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The lowest-degree stabilizer-free weak Galerkin finite element method for Poisson equation on rectangular and triangular meshes

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Abstract

Recently, the study on weak Galerkin (WG) methods with or without stabilizer parameters have received much attention. The WG methods are a discontinuous extension of the standard finite element methods in which classical differential operators are approximated on functions with discontinuity. A stabilizer term in the WG formulation is used to guarantee convergence and stability of the discontinuous approximations for a model problem. By removing this parameter, we can reduce the complexity of programming on this numerical method. Our goal in this paper is to introduce a new stabilizer-free WG (SFWG) method to solve the Poisson equation in which we use a new combination of WG elements. Numerical experiments indicate that our SFWG scheme is faster and more economical than the standard WG scheme. Errors and convergence rates on two types of mesh are presented for each of the considered methods, which show that our numerical scheme has $O(h^2)$ convergence rate while another method has $O(h)$ convergence rate in the energy norm and the L^2-norm.

Keywords: Stabilizer-free weak Galerkin, Discrete weak differential operators, The Poisson equation, Rectangular mesh, Lowest-degree elements.
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1 Introduction

In this work, we focus to seeks an unknown function $u = u(x)$ for the Poisson equation which is satisfying,

$$-\Delta u = f, \quad in \quad \Omega \subset \mathbb{R}^2, \tag{1.1}$$
$$u = 0, \quad on \quad \partial\Omega, \tag{1.2}$$

where Δu denotes the Laplacian operator of the function u and Ω is a polygonal domain in $\mathbb{R}^2$ with boundary $\partial\Omega$.

The Poisson equation has wide applications in many areas such as modeling various problems in mechanics and physics. Evaluation and analysis of numerical approximations of mathematical models has been one of the topics of interest for researchers in recent years. In order to achieve this goal, many numerical methods have been studied.

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Numerical methods that provide discontinuous approximations of PDEs are one of the methods considered in this field, such as local discontinuous Galerkin (DG) methods [2, 6, 5], hybridizable DG methods[8, 4] and WG methods [12, 13, 14, 10]. One of the obvious disadvantages of these discontinuous approximation tools is the use of different parameters in their Galerkin formulations to enforce weak continuity across element boundaries due to the nature of the discontinuity of the solutions. The presence of stabilizing parameters in the finite element formulation will increase the complexity of the implementation.

Recently, WG methods on PDEs have been investigated [3, 7, 15, 9]. The most obvious feature of this numerical method is the presence of weak functions and weak partial derivatives of them. For each element T, a weak function v is a piecewise function in the form $v = \{v_0, v_b\}$, where we consider the first and second components as the interior and edges value of v on T, respectively.

The standard WG method for the problem (1.1)-(1.2) has the following form

$$(\nabla_w u_h, \nabla_w v) + s(u_h, v) = (f, v), \tag{1.3}$$

where ∇_w is a weak version of ∇ and $s(\cdot,\cdot)$ is a stabilizer term that has the following usual form

$$s(u_h, v) = \sum_{T\in\mathcal{T}_h} h_T^r \langle Q_b u_0 - u_b, Q_b v_0 - v_b\rangle_{\partial T}, \tag{1.4}$$

where by choosing $r = \infty$, we will have $s(u_h, v) = 0$ [18].

The study of WG methods without any stabilizer parameters on PDEs has received much attention recently. The main attitude of this numerical tool is to increase the degree of approximation space for ∇_w. By increasing the degree of weak gradient space, the additional degrees of freedom will not be defined. In [17], the optimal order of convergence for SFWG method by choosing $(P_k(T), P_k(e), [P_j(T)]^d)$ elements for $j > k \geq 1$ and $d = 2, 3$ on polytopal meshes has been investigated. After that, Ye and Zhang [19] improved the convergence rates presented in [17] and achieved superconvergence on polytopal mesh. In [1], A lowest-order SFWG scheme on triangular meshes for the solution of (1.1)-(1.2) is proposed where $(P_0(T), P_1(e), [P_1(T)]^2)$ elements is used.

In this paper, we focus on the standard WG method (1.3) for seeking WG finite element solution $u_h = \{u_0, u_b\}$ and propose a lowest-degree possible WG scheme on rectangular and triangular meshes where a new combination $(P_0(T), P_0(e), [P_1(T)]^2)$ of elements will be used. For $r = \infty$ and 0, we compare our new SFWG scheme with the standard WG scheme (with stabilizer parameter). Some numerical examples are tested which show that our new scheme is faster and more efficient than the standard WG method.

The rest of this paper is organized as follows. In Section 2, we introduce the WG spaces and the weak differential operator to present our new scheme. The process of obtaining the error equation and some important inequalities is discussed in Section 3. The theoretical results related to the L^2 error estimates for our SFWG finite element method are established in Section 4. Several numerical examples are presented in Section 5 to confirm the presented theoretical results in Section 3 and Section 4. Section 6 is devoted to Conclusions of this paper.

2 SFWG schemes

Suppose that T be an element of the partition $\mathcal{T}_h$ created of $\Omega \subset \mathbb{R}^2$. For each element $T \in \mathcal{T}_h$, let $h = \max_{T\in\mathcal{T}_h} h_T$ be the mesh size of $\mathcal{T}_h$ where h_T is the diameter of each element T. For each element $T \in \mathcal{T}_h$, we define the WG space and its subspace as follows:

$$V_h = \{v = \{v_0, v_b\} \in P_0(T) \times P_0(e), e \in \partial T, \mathcal{T}_h = \{T\}\},$$

and

$$V_h^0 = \{v : v \in V_h, v_b|_e = 0, \quad e \in \partial\Omega \cap \partial T\}.$$

For each $T \in \mathcal{T}_h$, we define the L^2-projections as follows:

$$Q_0 : L^2(T) \to P_0(T), \tag{2.1}$$

$$Q_b : L^2(e) \to P_0(e), \tag{2.2}$$

$$\mathbb{Q}_h : [L^2(T)]^2 \to [P_1(T)]^2. \tag{2.3}$$

By combining the L^2-projections Q_0 and Q_b, we define another L^2-projection onto V_h which is defined as $Q_h = \{Q_0, Q_b\}$.

On each element T and for any $\phi = \{\phi_0, \phi_b\} \in V_h$, the weak gradient $\nabla_w \in [P_1(T)]^2$ is defined as follows:

$$(\nabla_w \phi, \boldsymbol{\tau})_T = -(\phi_0, \nabla \cdot \boldsymbol{\tau})_T + \langle \phi_b, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T}, \qquad \forall \boldsymbol{\tau} \in [P_1(T)]^2, \tag{2.4}$$

where $\mathbf{n}$ is the unit outward normal vector of ∂T.

Throughout this paper, we applied these notations for simplicity

$$(\phi, \psi)_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} (\phi, \psi)_T = \sum_{T \in \mathcal{T}_h} \int_T \phi \cdot \psi dx,$$
$$\langle \phi, \psi \rangle_{\partial \mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \langle \phi, \psi \rangle_{\partial T} = \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \cdot \psi ds.$$

2.1 The lowest-degree SFWG

By finding $u_h = \{u_0, u_b\} \in V_h^0$, a numerical approximation for (1.1)-(1.2) can be obtained such that

$$(\nabla_w u_h, \nabla_w v) = (f, v_0), \qquad \forall v = \{v_0, v_b\} \in V_h^0. \tag{2.5}$$

An energy norm according to our SFWG scheme is defined as

$$|||v|||^2 = (\nabla_w v, \nabla_w v)_{\mathcal{T}_h},$$

and H^1 semi-norm on V_h is defined as

$$\|v\|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} \|\nabla v_0\|_T^2 + \sum_{T \in \mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2.$$

It can be easily checked that $\|\cdot\|_{1,h}$ defines a norm on V_h^0. Below, we indicate the equivalence of the two norms $\|\cdot\|_{1,h}$ and $|||\cdot|||$.

Lemma 2.1. Let $v = \{v_0, v_b\} \in V_h$. There exist two constants C_1 and C_2 such that

$$C_1 \|v\|_{1,h} \le |||v||| \le C_2 \|v\|_{1,h}.$$

Proof . We provide only details of proof $|||v||| \le C_2 \|v\|_{1,h}$. We can referred to [16] for the next part.
From (2.4) and integration by parts, we have

$$\begin{aligned}(\nabla_w v, \boldsymbol{\tau})_T &= -(v_0, \nabla \cdot \boldsymbol{\tau})_T + \langle v_b, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla v_0, \boldsymbol{\tau})_T - \langle v_0 - v_b, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial T}, \qquad \forall \boldsymbol{\tau} \in [P_1(T)]^2.\end{aligned} \tag{2.6}$$

It follows from the trace inequality and the inverse inequality that

$$\begin{aligned}(\nabla_w v, \boldsymbol{\tau})_T &\le \|\nabla v_0\|_T \|\boldsymbol{\tau}\|_T + \|v_0 - v_b\|_{\partial T} \|\boldsymbol{\tau}\|_{\partial T} \\ &\le \|\nabla v_0\|_T \|\boldsymbol{\tau}\|_T + C h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\boldsymbol{\tau}\|_T.\end{aligned} \tag{2.7}$$

By letting $\boldsymbol{\tau} = \nabla_w v$ in the above inequality, we can get the desired result. □

3 Error Equation

In the following, we will provide the basic lemmas to achieve the error equation. At first, we introduce the crucial property of ∇_w and ∇ by the definition of L^2-projections.

Lemma 3.1. For each $T \in \mathcal{T}_h$ and $\phi \in H^1(\Omega)$, we have

$$\mathbb{Q}_h(\nabla\phi) = \nabla_w Q_h \phi. \tag{3.1}$$

Proof . From definition (2.4), the property of L^2-projections and integration by parts, we can write

$$\begin{aligned}(\nabla_w Q_h\phi, \boldsymbol{\tau})_T &= -(Q_0\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle Q_b\phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle \phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla\phi, \boldsymbol{\tau})_T + \langle \phi-\phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\mathbb{Q}_h\nabla\phi, \boldsymbol{\tau})_T.\end{aligned}$$

The desired property (3.1) has been proven. □

Lemma 3.2. Let $\phi \in H^1(\Omega)$. For $\boldsymbol{\tau} \in [P_1(T)]^2$ the following equation holds

$$(\nabla_w Q_h\phi, \boldsymbol{\tau})_T = (\mathbb{Q}_h\nabla\phi, \boldsymbol{\tau})_T - (\phi - \mathbb{Q}_h\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle \phi - \mathbb{Q}_h\phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}. \tag{3.2}$$

Proof . It follows from (2.4), (2.1), (2.2), (2.3) and integration by parts that

$$\begin{aligned}(\nabla_w Q_h\phi, \boldsymbol{\tau})_T &= -(Q_0\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle Q_b\phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}\\ &= -(\mathbb{Q}_h\phi, \nabla\cdot\boldsymbol{\tau})_T - (\phi-\mathbb{Q}_h\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle \phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\mathbb{Q}_h\nabla\phi, \boldsymbol{\tau})_T - (\phi-\mathbb{Q}_h\phi, \nabla\cdot\boldsymbol{\tau})_T + \langle \phi-\mathbb{Q}_h\phi, \boldsymbol{\tau}\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned}$$

The desired result is visible. □

Lemma 3.3. For all $v = \{v_0, v_b\} \in V_h^0$ and the solution u of (1.1)-(1.2), we have

$$-(\Delta u, v_0)_T = (\nabla_w Q_h u, \nabla_w v)_T + \mathcal{I}_1(u,v) - \mathcal{I}_2(u,v) - \mathcal{I}_3(u,v), \tag{3.3}$$

where

$$\begin{cases}\mathcal{I}_1(u,v) &= (u-\mathbb{Q}_h u, \nabla\cdot\nabla_w v)_T,\\ \mathcal{I}_2(u,v) &= \langle u-\mathbb{Q}_h u, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T},\\ \mathcal{I}_3(u,v) &= \langle (\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}.\end{cases}$$

Proof . Since $\sum_{T\in\mathcal{T}_h}\langle \nabla u\cdot\mathbf{n}, v_b\rangle_{\partial T} = 0$, using integration by parts, we have

$$-(\Delta u, v_0)_T = (\nabla u, \nabla v_0)_T - \langle v_0 - v_b, \nabla u\cdot\mathbf{n}\rangle_{\partial T}. \tag{3.4}$$

Letting $\boldsymbol{\tau} = \nabla_w v$ in (3.2), we get

$$\begin{aligned}(\nabla u, \nabla v_0)_T &= (\mathbb{Q}_h\nabla u, \nabla v_0)_T\\ &= -(v_0, \nabla\cdot(\mathbb{Q}_h\nabla u))_T + \langle v_0, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla_w v, \mathbb{Q}_h\nabla u)_T + \langle v_0 - v_b, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial T}\\ &= (\nabla_w Q_h u, \nabla_w v)_T + (u-\mathbb{Q}_h u, \nabla\cdot\nabla_w v)_T - \langle u-\mathbb{Q}_h u, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T}\\ &+ \langle v_0 - v_b, \mathbb{Q}_h\nabla u\cdot\mathbf{n}\rangle_{\partial T}.\end{aligned} \tag{3.5}$$

By combining (3.4) and (3.5), we can get

$$\begin{aligned}-(\Delta u, v_0)_T &= (\nabla_w Q_h u, \nabla_w v)_T + (u-\mathbb{Q}_h u, \nabla\cdot\nabla_w v)_T\\ &- \langle u-\mathbb{Q}_h u, \nabla_w v\cdot\mathbf{n}\rangle_{\partial T} - \langle (\nabla u - \mathbb{Q}_h\nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}.\end{aligned}$$

The desired result is achieved. □

In the next lemma, we provide the error equation for the error function $e_h = \{e_0, e_b\} = Q_h u - u_h$.

Lemma 3.4. Let $e_h = \{e_0, e_b\} = Q_h u - u_h$ be the error between the L^2-projection of the exact solution u and the SFWG solution u_h. Then the following error equation holds

$$(\nabla_w e_h, \nabla_w v)_{\mathcal{T}_h} = -\mathcal{I}_1(u,v) + \mathcal{I}_2(u,v) + \mathcal{I}_3(u,v). \tag{3.6}$$

Proof . By testing (1.1) by the first component v_0 of $v = \{v_0, v_b\} \in V_h^0$, we have

$$-(\Delta u, v_0)_T = (f, v_0).$$

From Lemma 3.3 we have

$$(\nabla_w Q_h u, \nabla_w v)_T = (f, v_0) - \mathcal{I}_1(u,v) + \mathcal{I}_2(u,v) + \mathcal{I}_3(u,v). \tag{3.7}$$

Subtracting (2.5) from (3.7) and summing over $T \in \mathcal{T}_h$ yields the desired result. □

In the following, we will provide error estimates for our SFWG scheme.

Lemma 3.5. Assume that $u \in H_0^{2+i}(\Omega)$, for $i = 0, 1$, be the exact solution of (1.1)-(1.2). Then for all $v \in V_h^0$,

$$|\mathcal{I}_1(u,v)| \le Ch^{1+i}\|u\|_{2+i}|\!|\!|v|\!|\!|, \tag{3.8}$$
$$|\mathcal{I}_2(u,v)| \le Ch^{1+i}\|u\|_{2+i}|\!|\!|v|\!|\!|, \tag{3.9}$$
$$|\mathcal{I}_3(u,v)| \le Ch^{1+i}\|u\|_{2+i}|\!|\!|v|\!|\!|, \quad i = 0, 1. \tag{3.10}$$

Proof . It follows from the Cauchy-Schwarz inequality and the inverse inequality that

$$\begin{aligned}
|\mathcal{I}_1(u,v)| &= \left| \sum_{T\in\mathcal{T}_h} (u - \mathbb{Q}_h u, \nabla\cdot\nabla_w v)_T \right| \\
&\le C \sum_{T\in\mathcal{T}_h} \|u - \mathbb{Q}_h u\|_T \|\nabla\cdot\nabla_w v\|_T \\
&\le C \left(\sum_{T\in\mathcal{T}_h} \|u - \mathbb{Q}_h u\|_T^2 \right)^{1/2} \cdot \left(\sum_{T\in\mathcal{T}_h} \|\nabla\cdot\nabla_w v\|_T^2 \right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i}|\!|\!|v|\!|\!|, \ \ i = 0, 1.
\end{aligned}$$

Similarly, from the trace inequality, we have

$$\begin{aligned}
|\mathcal{I}_2(u,v)| &\le \sum_{T\in\mathcal{T}_h} |\langle u - \mathbb{Q}_h u, \nabla_w v \cdot \mathbf{n}\rangle_{\partial T}| \\
&\le C \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|u - \mathbb{Q}_h u\|_{\partial T}^2 \right)^{1/2} \cdot \left(\sum_{T\in\mathcal{T}_h} h_T \|\nabla_w v\|_{\partial T}^2 \right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i},
\end{aligned}$$

and for the third inequality, we can write

$$\begin{aligned}
|\mathcal{I}_3(u,v)| &\le \sum_{T\in\mathcal{T}_h} |\langle (\nabla u - \mathbb{Q}_h \nabla u)\cdot\mathbf{n}, v_0 - v_b\rangle_{\partial T}| \\
&\le C \left(\sum_{T\in\mathcal{T}_h} h_T \|\nabla u - \mathbb{Q}_h \nabla u\|_{\partial T}^2 \right)^{1/2} \cdot \left(\sum_{T\in\mathcal{T}_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i}|\!|\!|v|\!|\!|, \ \ i = 0, 1.
\end{aligned}$$

□

Theorem 3.6. Let $u_h = \{u_0, u_b\} \in V_h$ and $u \in H_0^{2+i}(\Omega)$ for i = 0, 1 be the SFWG solution of (2.5) and the exact solution of (1.1)-(1.2), respectively. Then, there exists a constant C such that

$$|||Q_h u - u_h||| \leq Ch^{1+i}\|u\|_{2+i}, \quad i = 0, 1, \tag{3.11}$$

$$|||u - u_h||| \leq C\|u\|_2. \tag{3.12}$$

Proof . Taking $v = e_h$ in the error equation (3.6), we obtain

$$|||e_h|||^2 = (\nabla_w e_h, \nabla_w e_h)_{\mathcal{T}_h} \leq |\mathcal{I}_1(u, e_h)| + |\mathcal{I}_2(u, e_h)| + |\mathcal{I}_3(u, e_h)|.$$

From the provided estimates in Lemma 3.5, we have

$$|||Q_h u - u_h||| = |||e_h||| \leq Ch^{1+i}\|u\|_{2+i}, \quad i = 0, 1.$$

We follow the idea of (2.7) by taking $v = u - Q_h u$ and using the trace inequality, to get

$$|||u - Q_h u||| \leq Ch\|u\|_2, \tag{3.13}$$

which, together with triangle inequality, (3.12) follows. □

4 L^2-Error Estimate

Based on a duality argument, we will provide a L^2-error estimate for SFWG scheme (2.5). For this reason, the duality problem is considered to seeks $\Psi \in H_0^2$ as follows:

$$\Delta\Psi = e_0, \quad in \quad \Omega. \tag{4.1}$$

Also, we consider the H^2-regularity for (4.1) as

$$\|\Psi\|_2 \leq C\|e_0\|. \tag{4.2}$$

Theorem 4.1. With H^2-regularity (4.2), assume that $u_h = \{u_0, u_b\} \in V_h$ and $u \in H_0^{2+i}$ for $i = 0, 1$ be the solutions of (2.5) and (1.1)-(1.2), respectively. Then, there exists a constant C such that

$$\|Q_0 u - u_0\| \leq Ch^{1+i}\|u\|_{2+i}, \quad i = 0, 1, \tag{4.3}$$

$$\|u - u_0\| \leq C\|u\|_2. \tag{4.4}$$

Proof . Taking $u = \Psi$ and $v = e_h$ in (3.3), we will have

$$-(\Delta\Psi, e_0)_T = (\nabla_w Q_h \Psi, \nabla_w e_h)_T + \mathcal{I}_1(\Psi, e_h) - \mathcal{I}_2(\Psi, e_h) - \mathcal{I}_3(\Psi, e_h). \tag{4.5}$$

Testing (4.1) by the first component e_0 of the error function $e_h = \{e_0, e_b\}$, we have

$$\|e_0\|^2 = -(\Delta\Psi, e_0). \tag{4.6}$$

Summing over $T \in \mathcal{T}_h$ and then substituting (4.5) into (4.6) gives

$$\|e_0\|^2 = (\nabla_w Q_h \Psi, \nabla_w e_h)_{\mathcal{T}_h} + \mathcal{I}_1(\Psi, e_h) - \mathcal{I}_2(\Psi, e_h) - \mathcal{I}_3(\Psi, e_h).$$

Using the error equation (3.6), we can write

$$\begin{aligned} \|e_0\|^2 = &-\mathcal{I}_1(u, Q_h\Psi) + \mathcal{I}_2(u, Q_h\Psi) + \mathcal{I}_3(u, Q_h\Psi) \\ &+ \mathcal{I}_1(\Psi, e_h) - \mathcal{I}_2(\Psi, e_h) - \mathcal{I}_3(\Psi, e_h). \end{aligned} \tag{4.7}$$

In the next step, we will provide the estimate of the six terms on the right hand side of (4.7). Using the Cauchy-Schwarz inequality and the inverse inequality, we have

$$\begin{aligned}
|\mathcal{I}_1(u,Q_h\Psi)| &\le \sum_{T\in\mathcal{T}_h} |(u-\mathbb{Q}_h u, \nabla\cdot\nabla_w Q_h\Psi)_T| \\
&\le C\left(\sum_{T\in\mathcal{T}_h}\|u-\mathbb{Q}_h u\|_T^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}\|\nabla\cdot\mathbb{Q}_h\nabla\Psi\|_T^2\right)^{1/2} \\
&\le C\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|u-\mathbb{Q}_h u\|_T^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla\cdot(\mathbb{Q}_h\nabla\Psi-\nabla\Psi)\|_T^2\right)^{1/2} \\
&+ C\left(\sum_{T\in\mathcal{T}_h}h_T^{-2}\|u-\mathbb{Q}_h u\|_T^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}h_T^2\|\nabla\cdot\nabla\Psi\|_T^2\right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \;\; i=0,1.
\end{aligned} \tag{4.8}$$

For the second term, from the trace inequality, we get

$$\begin{aligned}
|\mathcal{I}_2(u,Q_h\Psi)| &\le \sum_{T\in\mathcal{T}_h} |\langle u-\mathbb{Q}_h u, \nabla_w Q_h\Psi\cdot\mathbf{n}\rangle_{\partial T}| \\
&\le \sum_{T\in\mathcal{T}_h} |\langle u-\mathbb{Q}_h u, (\mathbb{Q}_h\nabla\Psi-\nabla\Psi)\cdot\mathbf{n}\rangle_{\partial T}| \\
&+ \sum_{T\in\mathcal{T}_h} |\langle u-\mathbb{Q}_h u, \nabla\Psi\cdot\mathbf{n}\rangle_{\partial T}| \\
&\le C\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|u-\mathbb{Q}_h u\|_{\partial T}^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}h_T\|\mathbb{Q}_h\nabla\Psi-\nabla\Psi\|_{\partial T}^2\right)^{1/2} \\
&+ C\left(\sum_{T\in\mathcal{T}_h}\|u-\mathbb{Q}_h u\|_{\partial T}^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}\|\nabla\Psi\|_{\partial T}^2\right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \;\; i=0,1.
\end{aligned} \tag{4.9}$$

Similarly, from the definition of Q_b, we can write

$$\begin{aligned}
|\mathcal{I}_3(u,Q_h\Psi)| &\le \sum_{T\in\mathcal{T}_h} |\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}, Q_0\Psi-Q_b\Psi\rangle_{\partial T}| \\
&\le \sum_{T\in\mathcal{T}_h} |\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}, Q_0\Psi-\Psi\rangle_{\partial T}| \\
&+ \sum_{T\in\mathcal{T}_h} |\langle(\nabla u-\mathbb{Q}_h\nabla u)\cdot\mathbf{n}, \Psi-Q_b\Psi\rangle_{\partial T}| \\
&\le C\left(\sum_{T\in\mathcal{T}_h}h_T\|\nabla u-\mathbb{Q}_h\nabla u\|_{\partial T}^2\right)^{1/2}\cdot\left(\sum_{T\in\mathcal{T}_h}h_T^{-1}\|Q_0\Psi-\Psi\|_{\partial T}^2\right)^{1/2} \\
&\le Ch^{1+i}\|u\|_{2+i}\|\Psi\|_1,
\end{aligned}$$

which clearly, we have

$$|\mathcal{I}_3(u,Q_h\Psi)| \le Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \;\; i=0,1. \tag{4.10}$$

The estimate (3.8) and (3.11) give

$$\begin{aligned}
|\mathcal{I}_1(\Psi,e_h)| &\le \sum_{T\in\mathcal{T}_h} |(\Psi-\mathbb{Q}_h\Psi, \nabla\cdot\nabla_w e_h)_T| \\
&\le Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2,
\end{aligned} \tag{4.11}$$

and from the estimates (3.9), (3.10) and (3.11), we have

$$
\begin{aligned}
|\mathcal{I}_2(\Psi, e_h)| &\leq \sum_{T\in\mathcal{T}_h} |\langle \Psi - \mathbb{Q}_h\Psi, \nabla_w e_h \cdot \mathbf{n}\rangle_{\partial T}| \\
&\leq Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \qquad (4.12)
\end{aligned}
$$

and

$$
\begin{aligned}
|\mathcal{I}_3(\Psi, e_h)| &\leq \sum_{T\in\mathcal{T}_h} |\langle (\nabla\Psi - \mathbb{Q}_h\nabla\Psi)\cdot\mathbf{n}, e_0 - e_b\rangle_{\partial T}| \\
&\leq Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \;\; i = 0, 1. \qquad (4.13)
\end{aligned}
$$

Combining (4.7) with the estimates (4.8)-(4.13), we will have

$$\|e_0\|^2 \leq Ch^{1+i}\|u\|_{2+i}\|\Psi\|_2, \;\; i = 0, 1.$$

In the last step, from the H^2-regularity (4.2) and then dividing $\|e_0\|$, we have the following desired error estimate

$$\|Q_0 u - u_0\| \leq Ch^{1+i}\|u\|_{2+i}, \;\; i = 0, 1.$$

Using the triangle inequality, we get

$$\|u - u_0\| \leq \|u - Q_0 u\| + \|Q_0 u - u_0\| \leq C\|u\|_2.$$

□

5 Numerical Tests

Our goal in this section is to evaluate the flexibility and efficiency of our SFWG scheme (2.5). Based on the lowest-degree possible $(P_0(T), P_0(e), [P_1(T)]^2)$ elements, we will compare our new SFWG method with the standard WG method in which we consider the stabilizer parameter (1.4) with $r = 0$. The comparison process will show that the new SFWG method is faster and more economical than the standard WG method (1.3). We denote $e_h = \{e_0, e_b\} = Q_h u - u_h$ and $\epsilon_h = \{\epsilon_0, \epsilon_b\} = u - u_h$ be the error functions. Our numerical calculations are supported by MATLAB R2017a and performed on a Laptop computer with 8.0 GB memory and Intel(R) CPU @ 2.13 GHz. To achieve our goal in this section, we consider the following basic steps:

- Create a uniform rectangular and triangular mesh on the desired domain $\Omega = [0, 1]^2$,
- The exact solutions $u_1, u_2 \in H^2(\Omega) \cup H_0^2(\Omega)$ to the problem (1.1)-(1.2) are chosen as follows:

$$
\begin{aligned}
u_1(x, y) &= \sin(\pi x)\sin(\pi y), \\
u_2(x, y) &= (1 + x^2)\sin(\pi y),
\end{aligned}
$$

  where the source term $f(x, y)$ and the boundary conditions are accessible accordingly,
- For different values of the parameter h, we consider only the WG scheme (1.3) with $r = 0, \infty$ on rectangular mesh. Also, we apply the SFWG scheme (2.5) on triangular mesh.

Based on the exact solutions u_1 and u_2, the numerical outcome are reported in Tables 1-4. In Table 1 and Table 3, the errors and convergence rates for with or without lowest-degree possible $(P_0(T), P_0(e), [P_1(T)]^2)$ WG elements on a rectangular mesh are presented which indicate that our SFWG method (2.5) has $O(h^2)$ convergence rate in both energy norm and L^2-norm while the standard WG method (with stabilizer parameter) has a convergence rate of $O(h)$ in the same norms. The numerical results in these two tables show that our method is more efficient. Also, we investigated our method on a triangular mesh that the corresponding numerical results in Tables 2 and 4 confirm the theory discussed in the previous sections. Our SFWG solutions (discontinuous solutions) and the exact solutions on rectangular and triangular meshes are plotted for u_1 in Figures 2 and 3 which show the accuracy of the proposed method. In Figure 1, we compare the computation times (in seconds) between the SFWG method (2.5) and the standard WG method related to Table 1. As the number of elements increases, the computational time increases and the SFWG finite element method (2.5) is faster and more efficient than the standard WG method.

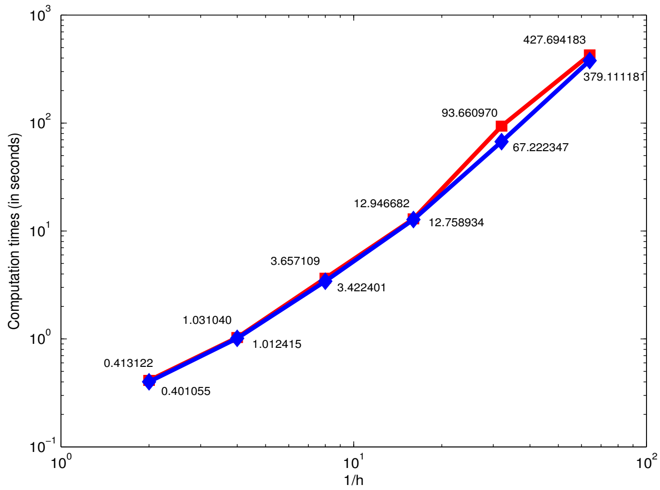

Figure 1: Comparison of computation times between the SFWG (blue color) and the WG (red color) methods.

Table 1: Errors and convergence rates on rectangular mesh for u_1.

r	$1/h$	$\|\|\|e_h\|\|\|$	Rate	$\|\|e_0\|\|$	Rate
	2	$3.8801e-01$	–	$7.1953e-02$	–
	4	$1.0969e-01$	1.82	$2.3636e-02$	1.61
∞	8	$2.8263e-02$	1.96	$6.2942e-03$	1.90
	16	$7.1191e-03$	1.99	$1.5981e-03$	1.98
	32	$1.7831e-03$	2.00	$4.0108e-04$	1.99
	64	$4.4599e-04$	2.00	$1.0037e-04$	2.00
	2	$7.5523e-01$	–	$1.2836e-01$	–
	4	$3.5278e-01$	1.10	$7.2032e-02$	0.83
0	8	$1.6049e-01$	1.13	$3.4716e-02$	1.05
	16	$7.5163e-02$	1.09	$1.6621e-02$	1.06
	32	$3.6185e-02$	1.05	$8.0769e-03$	1.04
	64	$1.7729e-02$	1.03	$3.9743e-03$	1.02

Table 2: Errors and convergence rates of the SFWG method on triangular mesh for u_1.

$1/h$	$\|\|\|\epsilon_h\|\|\|$	Rate	$\|\|\epsilon_0\|\|$	Rate
2	2.0829	–	$2.8285e-01$	–
4	2.0021	0.06	$2.7581e-01$	0.04
8	1.9777	0.02	$2.7453e-01$	0.007
16	1.9713	0.005	$2.7428e-01$	0.001
32	1.9696	0.001	$2.7422e-01$	0.0003
64	1.9692	0.0003	$2.7421e-01$	0.0001

In the last step, we adopted the exact solution from [11] as follows:

$$u_3(r,\theta) = r^{1/2}\sin(\frac{\theta}{2}),$$

where $r = \sqrt{x^2+y^2}$ and $\theta = \arctan(\frac{y}{x})$. The singularity in the chosen solution $u_3 \in H^{1+1/2}(\Omega)$ will confer different rates of convergence. Table 5 indicates that the two considered methods have the convergence of rates $O(h^{1/2})$ and $O(h^{3/2})$ in the relevant norms on rectangular mesh.

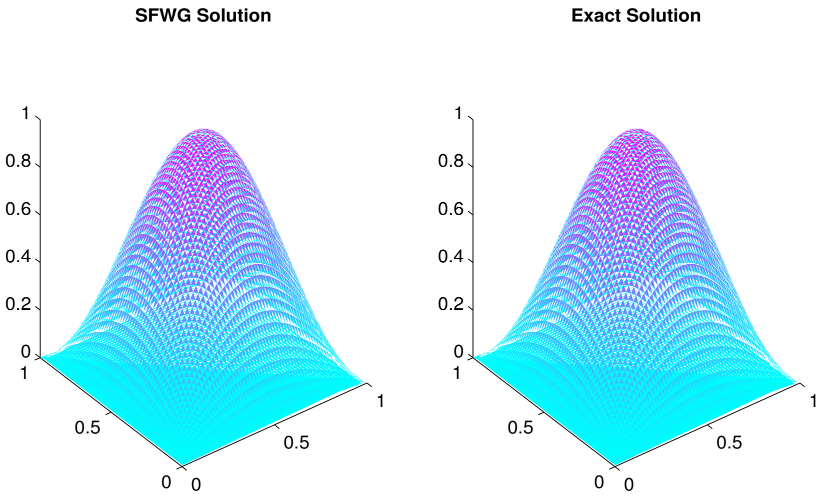

Figure 2: Numerical solutions based on u_1 on rectangular mesh with $h = 1/64$.

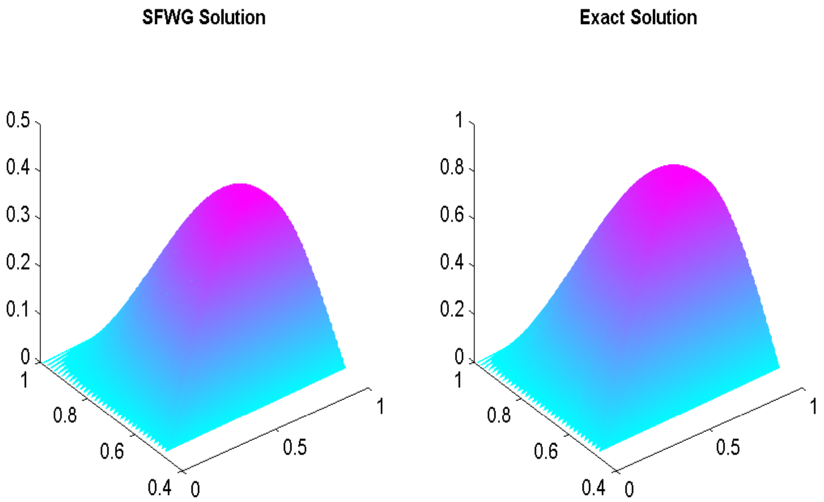

Figure 3: Numerical solutions based on u_1 on triangular mesh with $h = 1/64$.

6 Conclusion

In this paper, we proposed a new SFWG finite element method to solve the Poisson equations in which we used the lowest-degree possible $(P_0(T), P_0(e), [P_1(T)]^2)$ WG elements. Several numerical examples on two types of meshes indicated that our new method is more efficient than the standard WG finite element method (with stabilizer parameter). The numerical outcomes showed the accuracy of the claim made in the theoretical analysis section. By comparing the computational time between the two considered methods, it is shown that our numerical scheme is faster than the standard WG finite element method. Our next project is to extend theoretical and numerical analysis on higher dimensions of space.

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Table 3: Errors and convergence rates on rectangular mesh for u_2.

r	$1/h$	$\|\|\|e_h\|\|\|$	Rate	$\|\|e_0\|\|$	Rate
	2	$5.2249e-01$	$-$	$2.4020e-02$	$-$
	4	$1.2254e-01$	2.10	$1.1896e-02$	1.01
∞	8	$3.0008e-02$	2.03	$3.3323e-03$	1.84
	16	$7.4611e-03$	2.00	$8.5612e-04$	1.96
	32	$1.8627e-03$	2.00	$2.1547e-04$	1.99
	64	$4.6551e-04$	2.00	$5.3959e-05$	2.00
	2	$8.7634e-01$	$-$	$7.2637e-02$	$-$
	4	$3.0271e-01$	1.53	$4.4390e-02$	0.71
0	8	$1.2303e-01$	1.30	$2.2397e-02$	0.99
	16	$5.5353e-02$	1.15	$1.1111e-02$	1.01
	32	$2.6250e-02$	1.08	$5.5242e-03$	1.01
	64	$1.2782e-02$	1.04	$2.7527e-03$	1.00

Table 4: Errors and convergence rates of the SFWG method on triangular mesh for u_2.

$1/h$	$\|\|\|\epsilon_h\|\|\|$	Rate	$\|\|\epsilon_0\|\|$	Rate
2	2.0585	$-$	$2.3593e-01$	$-$
4	1.7773	0.21	$2.1025e-01$	0.17
8	1.6830	0.08	$2.0409e-01$	0.04
16	1.6567	0.02	$2.0258e-01$	0.01
32	1.6498	0.005	$2.0220e-01$	0.003
64	1.6481	0.001	$2.0211e-01$	0.0007

Table 5: Errors and convergence rates on rectangular mesh for u_3.

r	$1/h$	$\|\|\|e_h\|\|\|$	Rate	$\|\|e_0\|\|$	Rate
	2	$3.9892e-02$	$-$	$2.6884e-03$	$-$
	4	$2.8201e-02$	0.5	$1.3908e-03$	0.95
∞	8	$1.9943e-02$	0.5	$5.4776e-04$	1.34
	16	$1.4102e-02$	0.5	$2.0314e-04$	1.43
	32	$9.9714e-03$	0.5	$7.3502e-05$	1.47
	64	$7.0508e-03$	0.5	$2.5991e-05$	1.50
	2	$3.8598e-02$	$-$	$2.1421e-03$	$-$
	4	$2.7246e-02$	0.50	$1.2358e-03$	0.79
0	8	$1.9514e-02$	0.48	$5.1742e-04$	1.26
	16	$1.3933e-02$	0.49	$1.9756e-04$	1.39
	32	$9.9085e-03$	0.49	$7.2497e-05$	1.45
	64	$7.0384-03$	0.49	$2.5726e-05$	1.49

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