matrix of rank K of observable nonrandom variables (where $n > p > K$), B is a (p x 1) vector of unknown parameters, and (ϵ) is an (n x 1) non observable random vector. This model is defined to be a design model if and only if the elements of X consist of the numbers 0 and 1.

2. Design Matrix.

   An (n x p) matrix (X) is defined to be a design matrix if and only if (X) can be partitioned as $X_0, X_1, \dots, X_q$, where X_i is an (n x q_i), matrix and satisfies the following:

1. the elements of the matrix are the numbers 0 or 1.
2. for each susmallmatrix $X_i, i = 0, 1, ..., q$, every row contains exactly one element equal to 1 (the remaining elements in each row are zeros).
3. for each $X_i, i = 0, 1, ..., q$, every column contains at least one non-zero element. [3]

In matrix form the model with 2^3 and r=2 is:

$$Y = X\beta + \varepsilon \tag{6.1}$$

where:

$$X = \begin{array}{c} \begin{array}{ccccccccccccccccccccccccccc} \mu & \alpha_1 & \alpha_2 & \beta_1 & \beta_2 & \gamma_1 & \gamma_2 & \alpha\beta_{11} & \alpha\beta_{12} & \alpha\beta_{21} & \alpha\beta_{22} & \alpha\gamma_{11} & \alpha\gamma_{12} & \alpha\gamma_{21} & \alpha\gamma_{22} & \beta\gamma_{11} & \beta\gamma_{12} & \beta\gamma_{21} & \beta\gamma_{22} & \alpha\beta\gamma_{111} & \alpha\beta\gamma_{112} & \alpha\beta\gamma_{121} & \alpha\beta\gamma_{122} & \alpha\beta\gamma_{211} & \alpha\beta\gamma_{212} & \alpha\beta\gamma_{221} & \alpha\beta\gamma_{222} \end{array} \\ \begin{bmatrix} 1&1&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&1&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0 \\ 1&1&0&1&0&0&1&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0 \\ 1&1&0&1&0&0&1&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&1&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&1&0&0&0&0&0&0 \\ 1&1&0&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 1&1&0&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0 \\ 1&0&1&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0 \\ 1&0&1&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0 \\ 1&0&1&1&0&0&1&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0 \\ 1&0&1&1&0&0&1&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0 \\ 1&0&1&0&1&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 1&0&1&0&1&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 1&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1 \\ 1&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1 \end{bmatrix} \end{array} \tag{6.2}$$

$$\beta = \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \gamma_1 \\ \gamma_2 \\ \alpha\beta_{11} \\ \alpha\beta_{12} \\ \alpha\beta_{21} \\ \alpha\beta_{22} \\ \alpha\gamma_{11} \\ \alpha\gamma_{12} \\ \alpha\gamma_{21} \\ \alpha\gamma_{22} \\ \beta\gamma_{11} \\ \beta\gamma_{12} \\ \beta\gamma_{21} \\ \beta\gamma_{22} \\ \alpha\beta\gamma_{111} \\ \alpha\beta\gamma_{112} \\ \alpha\beta\gamma_{121} \\ \alpha\beta\gamma_{122} \\ \alpha\beta\gamma_{211} \\ \alpha\beta\gamma_{212} \\ \alpha\beta\gamma_{221} \\ \alpha\beta\gamma_{222} \end{bmatrix} \tag{6.3}$$

,

$$y = \begin{bmatrix} y_{1111} \\ y_{1112} \\ y_{1121} \\ y_{1122} \\ y_{1211} \\ y_{1212} \\ y_{1221} \\ y_{1222} \\ y_{2111} \\ y_{2112} \\ y_{2121} \\ y_{2122} \\ y_{2211} \\ y_{2212} \\ y_{2221} \\ y_{2222} \end{bmatrix} \tag{6.4}$$

,

$$\epsilon = \begin{bmatrix} \varepsilon_{1111} \\ \varepsilon_{1112} \\ \varepsilon_{1121} \\ \varepsilon_{1122} \\ \varepsilon_{1211} \\ \varepsilon_{1212} \\ \varepsilon_{1221} \\ \varepsilon_{1222} \\ \varepsilon_{2111} \\ \varepsilon_{2112} \\ \varepsilon_{2121} \\ \varepsilon_{2122} \\ \varepsilon_{2211} \\ \varepsilon_{2212} \\ \varepsilon_{2221} \\ \varepsilon_{2222} \end{bmatrix} \tag{6.5}$$

$$X'Y = \begin{bmatrix} y_{....} \\ y_{1...} \\ y_{2...} \\ y_{.1..} \\ y_{.2..} \\ y_{..1.} \\ y_{..2.} \\ y_{11..} \\ y_{12..} \\ y_{21..} \\ y_{22..} \\ y_{1.1.} \\ y_{1.2.} \\ y_{2.1.} \\ y_{2.2.} \\ y_{.11.} \\ y_{.12.} \\ y_{.21.} \\ y_{.22.} \\ y_{111.} \\ y_{112.} \\ y_{121.} \\ y_{122.} \\ y_{211.} \\ y_{212.} \\ y_{221.} \\ y_{222.} \end{bmatrix} \tag{6.6}$$

,

The matrix (X) is order (16x27), vector (β) is order (27x1), vector (y) is order (16x1), and vector (ϵ) is order (16x1), it is assumed that (ε) is a normally distributed random vector with mean (0) and variance $\sigma^2 I$.

We note that the matrix (X) has less than full rank, that is, the first column of the matrix is equal to the sum of the remaining columns, so the columns are linearly related, and thus neither the inverse nor the determinant of it can be found because it is unique. Therefore, the following equation (6.7) will be used to show the possibility of estimating and testing the vector (β).

$$LH = L \tag{6.7}$$

Where:

$$L = \begin{bmatrix}
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1
\end{bmatrix} \tag{6.8}$$

$$H = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&0&1&0&0&1&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0\\
1&1&0&0&1&1&0&0&1&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0\\
1&1&0&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&0&0&1&0&0&0&0\\
1&0&1&1&0&1&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0&0\\
1&0&1&1&0&0&1&0&0&1&0&0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0\\
1&0&1&0&1&1&0&0&0&0&1&0&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0\\
1&0&1&0&1&0&1&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1
\end{bmatrix} \tag{6.9}$$

$$H = \left(X'X\right)^c \left(X'X\right) \tag{6.10}$$

After applying the formula (6.10) it was found ($LH \neq L$), the vector (β) cannot be estimated, i.e. every part of the vector (parameters) is unestimable and untestable, in the full-rank model the system of regular equations has exactly one solution; In the less-than-full form, there are many solutions to the system. The immediate problem is to find a general method for solving ordinary equations in a less than complete order model. For the purpose of addressing this defect in the rank of the matrix, Statisticians have developed a number of techniques for obtaining parameter estimates that circumvent the rank deficiency of X. The three most widely used techniques involve the following:

1. Placing restrictions on the unknown parameters
2. Reparameterization solving for (r) linear combinations of the original parameters, where (r) is the rank of X
3. Using a generalized inverse. [7]

7. Reparameterization and Generalized Inverse

This method is used when a matrix (x) is of less than full rank, and thus the model cannot be estimated and therefore cannot be tested. The aim of this work is to obtain a matrix that is of full rank, and therefore the design model is a regression model, and thus the model can be used for estimation and testing, and to obtain a matrix with A full rank is by integrating the effect of the general trend in addition to the effect of the treatments, meaning that the model (3.1) is Reparameterization to the model (7.1) as follows [3, 12]

$$y_{ijkl} = \mu_{ijk} + \varepsilon_{ijkl} \tag{7.1}$$

where as

$$\mu_{ijk} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} \tag{7.2}$$

we can write the following identity in (μ_{ijk})

$$\begin{aligned} \mu_{ijk} = {} & \overline{\mu}_{...} + (\overline{\mu}_{i..} - \overline{\mu}_{...}) + \left(\overline{\mu}_{.j.} - \overline{\mu}_{...}\right) + (\overline{\mu}_{..k} - \overline{\mu}_{...}) + \left(\overline{\mu}_{ij.} - \overline{\mu}_{i..} - \overline{\mu}_{.j.} + \overline{\mu}_{...}\right) + \\ & (\overline{\mu}_{i.k} - \overline{\mu}_{i..} - \overline{\mu}_{.k.} + \overline{\mu}_{...}) + \left(\overline{\mu}_{.jk} - \overline{\mu}_{.j.} - \overline{\mu}_{..k} + \overline{\mu}_{...}\right) + \\ & \left(\mu_{ijk} - \overline{\mu}_{ij.} - \overline{\mu}_{i.k} - \overline{\mu}_{.jk} + \overline{\mu}_{i..} + \overline{\mu}_{.j.} + \overline{\mu}_{..k} - \overline{\mu}_{...}\right) \end{aligned} \tag{7.3}$$

$$\therefore \mu_{ijk} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_k^* + (\alpha\beta)^*_{ij} + (\alpha\gamma)^*_{ik} + (\beta\gamma)^*_{jk} + (\alpha\beta\gamma)^*_{ijk} \tag{7.4}$$

where
$$\widehat{\mu}^* = \overline{\mu}_{...} \tag{7.5}$$

$$\widehat{\alpha}_i^* = (\overline{\mu}_{i..} - \overline{\mu}_{...}) \tag{7.6}$$

$$\widehat{(\alpha\beta)}^*_{ij} = \overline{\mu}_{ij.} - \overline{\mu}_{i..} - \overline{\mu}_{.j.} + \overline{\mu}_{...} \tag{7.7}$$

$$\widehat{\beta}_j^* = \left(\overline{\mu}_{.j.} - \overline{\mu}_{...}\right) \tag{7.8}$$

$$\widehat{(\alpha\gamma)}^*_{ik} = \overline{\mu}_{i.k} - \overline{\mu}_{i..} - \overline{\mu}_{.k.} + \overline{\mu}_{...} \tag{7.9}$$

$$\widehat{\gamma}_k^* = (\overline{\mu}_{..k} - \overline{\mu}_{...}) \tag{7.10}$$

$$\widehat{(\beta\gamma)}^*_{jk} = \overline{\mu}_{.jk} - \overline{\mu}_{.j.} - \overline{\mu}_{..k} + \overline{\mu}_{...} \tag{7.11}$$

$$\widehat{(\alpha\beta\gamma)}^*_{ijk} = \mu_{ijk} - \overline{\mu}_{ij.} - \overline{\mu}_{i.k} - \overline{\mu}_{.jk} + \overline{\mu}_{i..} + \overline{\mu}_{.j.} + \overline{\mu}_{..k} - \overline{\mu}_{...} \tag{7.12}$$

and

$$\overline{\mu}_{...} = \frac{1}{abc}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\mu_{ijk} = \mu + \overline{\alpha}_. + \overline{\beta}_. + \overline{\gamma}_. + \overline{(\alpha\beta)}_{...} + \overline{(\alpha\gamma)}_{...} + \overline{(\beta\gamma)}_{...} + \overline{(\alpha\beta\gamma)}_{...} \tag{7.13}$$

$$\overline{\mu}_{i..} = \frac{1}{bk}\sum_{j=1}^{b}\sum_{k=1}^{c}\mu_{ijk} = \mu + \alpha_i + \overline{\beta}_. + \overline{\gamma}_. + \overline{(\alpha\beta)}_{i..} + \overline{(\alpha\gamma)}_{i..} + \overline{(\beta\gamma)}_{...} + \overline{(\alpha\beta\gamma)}_{i..} \tag{7.14}$$

$$\overline{\mu}_{.j.} = \frac{1}{ak}\sum_{i=1}^{a}\sum_{k=1}^{c}\mu_{ijk} = \mu + \overline{\alpha}_. + \beta_j + \overline{\gamma}_. + \overline{(\alpha\beta)}_{.j.} + \overline{(\alpha\gamma)}_{...} + \overline{(\beta\gamma)}_{.j.} + \overline{(\alpha\beta\gamma)}_{.j.} \tag{7.15}$$

$$\overline{\mu}_{..k} = \frac{1}{ab}\sum_{i=1}^{a}\sum_{j=1}^{b}\mu_{ijk} = \mu + \overline{\alpha}_. + \overline{\beta}_. + \gamma_k + \overline{(\alpha\beta)}_{...} + \overline{(\alpha\gamma)}_{..k} + \overline{(\beta\gamma)}_{..k} + \overline{(\alpha\beta\gamma)}_{..k} \tag{7.16}$$

$$\overline{\mu}_{ij.} = \frac{1}{c}\sum_{k=1}^{c}\mu_{ijk} = \mu + \alpha_i + \beta_j + \overline{\gamma}_{.} + \overline{(\alpha\beta)}_{ij.} + \overline{(\alpha\gamma)}_{i..} + \overline{(\beta\gamma)}_{.j.} + \overline{(\alpha\beta\gamma)}_{ij.} \tag{7.17}$$

$$\overline{\mu}_{i.k} = \frac{1}{b}\sum_{j=1}^{b}\mu_{ijk} = \mu + \alpha_i + \overline{\beta}_{.} + \gamma_k + \overline{(\alpha\beta)}_{i..} + \overline{(\alpha\gamma)}_{i.k} + \overline{(\beta\gamma)}_{..k} + \overline{(\alpha\beta\gamma)}_{i.k} \tag{7.18}$$

$$\overline{\mu}_{.jk} = \frac{1}{a}\sum_{i=1}^{a}\mu_{ijk} = \mu + \overline{\alpha}_{.} + \beta_{j.} + \gamma_k + \overline{(\alpha\beta)}_{.j.} + \overline{(\alpha\gamma)}_{..k} + \overline{(\beta\gamma)}_{.jk} + \overline{(\alpha\beta\gamma)}_{.jk} \tag{7.19}$$

where

$$\overline{\alpha}_{.} = \frac{\sum_{i=1}^{a}\alpha_{i..}}{a} \tag{7.20}$$

$$\overline{\beta}_{...} = \frac{\sum_{j=1}^{b}\beta_{.j.}}{b} \tag{7.21}$$

$$\overline{\gamma}_{...} = \frac{\sum_{k=1}^{c}\gamma_{..k}}{c} \tag{7.22}$$

$$\overline{(\alpha\beta)}_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\alpha\beta)_{ij}}{ab} \tag{7.23}$$

$$\overline{(\alpha\gamma)}_{...} = \sum_{i=1}^{a}\sum_{k=1}^{c}\frac{(\alpha\gamma)_{ik}}{ac} \tag{7.24}$$

$$\overline{(\beta\gamma)}_{...} = \sum_{j=1}^{b}\sum_{k=1}^{c}\frac{(\beta\gamma)_{jk}}{bc} \tag{7.25}$$

$$\overline{(\alpha\beta\gamma)}_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\frac{(\alpha\beta\gamma)_{ijk}}{abc} \tag{7.26}$$

$$\overline{(\alpha\beta)}_{i..} = \sum_{j=1}^{b}\frac{(\alpha\beta)_{ij}}{b} \tag{7.27}$$

$$\overline{(\alpha\gamma)}_{i..} = \sum_{j=1}^{b}\frac{(\alpha\gamma)_{ik}}{c} \tag{7.28}$$

$$\overline{(\alpha\beta)}_{.j.} = \sum_{i=1}^{a}\frac{(\alpha\beta)_{ij}}{a} \tag{7.29}$$

$$\overline{(\beta\gamma)}_{.j.} = \sum_{k=1}^{c}\frac{(\beta\gamma)_{jk}}{c} \tag{7.30}$$

$$\overline{(\alpha\beta\gamma)}_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{c}\frac{(\alpha\beta\gamma)_{ijk}}{ac} \tag{7.31}$$

$$\overline{(\alpha\gamma)}_{..k} = \sum_{i=1}^{a}\frac{(\alpha\gamma)_{i.k}}{a} \tag{7.32}$$

$$\overline{(\beta\gamma)}_{..k} = \sum_{j=1}^{b}\frac{(\beta\gamma)_{jk}}{b} \tag{7.33}$$

$$\overline{(\alpha\beta\gamma)}_{..k} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\alpha\beta\gamma)_{ijk}}{ab} \tag{7.34}$$

$$\overline{(\alpha\beta)}_{ij.} = \sum_{i=1}^{a}\sum_{j=1}^{b}\frac{(\alpha\beta)_{ij.}}{ab} \tag{7.35}$$

$$\overline{(\alpha\beta\gamma)}_{ij.} = \sum_{k=1}^{c}\frac{(\alpha\beta\gamma)_{ijk}}{c} \tag{7.36}$$

$$\overline{(\alpha\gamma)}_{i.k} = \sum_{i=1}^{a}\sum_{k=1}^{c}\frac{(\alpha\gamma)_{i.k}}{ac} \tag{7.37}$$

$$\overline{(\alpha\beta\gamma)}_{i.k} = \sum_{j=1}^{b}\frac{(\alpha\beta\gamma)_{ijk}}{b} \tag{7.38}$$

$$\overline{(\beta\gamma)}_{.jk} = \sum_{j=1}^{b}\sum_{i=1}^{c}\frac{(\beta\gamma)_{.jk}}{bc} \tag{7.39}$$

$$\overline{(\alpha\beta\gamma)}_{.jk} = \sum_{i=1}^{a}\frac{(\alpha\beta\gamma)_{ijk}}{a} \tag{7.40}$$

Thus, the model (7.1) can be expressed using equations from (7.3) to (7.40) as follows:

$$\begin{aligned} y_{ijkl} = &\left(\mu + \overline{\alpha}_{.} + \overline{\beta}_{.} + \overline{\gamma}_{.} + (\overline{\alpha\beta}_{...}) + (\overline{\alpha\gamma}_{...}) + (\overline{\beta\gamma}_{...}) + (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left(\alpha_i - \overline{\alpha}_{.} + (\overline{\alpha\beta}_{i..}) - (\overline{\alpha\beta}_{...}) + (\overline{\alpha\gamma}_{i..}) - (\overline{\alpha\gamma}_{...}) + (\overline{\alpha\beta\gamma}_{i..}) - (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left(\beta_j - \overline{\beta} + (\overline{\beta}_{.j.}) - (\overline{\alpha\beta}_{...}) + (\overline{\beta\gamma}_{.j.}) - (\overline{\beta\gamma}_{...}) + (\overline{\alpha\beta\gamma}_{.j.}) - (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left(\gamma_k - \overline{\gamma} + (\overline{\alpha\gamma}_{..k}) - (\overline{\alpha\gamma}_{...}) + (\overline{\beta\gamma}_{..k}) - (\overline{\beta\gamma}_{...}) + (\overline{\alpha\beta\gamma}_{..k}) - (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left(\overline{\alpha\beta}_{ij.} + (\overline{\alpha\beta\gamma}_{ij.}) - (\overline{\alpha\beta}_{i..}) - (\overline{\alpha\beta\gamma}_{i..}) - (\overline{\alpha\beta}_{.j.}) - (\overline{\alpha\beta\gamma}_{.j.}) + (\overline{\alpha\beta}_{...}) + (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left((\overline{\alpha\gamma}_{i.k}) + (\overline{\alpha\beta\gamma}_{i.k}) - (\overline{\alpha\gamma}_{i..}) - (\overline{\alpha\beta\gamma}_{i..}) - (\overline{\alpha\gamma}_{..k}) + (\overline{\alpha\gamma}_{...}) + (\overline{\alpha\beta\gamma}_{...})\right) \\ &+ \left((\overline{\beta\gamma}_{.jk}) + (\overline{\alpha\beta\gamma}_{.jk}) - (\overline{\beta\gamma}_{.j.}) - (\overline{\alpha\beta\gamma}_{.j.}) + (\overline{\beta\gamma}_{..k}) - (\overline{\alpha\beta\gamma}_{..k}) + (\overline{\beta\gamma}_{..}) + (\overline{\alpha\beta\gamma}_{...})\right) + \\ &\left((\alpha\beta\gamma)_{ijk} - (\overline{\alpha\beta\gamma}_{ij.}) - (\overline{\alpha\beta\gamma}_{i.k}) - \overline{\alpha\beta\gamma}_{.jk} + (\overline{\alpha\beta\gamma}_{i..}) + (\overline{\alpha\beta\gamma}_{.j.}) + (\overline{\alpha\beta\gamma}_{..k}) - (\overline{\alpha\beta\gamma}_{...})\right) \end{aligned} \tag{7.41}$$

$$\therefore\ y_{ijkl} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_k^* + (\alpha\beta)_{ij}^* + (\alpha\gamma)_{ik}^* + (\beta\gamma)_{jk}^* + (\alpha\beta\gamma)_{ijk}^* + \varepsilon_{ijkl} \tag{7.42}$$

Model (7.42) is the result of reparameterization model (3.1) as in the above steps.
The sum of the main effects of the factors and the binary and triple interactions are as follows:

$$\sum_{i=1}^{a}\alpha_i^* = 0, \qquad \sum_{j=1}^{b}\beta_j^* = 0, \qquad \sum_{k=1}^{c}\gamma_k^* = 0, \qquad \sum_{i=1}^{a}\sum_{j=1}^{b}(\alpha\beta)_{ij}^* = 0,$$

$$\sum_{i=1}^{a}\sum_{k=1}^{c}(\alpha\gamma)_{ik}^* = 0, \qquad \sum_{j=1}^{b}\sum_{k=1}^{c}(\beta\gamma)_{jk}^* = 0, \qquad \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}(\alpha\beta\gamma)_{ijk}^* = 0$$

After doing the reparameterization process for matrix (X) and model (3.1), if a full-rank model is obtained, the vector (β) will be testable and estimable. If a full-rank model is not obtained, we use the conditional inverse method, which states that the smallest matrix of (X) whose determinant is not equal to zero is chosen and is of full rank according to the following steps

Let $(X'X)$ be less than full rank. To find a conditional inverse $(X'X)$

1. Find any nonsingular (r*r) minor (M) from $(X'X)$
2. Find (M^{-1}) and $(M^{-1})\,'$.
3. In $(X'X)$ Replace each element of (M) by the corresponding element of $(M^{-1})\,'$.
4. Replace all other element in $(X'X)$ with zeros.
5. Transpose the resulting matrix.
6. The result is (G), a generalized inverse of $(X'X)$.[12][13]

As the rank of the matrix $(X'X)$ with dimensions (27*27) is less than the full rank and its rank is (8), we apply the steps of the conditional inverse.

1. We choose an matrix with rank (8) from the matrix $(X'X)$ let it be (M) be an nonsingular matrix.

$$M = \begin{bmatrix} 2&0&0&0&0&0&0&0 \\ 0&2&0&0&0&0&0&0 \\ 0&0&2&0&0&0&0&0 \\ 0&0&0&2&0&0&0&0 \\ 0&0&0&0&2&0&0&0 \\ 0&0&0&0&0&2&0&0 \\ 0&0&0&0&0&0&2&0 \\ 0&0&0&0&0&0&0&2 \end{bmatrix}$$

2. Finding the inverse and transpose matrix (M).

$$M^{-1} = \begin{bmatrix} 0.5&0&0&0&0&0&0&0 \\ 0&0.5&0&0&0&0&0&0 \\ 0&0&0.5&0&0&0&0&0 \\ 0&0&0&0.5&0&0&0&0 \\ 0&0&0&0&0.5&0&0&0 \\ 0&0&0&0&0&0.5&0&0 \\ 0&0&0&0&0&0&0.5&0 \\ 0&0&0&0&0&0&0&0.5 \end{bmatrix}, \quad \left(M^{-1}\right)' = \begin{bmatrix} 0.5&0&0&0&0&0&0&0 \\ 0&0.5&0&0&0&0&0&0 \\ 0&0&0.5&0&0&0&0&0 \\ 0&0&0&0.5&0&0&0&0 \\ 0&0&0&0&0.5&0&0&0 \\ 0&0&0&0&0&0.5&0&0 \\ 0&0&0&0&0&0&0.5&0 \\ 0&0&0&0&0&0&0&0.5 \end{bmatrix}$$

3. In $(X'X)$ Replace each element of (M) by the corresponding element of $\left(M^{-1}\right)'$. Replace all other element in $(X'X)$ with zeros. Transpose the resulting matrix.

4. The result is (G), a generalized inverse of $(X'X)$.

$$G = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.5
\end{bmatrix} = G'$$

Then $(\mu^*.\alpha_i^*.\beta_j^*.\gamma_k^*.(\alpha\beta)_{ij}^*.(\alpha\gamma)_{ik}^*.(\beta\gamma)_{jk}^* and (\alpha\beta\gamma)_{ijk}^*)$ are estimable and testable.

It is noted from the above that the whole previous process is a reparameterization and conditional inverse now it is possible to test the following hypotheses:

$$H_0 : \alpha_i^* = 0, \tag{7.43}$$

$$H_0 : (\alpha\beta)_{ij}^* = 0 \tag{7.44}$$

$$H_0 : \beta_j^* = 0, \tag{7.45}$$

$$H_0 : (\alpha\gamma)_{ik}^* = 0 \tag{7.46}$$

$$H_0 : \gamma_k^* = 0, \tag{7.47}$$

$$H_0 : (\beta\gamma)_{jk}^* = 0 \tag{7.48}$$

$$H_0 : (\alpha\beta\gamma)_{ijk}^* = 0 \tag{7.49}$$

The regression sum of squares for the reparametrized full rank model is thus given by [11]:

$$SS\,(Full) = \beta^{0'} X' y \tag{7.50}$$

Or

$$SS\,(Full) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} \tag{7.51}$$

$$\beta^{0'} = GX'Y \tag{7.52}$$

The reduced model is calculated according to the tested hypothesis, for example if the following null hypothesis is required to be tested:

$$H_0 : (\alpha\gamma)_{ik}^* = 0$$

The interference to be tested according to the above null hypothesis must be deleted from the model, so that the model (7.42) becomes as follows:

$$y_{ijkl} = \mu^* + \alpha_i^* + \beta_j^* + \gamma_k^* + (\alpha\beta)_{ij}^* + (\beta\gamma)_{jk}^* + (\alpha\beta\gamma)_{ijk}^* + \varepsilon_{ijkl} \tag{7.53}$$

And the sum of squares of the reduced model (7.53) contains all the coefficients and interactions except for those to be tested in the null hypotheses ($(\alpha\gamma)_{ik}^*$), as follows:

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2}{br} + \frac{\sum_{i=1}^{a} y_{i...}^2}{bcr} + \frac{\sum_{k=1}^{c} y_{..k.}^2}{abr} - \frac{y_{....}^2}{abcr} \tag{7.54}$$

And the sum of squares of the reduced model in the case of hypothesis testing ($H_0 : \alpha_i^* = 0$) is as follows:

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{i=1}^{a} y_{i...}^2}{bcr} + \frac{y_{....}^2}{abcr} \tag{7.55}$$

The sum of the reduced squares for the hypotheses ((7.45),(7.47),(7.44),(7.48),(7.49)) is as follows, respectively:

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{j=1}^{b} y_{.j..}^2}{acr} + \frac{y_{....}^2}{abcr} \tag{7.56}$$

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{k=1}^{c} y_{..k.}^2}{abr} + \frac{y_{....}^2}{abcr} \tag{7.57}$$

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2}{cr} + \frac{\sum_{i=1}^{a} y_{i...}^2}{bcr} + \frac{\sum_{j=1}^{b} y_{.j..}^2}{acr} - \frac{y_{....}^2}{abcr} \tag{7.58}$$

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2}{r} - \frac{\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2}{ar} + \frac{\sum_{j=1}^{b} y_{.j..}^2}{acr} + \frac{\sum_{k=1}^{c} y_{..k.}^2}{abr} - \frac{y_{....}^2}{abcr} \tag{7.59}$$

$$SS\,(Reduced) = \frac{\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2}{cr} + \frac{\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2}{br} + \frac{\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2}{ar} - \frac{\sum_{i=1}^{a} y_{i...}^2}{bcr} - \frac{\sum_{j=1}^{b} y_{.j..}^2}{acr} - \frac{\sum_{k=1}^{c} y_{..k.}^2}{abr} + \frac{y_{....}^2}{abcr} \tag{7.60}$$

The sum of squares of the tested hypothesis is the product of subtracting the sum of squares for the total regression and the sum of squares for the reduced model, as follows:

$$SS\,(Hypothesis) = SS\,(Full) - SS\,(Reduced) \tag{7.61}$$

$$SSe\,(error) = Y'Y - \beta^{0'}X'y \tag{7.62}$$

$$SST(corrected) = Y'Y - \frac{y_{....}^2}{abcr} \tag{7.63}$$

$$SST(uncorrected) = Y'Y \tag{7.64}$$

And the computed value (F_{cal}) is as follows:

$$F_{df(Hypothesis).df(error)} = \frac{\frac{SS(Hypothesis)}{df(Hypothesis)}}{\frac{SSe(error)}{df(error)}} \tag{7.65}$$

The analysis of variance tables for the tertiary, binary, and main interactions are as follows:

Table 2: Analysis of Variance for Interaction $(\alpha\beta\gamma)^*_{ijk}$

Sources of variation	***Degrees of freedom***	***Sum of squares***	***Mean square***	***F***
Regression	abc	***SS(Full)***	***MS(Full)***	
Reduced	$ab + ac + bc - a - b - c + 1$	***SS(Reduced)***	***MS(Reduced)***	
Hypothesis	$(a-1)(b-1)(c-1)$	***SS(Hypothesis)***	***MS(Hypothesis)***	$F = \frac{MS(Hypothesis)}{MSe(error)}$
Error	$abcr - abc$	***SSe(error)***	***Mse (error)***	
Total(Uncorrected)	$abcr$			
Total(corrected)	$abcr - 1$	***SST***		

Table 3: Analysis of Variance for Interaction $(\alpha\gamma)^*_{ik}$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	abc	***SS(Full)***	***MS(Full)***	
Reduced	$abc - ab + a + b - 1$	***SS(Reduced)***	***MS(Reduced)***	
Hypothesis	$(a-1)(b-1)$	***SS(Hypothesis)***	***MS(Hypothesis)***	$F = \frac{MS(Hypothesis)}{MSe(error)}$
Error	$abcr - abc$	***SSe(error)***	***Mse (error)***	
Total(Uncorrected)	$abcr$			
Total(corrected)	$abcr - 1$	***SST***		

Table 4: Analysis of Variance for main Factor $(\alpha)^*_i$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	abc	***SS(Full)***	***MS(Full)***	
Reduced	$abc - a + 1$	***SS(Reduced)***	***MS(Reduced)***	
Hypothesis	$a - 1$	***SS(Hypothesis)***	***MS(Hypothesis)***	$F = \frac{MS(Hypothesis)}{MSe(error)}$
Error	$abcr - abc$	***SSe(error)***	***Mse (error)***	
Total(Uncorrected)	$abcr$			
Total(corrected)	$abcr - 1$	***SST***		

And the analysis of variance tables for the hypotheses ((7.45),(7.47),(7.44),(7.48),(7.49)) are the same as the tables above, taking into account the degrees of freedom according to the coefficients used in the hypothesis.

8. Application

A 2^3 factorial experiment was carried out on methods of cultivating cauliflower. The factors were
1. Fertilizers applied in spring or summer
2. Spraying or nonspraying
3. Irrigation or lack of irrigation

The experiment was performed twice, and the percentage of poor quality cauliflower was observed in each case. The data for the experiment was as in the following Table [10].

Table 5: Data from 2^3 a factorial experiment on cauliflower cultivation methods

			A			
			Spring		Summer	
			B			
		Replicate	Spray	Nonspray	Spray	Nonspray
C	Irrigated	1	20.5	28.7	26.2	37.5
		2	19.7	31.3	29.9	35.0
	Nonirrigated	1	24.8	21.8	19.7	29.4
		2	26.5	26.0	27.0	26.6

The results of the application using equations ((7.50),(7.51),(7.52),(7.54)-(7.65)) and according to each hypothesis (7.43)-(7.49) are as in the following tables:

Table 6: Results of the analysis of variance for Interaction $(\alpha\beta\gamma)^*_{ijk}$

S.O.V	D.F	S.S	M.S	F
Regression	8	11926.46	MS(Full)	
Reduced	7	11910.06	MS(Reduced)	
Hypothesis	1	16.41	16.41	$2.41^{n.s}$
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 7: Results of the analysis of variance for Interaction $(\alpha\beta)^*_{ij}$

S.O.V	D.F	S.S	M.S	F
Regression	8	11926.46	MS(Full)	
Reduced	7	11920.94	MS(Reduced)	
Hypothesis	1	5.523	5.523	$0.81^{n.s}$
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 8: Results of the analysis of variance for Interaction $(\alpha\gamma)^*_{ik}$

S.O.V	D.F	S.S	M.S	F
Regression	8	11926.46	MS(Full)	
Reduced	7	11964.9	MS(Reduced)	
Hypothesis	1	38.44	38.44	5.64^*
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 9: Results of the analysis of variance for Interaction $(\beta\gamma)^*_{jk}$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	8	11926.46	***MS(Full)***	
Reduced	7	11984.22	***MS(Reduced)***	
Hypothesis	1	57.76	57.76	8.48*
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 10: Results of the analysis of variance for main Factor $(\alpha)^*_i$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	8	11926.46	***MS(Full)***	
Reduced	7	11862.46	***MS(Reduced)***	
Hypothesis	1	64	64	9.4*
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 11: Results of the analysis of variance for main Factor $(\beta)^*_j$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	8	11926.46	***MS(Full)***	
Reduced	7	11816.21	***MS(Reduced)***	
Hypothesis	1	110.250	110.250	16.18*
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

Table 12: Results of the analysis of variance for main Factor $(\gamma)^*_k$

S.O.V	***D.F***	***S.S***	***M.S***	***F***
Regression	8	11926.46	***MS(Full)***	
Reduced	7	11880.89	***MS(Reduced)***	
Hypothesis	1	45.56	45.56	6.69*
Error	8	54.50	6.81	
Total(Uncorrected)	16	11980.96		
Total(corrected)	15	392.438		

9. Results

1. From the Tables 6 and 7, we notice that the interactions $(\alpha\beta\gamma)^*_{ijk}$ and $(\alpha\beta)^*_{ij}$ are not significant, that is, the null hypothesis that assumes that the interaction is equal to zero is not rejected.

2. From the Tables 8, 9, 10, 11 and 12, we notice that the factories $(\alpha)^*_i$, $(\beta)^*_j$ and $(\gamma)^*_k$ and the interactions $(\alpha\gamma)^*_{ik}$, $(\beta\gamma)^*_{jk}$ have significant effects, i.e. rejecting the null hypothesis that imposes equality with zero.

10. Conclusion

The reparameterization and conditional inverse methods were used for a factorial experiment (2^3) applied in the complete random design to treat the less than full rank of the design matrix in the general linear model, and the results were obtained by estimation and testing after converting the design matrix to the full rank and the results of the effects of the main coefficients and the interaction between factors were obtained.

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