Int. J. Nonlinear Anal. Appl. 13 (2022) 1, 3749-3764
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.xxx

The smallest size of the arc of degree three in a projective plane of order sixteen

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(Communicated by Madjid Eshaghi Gordji)

Abstract

An $(n;3)$-arc in projective plane $PG(2,q)$ of size n and degree three is a set of n points such that no four of them collinear but some three of them are collinear.An $(n;r)$-arc is said to be complete if it is not contained in $(n+1;r)$-arc. The aim of this paper is to construct the projectively distinct $(n;3)$-arcs in $PG(2,16)$, determined the smallest complete arc in $PG(2,16)$ then the stabilizer group of these arcs are established and we have identified the group with which its isomorphic.

Keywords: Projective Plane, Complete Arc.

1. Introduction

The projective plane $PG(2,q)$ of order q over Galois field $GF(q)$, $q = p^t$ for some prime number p and with some integer $t \geq 1$, consists of a set of q^2+q+1 points and a set of q^2+q+1 lines, where each line contains exactly $q+1$ points lie on exactly one line. It follows from the definition that each point is contained in exactly $q+1$ lines and two distinct lines have exactly one common point. For an introduction in projective geometries over finite field, see [11, 12, 10, 3, 2, 17]. We have used Gap program [8].
The projective plane of order sixteen $PG(2,16)$ contains :

- 273 points and lines.
- 17 points on each line.
- 17 lines passage through each point.

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Received: June 2021 *Accepted:* October 2021

The study of $(n;r)$ for $r \geq 3$ in projective plane started by Barlotti [5] in 1956.The case $r = 3$ has been studied by Marcugini for $q = 7$ [16]. The researchers who studied in projective plane for all value of q rise the some questions such as what is for given n, r,what is the number of distinct $(n;r)$-arcs in $PG(2,q)$ and what is the largest and the smallest size of complete $(n;r)$-arc in $PG(2,q)$[4, 6]. The main aim of this paper is to construct the full numbers of distinct $(n;3)$-arcs in $PG(2,16)$ and then classify the arcs which of them is an equivelent or an inequivelent ,and determined the stabilizer group of each distinct arc. Many results of $PG(2,q)$, $q \leq 31$ have been satisfied see [11, 13, 1]. we depend on strategy that depends on choosing the number of inequivelent classes of $i-secand$ $\{l_0\ ,l_1,l_2,l_3\}$ distributions which represents the 0-secant ,1-secant ,2-secant, 3-secant in each procedure [9]. By using this strategy and another strategy we obtained that the smallest size of $(n;3)$-arcs in $PG(2,16)$ that has been constructed is equal to 17.

2. The projective plane (2,16)

The points of $PG(2,16)$ are generated by nonsingular matrix: $T = C(F)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^7 & 1 & 0 \end{bmatrix}$, where $F(X) = X^2+X+w^2$, is an irreducible polynomial in F_{16} ,such that $P_i = P(1,0,0)T^i$, $i = 0,\ldots,272$, they are given as follows :

$P_0 = P(1,0,0)$,$P_1 = P_0\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^7 & 1 & 0 \end{bmatrix} = P(0,1,0)$, $P_2 = P_0\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \omega^7 & 1 & 0 \end{bmatrix}^2 = P(0,0,1)$, so on ..., $P_{253} = P(1,1,1)$,..., $P_{272} = P(1,0,1)$.

To find the lines in $PG(2,16)$:Let l_1 contains of 17 points such that the third coordinate of it is equal to zero. Then the points and the lines l_i in $PG(2,16)$ can be represented by the following array.

Table 1: The lines of $PG(2,16)$

l_1	$\{0,1,3,7,15,31,63,90,116,127,136,181,194,204,233,138,255\}$
l_2	$\{1,2,4,8,16,32,64,91,117,128,137,182,195,205,234,139,256\}$
l_3	$\{2,3,5,9,17,33,65,92,118,129,138,183,196,206,235,140,257\}$
$\vdots$	$\vdots$
l_{273}	$\{272,0,2,6,14,30,62,89,115,126,135,180,193,203,232,237,254\}$

3. Some definitions and properties

Definition 3.1. *[13] A $(n;r)$-arc in $PG(2,q)$ is a set K of n points, no r+1 of which are collinear, but with at least one set of r points collinear, that is $|K \cap\ l| \ \leq r$ $for\ all$ $\ l\ \in PG(2,16)$.*

Definition 3.2. *[6] A $(n;\ r)$-arc K is a complete if it is not contained in a $(n+1;r)$-arc*

Definition 3.3. *[7] A line l of $PG(n,r)$; $n > 1$ is an i-secant of an (n,r) -arc K if l intersects K in **i** points .A 2-secsnt is called bisecant, 1-secsnt is an unisecant and 0-secsnt is an external line.*

Definition 3.4. *[11] Let K be k-arc in PG(n,q), and P a point of $PG(2,n)\backslash K$ .Then if exactly $i-bisecant$ of K pass through P it said to be a point of index i.The number of these point s is denoted by c_i.*

Notation 1.
(i) c_0 : denoted to the number of points with $0-bisecant$ pass through P
(ii) c_1 : denoted to the number of points with $1-bisecant$ pass through P.
(iii) c_2 : denoted to the number of points with $2-bisecant$ pass through P.
(iv) c_3 : denoted to the number of points with $3-bisecant$ pass through P.

Notation 2. Let ρ_i be the number of $i-secant$ of a $(n;r)$-arc K in $PG(2,q)$.

Definition 3.5. *[15] The secant distribution of a $(n;r)$ -arc K is the ordered $(r+1)$ tuple $(l_0, l_1, l_2, \ldots, l_r)$.*

Definition 3.6. *[14] Two $(n;r)$-arcs is said to be equivalent if they have the same $i-secant$ distribution.*

Definition 3.7 (Stabilizer Group). *[2] To determine the automorphism stabilizer group of (n,r) -arc , K calculate every projective $\beta \in PG(2,16)$ that maps K to itself , that is $K\beta = K$ .This achieved by finding G that maps (5,3)-arc onto K ,and then determining if $K\beta = K$.*

Lemma 3.8. *For a (n,r) -arc K ,the following equations hold:*

$$\sum_{i=0}^{r} \rho_i = q^2+q+1 \tag{3.1}$$

$$\sum_{i=0}^{r} i\rho_i = K(q+1) \tag{3.2}$$

$$\sum_{i=0}^{r} \frac{1}{2} i(i-1)\rho_i = \frac{1}{2}\ K(K-1) \tag{3.3}$$

Proof . See [11]. □

Theorem 3.9. *[1] A (n,r) -arc K in $PG(2,q)$ is a complete if an only if $c_0 = 0$. we will study and compute the arcs K of degree 3 and different size.*

4. The constructions and classifications of $(n;3)$-arc

4.1. The constructions and classifications of (5;3)-arc

Let $A = \{0,1,2,253\}$ be the set of the frame points in $PG(2,16)$, no three of them are collinear where $P_0 = (1,0,0)$, $P_1 = (0,1,0)$, $P_2 = (0,0,1)$, $P_{253} = (1,1,1)$. The (5,3) -arcs can be constructed by adding to A one point from the remaining 269 points of the projective plane $PG(2,16)$ which is three collinear as follows: $\boldsymbol{K_5} = AU\,\{3\}$ Where $P_3 = (\omega^7, 1, 0)$ .Then the K (5,3)-arc is the set $\{0,1,2,3,253\}$.

The size of intersection (5,3)-arc with the lines of $PG(2,16)$,as the following :

$$|l| = \begin{cases} 198 & \text{if}\,|K\cap l| = 0 \\ 68 & \text{if}\,|K\cap l| = 1 \\ 7 & \text{if}\,|K\cap l| = 2 \\ 1 & \text{if}\,|K\cap l| = 3 \end{cases}$$

Note that 198+68+7+1=273 that is the all lines of the $PG(2,16)$.

When we calculate the stabilizer group of (5,3)-arc in $PG(2,16)$ we obtained the group G of two elements $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *of order one and* $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ of order two.

It is clear that G isomorphic to $(Z_2, +_2)$. To show this we can define a mapping $f : G \to (Z_2, +_2)$ by $f(I) = 0$ and $f(T) = 1$, clear that f is homomorphism ,onto and one-one. So the stabilizer group with multiplicative of matrices, isomorphic to $(Z_2, +_2)$. Table 2 include the (5; 3)-arc with its stabilizer group.

Table 2: (5;3)-arc in $PG(2,16)$

(5; 3)-arc	**Stabilizer group**
$\{0, 1, 2, 3, 253\}$	Z_2

4.2. The constructions and classifications of (6;3)-arcs

To construct (6;3)-arc, finding the c_i , where $i = 0, 1, 2, 3$ for (5;3)-arc in $PG(2,16)$ and we obtained the following: c_0 =254 , c_1 =14 , c_2 =0 , c_3 =0. Then added one point separately of the points of c_0 to the (5;3)-arc. So the number of (6;3)-arc that have been constructed is 254. The secant distribution $\{l_0\ , l_1, l_2, l_3\}$ for each (6;3)-arc is calculated, and then the (6;3)-arcs separated into a number of sets of which have inequivalent secant distribution .Then choosing a (6;3)-arc from each set of the same secant distribution, the set R is constructed .Then the set as the following :

$$R = \{K; K \text{ is a } (6;3) \text{ arc }; \ K \ \text{ has distinct secant distribution}\}$$

4.2.1. The Groups of projectivities of (6;3)-arcs

1. The Groups of projectivities of $K_6^{(1)}$ have one projective.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The projective group contains : One matrix of order one

2. The Groups of projectivities of $K_6^{(2)}$ has three projectiveties .

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ T_2 = \begin{bmatrix} 0 & 0 & 1 \\ \omega^7 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix},\ T_3 = \begin{bmatrix} \omega^8 & \omega^8 & \omega^8 \\ 1 & \omega^8 & 1 \\ \omega^2 & 0 & 0 \end{bmatrix}$$

The projective group contains: One matrix of order one and two matrices of order two.

3. The Groups of projectivities of $K_6^{(3)}$ have one projective .

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The projective group contains :One matrix of order one.

Table 3: Distinct classes and inequivalent arcs of k_6 in $PG(2,16)$.

$K_6^{(i)}$;$i=1,2,3$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arc**	**Stabilize group**
$K_6^{(1)}$	$\{185,75,12,1\}$	$\{0,1,2,3,253,4\}$	I
$K_6^{(2)}$	$\{184,78,9,2\}$	$\{0,1,2,3,253,5\}$	Z_3
$K_6^{(3)}$	$\{183,81,6,3\}$	$\{0,1,2,3,253,10\}$	I

4.3. The constructions and classifications of (7;3)-arcs

The number of the points of index zero of each distinct (6;3)-arc in $PG(2,16)$ are shown in the table 4.

Table 4: The Number of c_0

Index	**Arc**	**The No. of c_0**
$K_6^{(1)}$	$\{0,1,2,3,253,4\}$	239
$K_6^{(2)}$	$\{0,1,2,3,253,5\}$	225
$K_6^{(3)}$	$\{0,1,2,3,253,10\}$	253

Hence, by adding one point from above c_0 of the (6;3)-arcs in $PG(2,16)$ to each (6;3)- arc to construct the (7;3)-arcs. The number of (7;3) -arc that have been constructed are (717). Among of these arcs, there are (5) distinct where each has the class of secant distribution that shown in table 5.

4.3.1. The Groups of projectivities of (7;3)-arcs

1. The Group of projectivities of $K_7^{(1)}$ has four **projectivities**.

$$T_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix},\quad T_2=\begin{bmatrix}1&0&0\\0&1&0\\1&1&1\end{bmatrix},\quad T_3=\begin{bmatrix}1&0&0\\0&1&0\\1&\omega^8&1\end{bmatrix},\quad T_4=\begin{bmatrix}1&0&0\\0&1&0\\0&\omega^2&1\end{bmatrix}$$

   The projective group contains : One matrix of order one and three matrices of order two.

2. The Group of projectivities of $K_7^{(i)}$ $i=2,\dots\ 5$ has one **projective**.
   The projective group contains: One matrix of order one .

   To show that the group of projectivities of $K_7^{(1)}$ isomorphic to $Z_2\times Z_2$.

   (a) Let $T=\{T_1,T_2,T_3,T_4\}$, $Z_2\times Z_2=\{(0,0),(1,0),(0,1),(1,1)\}$ and define $\alpha:(T,.)\to(Z_2\times Z_2,+_2)$ such that; $\alpha(T_1)=(0,0)$, $\alpha(T_2)=(1,0)$, $\alpha(T_3)=(0,1)$ and $\alpha(T_4)=(1,1)$.

   (b) Define T and $Z_2\times Z_2$ by the following tables:

$\cdot$	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2	T_1	T_4	T_3
T_3	T_3	T_4	T_1	T_2
T_4	T_4	T_3	T_2	T_1

$+_2$	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

It is clear that α is bijective function. To prove that α is homomorphism, we find
$\alpha(T_1.T_2) = \alpha(T_2) = (1,0)$ and $\alpha(T_1) +_2 \alpha(T_2) = (1,0)$
$\alpha(T_1.T_3) = \alpha(T_3) = (0,1)$ and $\alpha(T_1) +_2 \alpha(T_3) = (0,1)$
$\alpha(T_1.T_4) = \alpha(T_4) = (1,1)$ and $\alpha(T_1) +_2 \alpha(T_4) = (1,1)$
$\alpha(T_2.T_3) = \alpha(T_4) = (1,0)$ and $\alpha(T_2) +_2 \alpha(T_3) = (1,1)$
$\alpha(T_2.T_4) = \alpha(T_3) = (0,1)$ and $\alpha(T_2) +_2 \alpha(T_3) = (0,1)$
$\alpha(T_3.T_4) = \alpha(T_2) = (1,0)$ and $\alpha(T_3) +_2 \alpha(T_2) = (1,0)$
$\alpha(T_2.T_2) = \alpha(T_1) = (0,0)$ and $\alpha(T_2) +_2 \alpha(T_2) = (0,0)$
$\alpha(T_3.T_3) = \alpha(T_1) = (0,0)$ and $\alpha(T_3) +_2 \alpha(T_3) = (0,0)$
$\alpha(T_4.T_4) = \alpha(T_1) = (0,0)$ and $\alpha(T_1) +_2 \alpha(T_2) = (0,0)$.
Hence α is homomorphism implies that α is isomorphism. □

The related information of the (7; 3)-arc given in table 5.

Table 5: Distinct classes and inequivalent arcs of K_7

$\boldsymbol{K_7^{(i)}; i = 1,2,3,4,5}$	$\{\boldsymbol{l_0}, \boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent Arcs**	Stabilize group
$K_7^{(1)}$	$\{170, 92, 6, 5\}$	$\{0, 1, 2, 3, 253, 5, 32\}$	$Z_2 \times Z_2$
$K_7^{(2)}$	$\{171, 89, 9, 4\}$	$\{0, 1, 2, 3, 253, 5, 4\}$	I
$K_7^{(3)}$	$\{172, 86, 12, 3\}$	$\{0, 1, 2, 3, 253, 18, 4\}$	I
$K_7^{(4)}$	$\{173, 83, 15, 2\}$	$\{0, 1, 2, 3, 253, 4, 11\}$	I
$K_7^{(5)}$	$\{174, 80, 18, 1\}$	$\{0, 1, 2, 3, 253, 10, 12\}$	I

4.4. The constructions and classifications of(8;3)-arc

The number of c_0 of each distinct (7;3)-arc in $PG(2,16)$ are shown in the table 6.

Table 6: The of c_0 of K_7

Arc	$\boldsymbol{K_7^{(1)}}$	$\boldsymbol{K_7^{(2)}}$	$\boldsymbol{K_7^{(3)}}$	$\boldsymbol{K_7^{(4)}}$	$\boldsymbol{K_7^{(5)}}$
No. of $\boldsymbol{c_0}$	238	252	224	196	210

In the same way adding separately each point from above c_0 of the (7;3)-arcs in $PG(2,16)$ to each (7;3)-arc to construct the (8;3)-arcs .The number of (8;3)-arcs that have been constructed is (1120).The number of distinct (8;3)-arc is (7) arcs have the classes distributions that shown in the table 7:

4.4.1. The Groups of projectivities of (8;3)-arc

1. The Group of projectivities of $K_8^{(1)}, K_8^{(2)}$ and of $K_8^{(i)}$; $i = 4, \ldots,$ 7 have one projectivities.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The projective group contains : Only one matrix of order one .

2. The Group of projectivities of $K_8^{(3)}$ has two **projectivities**.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} \omega^9 & \omega^9 & \omega^7 \\ 0 & 0 & \omega^7 \\ 0 & \omega^7 & 0 \end{bmatrix}$$

The projective group contains : One matrix of order one and one matrix of order two.

Table 7: Distinct classes inequivalent arcs of k_8

$\boldsymbol{K_8^{(i)}}$;$\boldsymbol{i = 1, \ldots ,7}$	$\{\boldsymbol{l_0}\ ,\boldsymbol{l_1}, \boldsymbol{l_2}, \boldsymbol{l_3}\}$	**Inequivalent Arcs**	**Stabilizer group**
$K_8^{(1)}$	$\{158, 101, 7, 7\}$	$\{0, 1, 2, 3, 253, 4, 5, 6\}$	I
$K_8^{(2)}$	$\{159, 98, 10, 6\}$	$\{0, 1, 2, 3, 253, 4, 5, 36\}$	I
$K_8^{(3)}$	$\{160, 95, 13, 5\}$	$\{0, 1, 2, 3, 253, 4, 32, 224\}$	Z_2
$K_8^{(4)}$	$\{161, 92, 16, 4\}$	$\{0, 1, 2, 3, 253, 10, 12, 11\}$	I
$K_8^{(5)}$	$\{162, 89, 19, 3\}$	$\{0, 1, 2, 3, 253, 4, 11, 13\}$	I
$K_8^{(6)}$	$\{163, 86, 22, 2\}$	$\{0, 1, 2, 3, 253, 18, 4, 9\}$	I
$K_8^{(7)}$	$\{164, 83, 25, 1\}$	$\{0, 1, 2, 3, 253, 4, 11, 21\}$	I

4.5. The construction and classification of (9;3)-arcs

The number of $\boldsymbol{c_0}$ of each distinct (8;3)-arc in $PG(2, 16)$ are shown in the table 8.

Table 8: The number of c_0 of k_8

ARC	$\boldsymbol{K_8^{(1)}}$	$\boldsymbol{K_8^{(2)}}$	$\boldsymbol{K_8^{(3)}}$	$\boldsymbol{K_8^{(4)}}$	$\boldsymbol{K_8^{(5)}}$	$\boldsymbol{K_8^{(6)}}$	$\boldsymbol{K_8^{(7)}}$
N0. of $\boldsymbol{c_0}$	223	251	209	170	197	183	224

Hence, by the same above way. The number of (9;3)-arcs that have been constructed is (1457). Among these arcs, there are (9) distinct arcs where each one has the class of secant distribution shown in table 9.

Table 9: Distinct classes and inequivalent arcs of k_9

$K_9^{(i)}$;$i=1,\ldots,9$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent Arcs**	**Stabilize group**
$K_9^{(1)}$	$\{147, 108, 9, 9, \}$	$\{0, 1, 2, 3, 253, 10, 12, 11, 24\}$	I
$K_9^{(2)}$	$\{148, 105, 12, 7\}$	$\{0, 1, 2, 3, 253, 4, 5, 36, 6\}$	I
$K_9^{(3)}$	$\{149, 102, 15, 7\}$	$\{0, 1, 2, 3, 253, 5, 32, 224, 80\}$	I
$K_9^{(4)}$	$\{150, 99, 18, 6\}$	$\{0, 1, 2, 3, 253, 10, 12, 11, 19\}$	I
$K_9^{(5)}$	$\{151, 96, 21, 5\}$	$\{0, 1, 2, 3, 253, 4, 11, 13, 26\}$	I
$K_9^{(6)}$	$\{152, 93, 24, 4\}$	$\{0, 1, 2, 3, 253, 18, 4, 9, 12\}$	I
$K_9^{(7)}$	$\{153, 90, 27, 3\}$	$\{0, 1, 2, 3, 253, 4, 11, 13, 25\}$	I
$K_9^{(8)}$	$\{154, 87, 30, 2\}$	$\{0, 1, 2, 3, 253, 4, 11, 21, 5\}$	I
$K_9^{(9)}$	$\{155, 84, 33, 1\}$	$\{0, 1, 2, 3, 253, 5, 32, 224, 43\}$	I

4.6. The constructions and classifications of (10;3)-arcs

The number of c_0 of each distinct (9;3)-arc in $PG(2, 16)$ are shown in the table 15.

Table 10: The number of c_0 of k_9

Arc	$K_9^{(1)}$	$K_9^{(2)}$	$K_9^{(3)}$	$K_9^{(4)}$	$K_9^{(5)}$
No.of c_0	237	159	169	250	222
Arc	$K_9^{(6)}$	$K_9^{(7)}$	$K_9^{(8)}$	$K_9^{(9)}$	
No.of c_0	195	210	183	146	

Thenumber of (10;3)-arcs that have been made up is (1771) . Here there are (11) distinct (10;3)-arcs having the following distinct classes.

Table 11: Distinct classes and inequivalent arcs of K_{10}

$K_9^{(i)}$;$i=1,\ldots,11$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent Arcs**	Stabilizer group
$K_{10}^{(1)}$	$\{137, 113, 12, 11\}$	A$\cup\{5, 32, 224, 43, 102\}$	I
$K_{10}^{(2)}$	$\{138, 110, 15, 10\}$	A$\cup$ $\{4, 5, 36, 6, 21\}$	I
$K_{10}^{(3)}$	$\{139, 107, 18, 9\}$	A$\cup$ $\{4, 11, 21, 5, 40\}$	I
$K_{10}^{(4)}$	$\{140, 104, 21, 8\}$	A$\cup$ $\{4, 11, 13, 25, 23\}$	I
$K_{10}^{(5)}$	$\{141, 101, 24, 7\}$	A$\cup$ $\{5, 32, 224, 80, 6\}$	I
$K_{10}^{(6)}$	$\{142, 98, 27, 6\}$	A$\cup$ $\{10, 12, 11, 24, 14\}$	I
$K_{10}^{(7)}$	$\{143, 95, 30, 5\}$	A$\cup$ $\{18, 4, 9, 12, 11\}$	I
$K_{10}^{(8)}$	$\{144, 92, 33, 4\}$	A$\cup$ $\{10, 12, 11, 19, 5\}$	I
$K_{10}^{(9)}$	$\{145, 89, 36, 3\}$	A$\cup$ $\{4, 11, 13, 26, 24\}$	I
$K_{10}^{(10)}$	$\{146, 86, 39, 2\}$	A$\cup$ $\{10, 12, 11, 24, 48\}$	I
$K_{10}^{(11)}$	$\{147, 83, 42, 1\}$	A$\cup$ $\{10, 12, 11, 19, 71\}$	I

4.7. The constructions and classifications of (11;3)-arc

The number of $\boldsymbol{c_0}$ of each (10;3)-arc in $PG(2,16)$ are shown in table 12

Table 12: The number of c_0 of K_{10}

Arc	$\boldsymbol{K_{10}^{(1)}}$	$\boldsymbol{K_{10}^{(2)}}$	$\boldsymbol{K_{10}^{(3)}}$	$\boldsymbol{K_{10}^{(4)}}$	$\boldsymbol{K_{10}^{(5)}}$	$\boldsymbol{K_{10}^{(6)}}$
No.of $\boldsymbol{c_0}$	123	136	147	172	156	196
Arc	$\boldsymbol{K_{10}^{(7)}}$	$\boldsymbol{K_{10}^{(8)}}$	$\boldsymbol{K_{10}^{(9)}}$	$\boldsymbol{K_{10}^{(10)}}$	$\boldsymbol{K_{10}^{(11)}}$	
No.of $\boldsymbol{c_0}$	183	208	221	236	249	

The number of (11;3)-arcs that have been established is (2027) and the number of distinct arcs is (13) that have the following class of secant distribution shown in the table 12.

Table 13: Distinct classes and the inequivalent arcs of K_{11}

$\boldsymbol{K_{11}^{(i)};i=1,\ldots,13}$	$\{\boldsymbol{l_0}\ ,\boldsymbol{l_1},\boldsymbol{l_2},\boldsymbol{l_3}\}$	**Inequivalent arcs**	**S.G**
$K_{11}^{(1)}$	$\{127,119,13,14\}$	A$\cup$ $\{5,32,224,43,102,126\}$	I
$K_{11}^{(2)}$	$\{128,116,16,13\}$	A$\cup$ $\{4,5,36,6,21,37\}$	I
$K_{11}^{(3)}$	$\{129,113,19,12\}$	A$\cup$ $\{4,11,21,5,40,158\}$	I
$K_{11}^{(4)}$	$\{130,110,22,11\}$	A$\cup$ $\{4,11,13,25,23,124\}$	I
$K_{11}^{(5)}$	$\{131,107,25,10\}$	A$\cup$ $\{5,32,224,80,6,10\}$	I
$K_{11}^{(6)}$	$\{132,104,28,9\}$	A$\cup$ $\{10,12,11,24,14,137\}$	I
$K_{11}^{(7)}$	$\{133,101,31,8\}$	A$\cup$ $\{18,4,9,12,11,19\}$	I
$K_{11}^{(8)}$	$\{134,95,34,7\}$	A$\cup$ $\{10,12,11,19,5,6\}$	I
$K_{11}^{(9)}$	$\{135,95,37,6\}$	A$\cup$ $\{18,4,9,12,11,104\}$	I
$K_{11}^{(10)}$	$\{136,92,40,5\}$	A$\cup$ 10,12,11, ,24,48,4	I
$K_{11}^{(11)}$	$\{137,89,43,4\}$	A$\cup$ $\{10,12,11,\ ,24,48,8\}$	I
$K_{11}^{(12)}$	$\{138,86,46,3\}$	A$\cup$ $\{4,11,13,26,24,39\}$	I
$K_{11}^{(13)}$	$\{139,83,49,2\}$	A$\cup$ $\{10,12,11,19,71,29\}$	I

4.8. The constructions and classifications of (12;3)-arc

The number of $\boldsymbol{c_0}$ of each (11;3)-arc in $PG(2,16)$ are shown in table 14.

Table 14: The number of c_0 of K_{11}

Index	$K_{11}^{(1)}$	$K_{11}^{(2)}$	$K_{11}^{(3)}$	$K_{11}^{(4)}$	$K_{11}^{(5)}$	$K_{11}^{(6)}$	$K_{11}^{(7)}$
No. of c_0	94	105	116	126	136	149	158
Index	$K_{11}^{(8)}$	$K_{11}^{(9)}$	$K_{11}^{(10)}$	$K_{11}^{(11)}$	$K_{11}^{(12)}$	$K_{11}^{(13)}$	
No. of c_0	170	182	194	208	220	234	

The number of all (12;3)-arcs is 2092. Among of these arcs there are (15), where each has the following distinct class of secant distribution .

Table 15: Distinct classes and the inequivalent arcs of K_{12}

$K_{12}^{(i)}; i = 1, \ldots, 15$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	**S.G**
$K_{12}^{(1)}$	$\{118, 123, 15, 17\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57\}$	I
$K_{12}^{(2)}$	$\{119, 120, 18, 16\}$	A∪ $\{4, 5, 36, 6, 21, 37, 19\}$	I
$K_{12}^{(3)}$	$\{120, 117, 21, 15\}$	A∪ $\{4, 11, 21, 5, 40, 158, 126\}$	I
$K_{12}^{(4)}$	$\{121, 114, 24, 14\}$	A∪ $\{4, 11, 13, 25, 23, 124, 6\}$	I
$K_{12}^{(5)}$	$\{122, 111, 27, 13\}$	A∪ $\{5, 32, 224, 80, 6, 10, 22\}$	I
$K_{12}^{(6)}$	$\{123, 108, 30, 12\}$	A∪ $\{10, 12, 11, 24, 14, 137, 27\}$	I
$K_{12}^{(7)}$	$\{124, 105, 33, 11\}$	A∪ $\{18, 4, 9, 12, 11, 19, 20\}$	I
$K_{12}^{(8)}$	$\{125, 102, 36, 10, \}$	A∪ $\{10, 12, 11, 19, 5, 6, 21\}$	I
$K_{12}^{(9)}$	$\{126, 99, 39, 9\}$	A∪ $\{18, 4, 9, 12, 11, 104, 35\}$	I
$K_{12}^{(10)}$	$\{127, 96, 42, 8\}$	A∪ $\{10, 12, 11, \ , 24, 48, 4, 17\}$	I
$K_{12}^{(11)}$	$\{128, 93, 45, 7\}$	A∪ $\{10, 12, 11, \ , 24, 48, 8, 13\}$	I
$K_{12}^{(12)}$	$\{129, 90, 48, 6\}$	A∪ $\{4, 11, 13, 26, 24, 39, 9\}$	I
$K_{12}^{(13)}$	$\{130, 87, 51, 5\}$	A∪ $\{10, 12, 11, 19, 71, 29, 17\}$	I
$K_{12}^{(14)}$	$\{131, 84, 54, 4\}$	A∪ $\{4, 11, 13, 26, 24, 39, 77\}$	I
$K_{12}^{(15)}$	$\{132, 81, 57, 3\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38\}$	I

4.9. The constructions and classifications of (13;3)-arcs

The number of c_0 of each (12;3)-arc in $PG(2, 16)$ are shown in table 16.

Table 16: The number of c_0 of K_{12}

Index	$K_{12}^{(1)}$	$K_{12}^{(2)}$	$K_{12}^{(3)}$	$K_{12}^{(4)}$	$K_{11}^{(5)}$	$K_{12}^{(6)}$	$K_{12}^{(7)}$	$K_{12}^{(8)}$
No. of c_0	72	80	89	99	107	118	128	139
Index	$K_{12}^{(9)}$	$K_{12}^{(10)}$	$K_{12}^{(11)}$	$K_{12}^{(12)}$	$K_{12}^{(13)}$	$K_{12}^{(14)}$	$K_{12}^{(15)}$	
No. of c_0	149	160	172	183	195	207	221	

Here the number of (13;3)-arcs that has been constructed is (2118) and the number of distinct arc is (17) ,The related information is shown in table 17.

Table 17: Distinct classes and the inequivalent arcs of K_{13}

$K_{13}^{(i)}; i = 1,\dots,17$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	**S.G**
$K_{13}^{(1)}$	$\{110, 125, 18, 20\}$	A∪ $\{32, 224, 43, 102, 126, 57, 53\}$	I
$K_{13}^{(2)}$	$\{111, 122, 21, 19\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 10\}$	I
$K_{13}^{(3)}$	$\{112, 119, 24, 18\}$	A∪ $\{4, 5, 36, 6, 21, 37, 19, 46\}$	I
$K_{13}^{(4)}$	$\{113, 116, 27, 17\}$	A∪ $\{4, 11, 21, 5, 40, 158, 126, 10\}$	I
$K_{13}^{(5)}$	$\{114, 113, 30, 16\}$	A∪ $\{4, 11, 13, 25, 23, 124, 6, 24\}$	I
$K_{13}^{(6)}$	$\{115, 110, 33, 15\}$	A∪ $\{5, 32, 224, 80, 6, 10, 22, 12\}$	I
$K_{13}^{(7)}$	$\{116, 107, 36, 14\}$	A∪ $\{10, 12, 11, 24, 14, 137, 27, 19\}$	I
$K_{13}^{(8)}$	$\{117, 104, 39, 13\}$	A∪ $\{18, 4, 9, 12, 11, 19, 20, 13\}$	I
$K_{13}^{(9)}$	$\{118, 101, 42, 12\}$	A∪ $\{10, 12, 11, 19, 5, 6, 21, 16\}$	I
$K_{13}^{(10)}$	$\{119, 98, 45, 11\}$	A∪ $\{18, 4, 9, 12, 11, 104, 35, 13\}$	I
$K_{13}^{(11)}$	$\{120, 95, 48, 10\}$	A∪ $\{10, 12, 11, 24, 48, 4, 17, 19\}$	I
$K_{13}^{(12)}$	$\{121, 92, 51, 9\}$	A∪ $\{10, 12, 11, 24, 48, 8, 13, 19\}$	I
$K_{13}^{(13)}$	$\{122, 89, 54, 8\}$	A∪ $\{10, 12, 11, 19, 71, 29, 17, 53\}$	I
$K_{13}^{(14)}$	$\{123, 86, 57, 7\}$	A∪ $\{4, 11, 13, 26, 24, 39, 9, 30\}$	I
$K_{13}^{(15)}$	$\{124, 83, 60, 6\}$	A∪ $\{4, 11, 13, 26, 24, 39, 77, 33\}$	I
$K_{13}^{(16)}$	$\{125, 80, 63, 5\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 23\}$	I
$K_{13}^{(17)}$	$\{126, 77, 66, 4\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121\}$	I

4.10. The constructions and classifications of (14;3)-arcs

The number of c_0 of each distinct (13;3)-arc in $PG(2, 16)$ are shown in table 18

Table 18: The number of c_0 of K_{13}

Index	$K_{13}^{(1)}$	$K_{13}^{(2)}$	$K_{12}^{(3)}$	$K_{13}^{(4)}$	$K_{13}^{(5)}$	$K_{13}^{(6)}$	$K_{13}^{(7)}$	$K_{13}^{(8)}$	$K_{13}^{(9)}$
No. of c_0	55	66	67	78	83	91	103	110	119
Index	$K_{13}^{(10)}$	$K_{13}^{(11)}$	$K_{13}^{(12)}$	$K_{13}^{(13)}$	$K_{13}^{(14)}$	$K_{13}^{(15)}$	$K_{13}^{(16)}$	$K_{13}^{(17)}$	
No.of c_0	129	141	150	193	161	182	196	208	

In this process, the number of (14;3)-arc established is (2132) .also there are (18) distinct (14;3)-arc, The related information is shown in table 19.

Table 19: Distinct classes and the inequivalent arcs of K_{14}

$K_{14}^{(i)}$;$i = 1, \ldots, 18$	$\{l_0, l_1, l_2, l_3\}$	Inequivalent arcs	S.G
$K_{14}^{(1)}$	$\{103, 15, 22, 23\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 53, 19\}$	I
$K_{14}^{(2)}$	$\{104, 122, 25, 22\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 10, 107\}$	I
$K_{14}^{(3)}$	$\{105, 119, 28, 21\}$	A∪ $\{4, 5, 36, 6, 21, 37, 19, 46, 50\}$	I
$K_{14}^{(4)}$	$\{106, 116, 31, 20\}$	A∪ $\{4, 11, 21, 5, 40, 158, 126, 10, 49\}$	I
$K_{14}^{(5)}$	$\{107, 113, 34, 19\}$	A∪ $\{4, 11, 13, 25, 23, 124, 6, 24, 43\}$	I
$K_{14}^{(6)}$	$\{108, 110, 37, 18\}$	A∪ $\{5, 32, 224, 80, 6, 10, 22, 12, 24\}$	I
$K_{14}^{(7)}$	$\{108, 107, 40, 17\}$	A∪ $\{10, 12, 11, 24, 14, 137, 27, 19, 34\}$	I
$K_{13}^{(8)}$	$\{110, 104, 43, 16\}$	A∪ $\{18, 4, 9, 12, 11, 19, 20, 13, 36\}$	I
$K_{14}^{(9)}$	$\{111, 101, 46, 15\}$	A∪ $\{10, 12, 11, 19, 5, 6, 21, 16, 28\}$	I
$K_{14}^{(10)}$	$\{112, 98, 49, 14\}$	A∪ $\{18, 4, 9, 12, 11, 104, 35, 13, 22\}$	I
$K_{14}^{(11)}$	$\{113, 95, 52, 13\}$	A∪ $\{10, 12, 11, 24, 48, 4, 17, 19, 23\}$	I
$K_{14}^{(12)}$	$\{114, 92, 55, 12\}$	A∪ $\{10, 12, 11, 24, 48, 8, 13, 19, 14\}$	I
$K_{14}^{(13)}$	$\{115, 89, 58, 11\}$	A∪ $\{10, 12, 11, 19, 71, 29, 17, 53, 20\}$	I
$K_{14}^{(14)}$	$\{116, 86, 61, 10\}$	A∪ $\{4, 11, 13, 26, 24, 39, 9, 30, 48\}$	I
$K_{14}^{(15)}$	$\{117, 83, 64, 9\}$	A∪ $\{4, 11, 13, 26, 24, 39, 77, 33, 22\}$	I
$K_{14}^{(16)}$	$\{118, 80, 67, 8\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 23, 33\}$	I
$K_{14}^{(17)}$	$\{119, 77, 67, 8\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 22\}$	I
$K_{14}^{(18)}$	$\{120, 74, 73, 6\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 28\}$	I

4.11. The constructions and classifications of (15;3)-arc

The number of c_0 of each distinct (14;3)-arc in $PG(2, 16)$ are shown in table 20.

Table 20: The number of c_0 of K_{14}

Index	$K_{14}^{(1)}$	$K_{14}^{(2)}$	$K_{14}^{(3)}$	$K_{14}^{(4)}$	$K_{14}^{(5)}$	$K_{14}^{(6)}$	$K_{13}^{(7)}$	$K_{14}^{(8)}$	$K_{14}^{(9)}$
No. of c_0	41	43	50	58	61	71	82	84	92
Index	$K_{14}^{(10)}$	$K_{14}^{(11)}$	$K_{14}^{(12)}$	$K_{14}^{(13)}$	$K_{14}^{(14)}$	$K_{14}^{(15)}$	$K_{14}^{(16)}$	$K_{14}^{(17)}$	$K_{14}^{(18)}$
No. of c_0	105	113	112	119	140	145	161	170	184

The number of (15;3)-arc is (1838) and there are (18) (15;3)-arcs, having the following distinct classes of secant distribution shown in table 21.

Table 21: Distinct classes and the inequivalent arcs of K_{15}

$K_{15}^{(i)}$; $i=1,\ldots,17$	$\{l_0, l_1, l_2, l_3\}$	**Inequivalent arcs**	S.G
$K_{15}^{(1)}$	$\{97, 123, 27, 26\}$	A∪ 5,32,224,43,102,126,57,53,19,11	I
$K_{15}^{(2)}$	$\{98, 120, 30, 25\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 10, 107, 48\}$	I
$K_{15}^{(3)}$	$\{99, 117, 33, 24\}$	A∪ $\{4, 5, 36, 6, 21, 37, 19, 46, 50, 45\}$	I
$K_{15}^{(4)}$	$\{100, 114, 36, 24\}$	A∪ $\{4, 11, 21, 5, 40, 158, 126, 10, 49, 28\}$	I
$K_{15}^{(5)}$	$\{101, 111, 39, 22\}$	A∪ $\{4, 11, 13, 25, 23, 124, 6, 24, 43, 26\}$	I
$K_{15}^{(6)}$	$\{102, 108, 42, 21\}$	A∪ $\{5, 32, 224, 80, 6, 10, 22, 12, 24, 19\}$	I
$K_{15}^{(7)}$	$\{103, 105, 45, 20\}$	A∪ $\{10, 12, 11, 24, 14, 137, 27, 19, 34, 29\}$	I
$K_{15}^{(8)}$	$\{104, 102, 48, 19\}$	A∪ $\{18, 4, 9, 12, 11, 19, 20, 13, 36, 30\}$	I
$K_{15}^{(9)}$	$\{105, 99, 51, 18\}$	A∪ $\{10, 12, 11, 19, 5, 6, 21, 16, 28, 37\}$	I
$K_{15}^{(10)}$	$\{106, 96, 54, 17\}$	A∪ $\{18, 4, 9, 12, 11, 104, 35, 13, 22, 24\}$	I
$K_{15}^{(11)}$	$\{107, 93, 57, 16\}$	A∪ $\{10, 12, 11, 24, 48, 4, 17, 19, 23, 22\}$	I
$K_{15}^{(12)}$	$\{108, 90, 60, 15\}$	A∪ $\{10, 12, 11, 24, 48, 8, 13, 19, 14, 35\}$	I
$K_{15}^{(13)}$	$\{109, 87, 63, 14\}$	A∪ $\{10, 12, 11, 19, 71, 29, 17, 53, 20, 57\}$	I
$K_{15}^{(14)}$	$\{110, 84, 66, 13\}$	A∪ $\{4, 11, 13, 26, 24, 39, 9, 30, 48, 35\}$	I
$K_{15}^{(15)}$	$\{111, 81, 69, 12\}$	A∪ $\{4, 11, 13, 26, 24, 39, 77, 33, 22, 12\}$	I
$K_{15}^{(16)}$	$\{112, 78, 72, 11\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 23, 33, 28\}$	I
$K_{15}^{(17)}$	$\{113, 75, 75, 10\}$	A∪$\{10, 12, 11, 19, 71, 29, 38, 121, 22, 23\}$	I
$K_{15}^{(18)}$	$\{114, 72, 78, 9\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 28, 45\}$	I

4.12. The constructions and classifications of (16;3)-arc

The number of c_0 of each distinct (15;3)-arc in $PG(2,16)$ are shown in table 22

Table 22: The number of c_0 of K_{15}

ndex	$K_{15}^{(1)}$	$K_{15}^{(2)}$	$K_{15}^{(3)}$	$K_{15}^{(4)}$	$K_{15}^{(5)}$	$K_{15}^{(6)}$	$K_{15}^{(7)}$	$K_{15}^{(8)}$	$K_{14}^{(9)}$
No. of c_0	30	34	36	45	53	53	63	67	70
Index	$K_{15}^{(10)}$	$K_{15}^{(11)}$	$K_{15}^{(12)}$	$K_{14}^{(13)}$	$K_{15}^{(14)}$	$K_{14}^{(15)}$	$K_{15}^{(16)}$	$K_{15}^{(17)}$	$K_{15}^{(18)}$
No. of c_0	87	87	98	102	109	123	134	140	150

The number of (16;3)-arcs in $PG(2,16)$ is (1481) and the number of distinct (16;3)-arcs is (20), having the following classes of secant distribution .

Table 23: Distinct classes and the inequivalent arcs of K_{16}

$K_{16}^{(i)}$;$i = 1, \ldots ,20$	$\{l_0 ,l_1, l_2, l_3\}$	**Inequivalent arcs**	**S.G**
$K_{16}^{(1)}$	$\{90, 125, 27, 3\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 53, 19, 11, 164\}$	I
$K_{16}^{(2)}$	$\{91, 122, 30, 30\}$	A∪ $\{5, 32, 224, 43, 102, 126, 57, 10, 107, 48, 265\}$	I
$K_{16}^{(3)}$	$\{92, 119, 33, 29\}$	A∪ $\{4, 5, 36, 6, 21, 37, 19, 46, 50, 45, 53\}$	I
$K_{16}^{(4)}$	$\{93, 116, 36, 28\}$	A∪ $\{4, 11, 21, 5, 40, 158, 126, 10, 49, 28, 72\}$	I
$K_{16}^{(5)}$	$\{94, 113, 39, 27\}$	A∪ $\{4, 11, 13, 25, 23, 124, 6, 24, 43, 26, 59\}$	I
$K_{16}^{(6)}$	$\{95, 110, 42, 26\}$	A∪ $\{5, 32, 224, 80, 6, 10, 22, 12, 24, 19, 45\}$	I
$K_{16}^{(7)}$	$\{96, 107, 45, 25\}$	A∪ $\{10, 12, 11, 24, 14, 137, 27, 19, 34, 29, 55\}$	I
$K_{16}^{(8)}$	$\{97, 104, 48, 24\}$	A∪ $\{18, 4, 9, 12, 11, 19, 20, 13, 36, 30, 41\}$	I
$K_{16}^{(9)}$	$\{98, 101, 51, 23\}$	A∪ $\{10, 12, 11, 19, 5, 6, 21, 16, 28, 37, 43\}$	I
$K_{16}^{(10)}$	$\{99, 98, 54, 22\}$	A∪ $\{18, 4, 9, 12, 11, 104, 35, 13, 22, 24, 37\}$	I
$K_{16}^{(11)}$	$\{100, 95, 57, 21\}$	A∪ $\{10, 12, 11, 24, 48, 4, 17, 19, 23, 22, 43\}$	I
$K_{16}^{(12)}$	$\{101, 92, 60, 20\}$	A∪ $\{10, 12, 11, 24, 48, 8, 13, 19, 14, 35, 28\}$	I
$K_{16}^{(13)}$	$\{102, 89, 63, 19\}$	A∪ $\{10, 12, 11, 19, 71, 29, 17, 53, 20, 57, 4\}$	I
$K_{16}^{(14)}$	$\{103, 86, 66, 18\}$	A∪ $\{4, 11, 13, 26, 24, 39, 9, 30, 48, 35, 10\}$	I
$K_{16}^{(15)}$	$\{104, 93, 69, 17\}$	A∪ $\{4, 11, 13, 26, 24, 39, 77, 33, 22, 12, 6\}$	I
$K_{15}^{(16)}$	$\{105, 80, 72, 16\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 23, 33, 28, 16\}$	I
$K_{16}^{(17)}$	$\{106, 77, 75, 15\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 22, 23, 17\}$	I
$K_{16}^{(18)}$	$\{107, 74, 78, 14\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 28, 45, 14\}$	I
$K_{16}^{(19)}$	$\{108, 71, 81, 13\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 22, 23, 28\}$	I
$K_{16}^{(20)}$	$\{109, 68, 84, 12\}$	A∪ $\{10, 12, 11, 19, 71, 29, 38, 121, 28, 45, 23\}$	I

5. The Smallest size of arc of degree three in $PG(2, 16)$

According to the previous calculations which is used to construct the $(n; 3)$-arc in $PG(2, 16)$ for some n, we obtained the following:

Theorem 5.1. *The cubic curve of the canonical form $F = z^2y + zy^2 + x^3 + \omega xy^2$ which has 17 points is the smallest complete$(n; 3)$-arc in $PG(2, 16)$.*

Proof . When we found the point on this curve we obtained the following points.

$$\left\{\begin{array}{c} [0,1,0],\ [0,0,1],\ [\omega^9,\omega^{13},1],\ [\omega^9,\omega^9,1],\ [\omega^8,1,1],\ [\omega^{14},\omega^{12},1],\ [1,\omega^9,1],\ [\omega^8,\omega^2,1], \\ [\omega^7,\omega^{10},1],\ [0,1,1],\ [\omega^7,\omega^9,1],\ [\omega^3,\omega,1],\ [1,\omega^2,1],\ [\omega^2,\omega^2,1],\ [w^8,1,0],\ [\omega^2,\omega^5,1], \\ [\omega^3,\omega^7,1\ ] \end{array}\right\}$$

which are 17 points, and then the number of c_0 of this curve are computed and the result was $c_0 = 0$. Then the secant distribution $\{l_0, l_1, l_2, l_3\} = \{56, 101, 24, 92\}$ for this curve is calculated.

So from this accounts the (17; 3)-arc is the smallest complete (n ;3)-arc in $PG(2, 16)$ □

The related information of (17; 3)-arc is given in table 24.

Table 24: The (17; 3)-arc details

Canonical form	**The No. of c_0**	$\{l_0, l_1, l_2, l_3\}$	**The points on the curve**	**Size**
$F = z^2y + zy^2 + x^3 + \omega yz^2$	0	$\{56, 101, 24, 92\}$	$\left\{ \begin{matrix} 1, 2, 19, 20, 58, 72, 88, \\ 6, 98, 137, 198, 139, 202 \\ , 224, 233, 259, 270 \end{matrix} \right\}$	17

Thus from above table and the our previous calculation in section four the size of smallest complete $(n; 3)$ in $PG(2, 16)$ is 17.

5.1. The Groups of projectivities of (17;3)-arc

The Group of projectivities of K_{17} has four projectivities.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, T_2 = \begin{bmatrix} \omega^9 & 0 & \omega^7 \\ \omega^9 & \omega^9 & 1 \\ 0 & 0 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} \omega^4 & 0 & 0 \\ 0 & \omega & \omega^4 \\ 0 & 0 & \omega^4 \end{bmatrix}, \ T_4 = \begin{bmatrix} \omega^6 & 0 & \omega^6 \\ \omega^6 & \omega^6 & \omega^4 \\ 0 & 0 & \omega^6 \end{bmatrix}$$

The projective group contains:
1- one matrix of order one.
2- one matrix of order two.
3- Two matrices of order four.
S0 the stabilize group of (17; 3) isomorphic to Z_4. To show that The group $T = \{T_1, T_2, T_3, T_4\} \cong Z_4$.

1. Let $\beta : T \to Z_4$ by $\beta(T_1) = [0]$, $\beta(T_2) = [1]$, $\beta(T_3) = [2]$ and $\alpha(T_4) = [3]$.

2. Define T and Z_4 by the following tables:

$\cdot$	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2	T_3	T_4	T_1
T_3	T_3	T_4	T_1	T_2
T_4	T_4	T_1	T_2	T_3

$+_4$	[0]	[1]	[2]	[3]
[0]	[0]	[1]	[2]	[3]
[1]	[1]	[2]	[3]	[0]
[2]	[2]	[3]	[0]	[1]
[3]	[3]	[0]	[1]	[2]

It is clear that β is bijective function. To prove that α is homomorphism, we find $\beta(T_1.T_2) = \beta(T_2) = [1]$ and $\beta(T_1) +_4 \beta(T_2) = [0[+_4 [1] = [1]$

$\beta(T_1.T_3) = \beta(T_3) = [2]$ and $\boldsymbol{\alpha}(T_1) +_4 \beta(T_3) = [0] +_4 [2] = [2]$

$\beta(T_1.T_4) = \beta(T_4) = [3]$ and $\beta(T_1) +_4 \beta(T_4) = [0] +_4 [3] = [3]$

$\beta(T_2.T_2) = \alpha(T_3) = [2]$ and $\beta(T_2) +_4 \beta(T_2) = [1] +_4 [1] = [2]$

$\beta(T_2.T_3) = \beta(T_4) = [3]$ and $\beta(T_2) +_4 \beta(T_3) = [1] +_4 [2] = [3]$

$\beta(T_2.T_4) = \beta(T_1) = [0]$ and $\beta(T_2) +_4 \beta(T_3) = [1] +_4 [3] = [0]$

$\beta(T_3.T_3) = \beta(T_1) = [0]$ and $\beta(T_3) +_4 \beta(T_3) = [2] +_4 [2] = [0]$

$\beta(T_3.T_4) = \beta(T_2)$ =[1] and $\beta(T_3) +_4 \beta(T_2)$ =[2] $+_4$ [3] = [1]

$\beta(T_4.T_4)$=$\beta(T_3)$= [2] and $\beta(T_4)$+$_4\beta(T_4)$=[3] $+_4$ [3] = [2].

Hence β isomorphism.

6. Conclusion

In this paper we study the number of $(n;3)$-arcs, which have the distinct secant distributions and the small size of $(n;3)$-arcs with property of completeness, from the previous calculation the following results obtained :-
1-The number of $(n;3)$-arcs in $PG(2,16)$ is classified.
2-The number of distinct $(n;3)$-arcs in $PG(2,16)$ is classified.
3-The small complete $(n;3)$-arcs of size n=17 is established.
4-The stabilizer group of these arcs are computed.

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