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On some topological concepts via grill

Saad S. Suliman[a,*], R. B. Esmaeel[a]

[a] *Department of Mathematics College of Education for Pure Science, Ibn Al-Haitham University of Baghdad, Iraq*

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Abstract

In this work, the new grill concepts are studied using grill topological spaces and by using some defined sets where the set α-open sets are defined. Properties of this set and some relationships were presented, in addition to studying a set of functions, including open, closed and continuous functions, finding the relationship between them and giving examples and properties that belong to this set, which will be a starting point for studying many topological properties using this set.

Keywords: Grill, α-open sets, α-closed sets, $\mathfrak{C}^*\alpha$-open function, $\mathfrak{C}^*\alpha$-c function.

1. Introduction

Choquet [1] developed the concept of a grill on a topological space, and it has proven to be a useful tool for exploring several topological problems. A grill on $\mathfrak{Q}$ is a family of non-empty subsets of a topological space ($\mathfrak{Q}$, τ). If (i) $\mathcal{A} \in \mathfrak{C}$ and $\mathcal{A} \subseteq \mathcal{B}$ so $\mathcal{B} \in \mathfrak{C}$, and (ii) $\mathcal{A}, \mathcal{B} \subseteq \mathfrak{Q}$ and $\mathcal{A} \cup \mathcal{B} \in \mathfrak{C}$, then $\mathcal{A} \in \mathfrak{C}$ or $\mathcal{B} \in \mathfrak{C}$. A triple ($\mathfrak{Q}$, τ, $\mathfrak{C}$) is said to be a grill topological space.

Roy and Mukherjee [6] used a grill to define a unique topology and researched topological ideas. For any topological space point x, ($\mathfrak{Q}$, τ), τ(q) represents a compilation of x's open neighborhoods. A mapping ϕ: P($\mathfrak{Q}$) $\rightarrow$ P($\mathfrak{Q}$) is defined as $\phi(\mathcal{A}) = \{$q$\in \mathfrak{Q}$: $\mathcal{A} \cap$ ş $\in \mathfrak{C}$ for all ş $\in \tau$(q)$\}$ for each $\mathcal{A} \in$ P($\mathfrak{Q}$). A mapping Ψ: P($\mathfrak{Q}$) $\rightarrow$ P($\mathfrak{Q}$) is defined as $\Psi(\mathcal{A}) = \mathcal{A} \cup \phi(\mathcal{A})$ for all $\mathcal{A} \in$ P($\mathfrak{Q}$). The map Ψ satisfies Kuratowski closure axioms:

i. $\Psi(\phi) = \phi$,

ii. If $\mathcal{A} \subseteq \mathcal{B}$ so $\Psi(\mathcal{A}) \subseteq \Psi$,

iii. If $\mathcal{A} \subseteq X$, so $\Psi(\Psi(\mathcal{A})) = \Psi(\mathcal{A})$,

*Corresponding author

Email address: saadsadeq05@gmail.com (Saad S. Suliman)

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iv. If $\mathcal{A}, \mathcal{B} \subseteq X$, so $\Psi(\mathcal{A} \cup \mathcal{B}) = \Psi(\mathcal{A}) \cup (\mathcal{B})$.

There are some types of a grill topological space as like a cofinite topology and discrete topology [6]. In the shape of a grill $\mathfrak{C}$ on a topological space ($\mathfrak{q}$, τ), there is a one kind of a topology. $\tau_{\mathfrak{C}}$ on $\mathfrak{q}$ a gift $\tau_{\mathfrak{C}} = \{ş \subseteq \mathfrak{q}$: Ψ ($\mathfrak{q}$- ş) = $\mathfrak{q}$-ş$\}$, for any reason $\mathcal{A} \subseteq \mathfrak{q}$, $\Psi(\mathcal{A}) = \mathcal{A} \cup \phi(\mathcal{A}) = \tau_{\mathfrak{C}}$-cl$(\mathcal{A})$ and $\tau \subseteq \tau_{\mathfrak{C}}$. We can find $\tau_{\mathfrak{C}}$ by used the base as following $\beta(\tau_{\mathfrak{C}}, \mathfrak{q}\ ) = \{V - \mathcal{A}; V \in \tau, \mathcal{A} \notin \}$ [6].

In any topological space ($\mathfrak{q}$,τ), there is a grill $\tau \subseteq \beta(\ \mathfrak{C},\tau) \subseteq \tau_{\mathfrak{C}}$, where $\beta(\ \mathfrak{C},\tau) = \{V - \mathcal{A} : V \in \tau, \mathcal{A} \notin \tau, \mathcal{A} \notin \mathfrak{C}\ \}$ is open base for $\tau_{\mathfrak{C}}$.

As an example, let ($\mathfrak{q}$,τ) be a topological space, if $\mathfrak{C}$ = P $\{\phi\}$, then, $\tau_{\mathfrak{C}} = \tau$, because for any $\tau_{\mathfrak{C}}$ basic open set V = $\mathfrak{q}$ - $\mathcal{A}$ with $u \in \tau$ and $\mathcal{A} \notin \tau_{\mathfrak{C}}$. We have $\mathcal{A} = \ \phi$, in order for V = ş $\in \tau$, so we have this case $\tau = \ \beta(\ \mathfrak{C},\tau) = \tau_{\mathfrak{C}}$. A subset $\mathcal{A}$ of a topological space $\mathfrak{q}$ is alleged to be: α-open [2] if $\mathcal{A} \subseteq$ Int(cl(Int($\mathcal{A}$))). The family of all α- open set denoted by τ_{α}.

There are many researchers who have used these combinations to obtain new generalizations [3, 4]. In this research used the symbol Int($\mathcal{A}$) to interior of the set $\mathcal{A}$ and the symbol cl($\mathcal{A}$) is the closure of $\mathcal{A}$.

2. On α-open sets in topological spaces

Definition 2.1. *The set $\mathcal{A}$ is said to be grill α-open if there exists ş $\in \tau$ such that ş-$\mathcal{A} \notin \mathfrak{C}$ and $\mathcal{A} -$ Intςl $_{\mathfrak{C}}$(ş) $\notin \mathfrak{C}$, and as indicated by $\mathfrak{C}^*\alpha$-open the complement $\mathfrak{C}^*\alpha$-open is $\mathfrak{C}^*\alpha$-closed. The set of all $\mathfrak{C}^*\alpha$-open symbolized by $\mathfrak{C}^*\alpha$ o(q) and the ensemble first and foremost $\mathfrak{C}^*\alpha$-closed shortly $\mathfrak{C}^*\alpha$ c(q).*

Example 2.2. *Let ($\mathfrak{q}$, τ, $\mathfrak{C}$) be a grill topological space, and let $\mathfrak{q}$ = $\{ q_1,q_2,q_3 \}$,τ = $\{ \mathfrak{q},\phi,\{ q_1 \},\{ q_1,q_2 \} \}$,$\mathcal{F}$ = $\{ \mathfrak{q}, \phi,\{ q_3 \},\{ q_2,q_3 \} \}$, $\mathfrak{C}$ = $\{$ ş $\subseteq \mathfrak{q}$;$q_2 \in$ ş $\}$, ϕ: P($\mathfrak{q}$) $\to$ P($\mathfrak{q}$), $\phi(\mathcal{A})$ =$\{ q \in \mathfrak{q}; \forall$ ş $\in \tau_x$; ş $\cap\mathcal{A} \in \mathfrak{C}$ $\}$, $\Psi(\mathcal{A}) = \mathcal{A} \cup \phi, \tau_{\mathfrak{C}}$ =$\{ \mathfrak{q}, \phi, \{ q_2,q_1\},\{q_3, q_2\}, \{q_2\}\}$,$\mathcal{F}_{\mathfrak{C}}$ = $\{ \mathfrak{q}, \phi,\{ q_3\},\{q_1\},\{q_2, q_3\}\}$, then $\mathfrak{C}^*\alpha$ o(q) = $\{\mathfrak{q},\phi,\{q_3\},\{q_2\},\{q_1\},\{q_2,q_3\}, \{q_1,q_2\},\{q_3,q_1\}\}$.*

Remark 2.3.

(i) Every set that is open is $\mathfrak{C}^*\alpha$-open set.

(ii) Every set that is closed is $\mathfrak{C}^*\alpha$-closed set.

Proof . *(i) Let $\mathcal{A} \in \tau$, then there exists ş $\in \mathcal{A}$ such that $u \subseteq$ Int cl$_{\mathfrak{C}}$(ş), but ş = $\mathcal{A} \in \tau$, so ş - $\mathcal{A} = \phi \notin \mathfrak{C} \wedge \mathcal{A}$ - Int cl$_{\mathfrak{C}}$ ($\mathcal{A}$) = $\phi \notin \mathfrak{C}$.*

(ii) Let $\mathcal{A}$ be a closed set. Then $\mathcal{A}^c \in \tau, \mathcal{A}^c$ is $\mathfrak{C}^*\alpha$o(q) so, $\mathcal{A}$ is $\mathfrak{C}^*\alpha c(q)$. □

The converse of Remark 2.3 is not true, see Example 2.4.

Example 2.4. *Let $\mathfrak{q}$ = $\{ q_1,q_2,q_3,q_4\}$, τ = $\{ \mathfrak{q},\phi,\{ q_3 \}, \{q_3, q_2\},\{q_2\} \}$, $\mathfrak{C}$ = P($\mathfrak{q}$) $\setminus\{\varnothing\}$. It's clear that $\{q_4, q_2 \} \in \mathfrak{C}^*\alpha$ o(Ꝃ), but $\{ q_4, q_2 \} \notin \tau$. And $\{ \{ q_1, q_3 \} \} \in \mathfrak{C}^*\alpha$ c(Ꝃ) but $\{ q_1, q_3 \}$ it is not a closed set.*

Theorem 2.5. *Every $\mathfrak{C}^*\alpha$-open set is a $\mathfrak{C}\alpha$-open.*

Proof . *Let ş-$\mathcal{A} \notin \mathfrak{C}$, so,* Intςl$_{\mathfrak{C}}$(ş) $\subseteq$ Intςl(ş) *and* $\mathcal{A} -$ Intςl(ş)$\subseteq \mathcal{A} -$ Intςl$_{\mathfrak{C}}$(ş)$\notin \mathfrak{C}$. *Let $\mathcal{A} -$ Intςl (ş) $\in \mathfrak{C}$, thus by the definition of a grill we will have $\mathcal{A} -$ Intςl $_{\mathfrak{C}}$(ş)$\in \mathfrak{C}$, a contradiction, so $\mathcal{A} -$ Intςl$_{\mathfrak{C}}$(ş)$\notin \mathfrak{C}$.* □

Proposition 2.6. *For any grill topology ($\mathcal{Q}$,τ, $\mathfrak{G}$), $\mathcal{A}$ is a $\mathfrak{G}\alpha$-open set if and only if $\mathcal{A}$ is a $\mathfrak{G}^*\alpha$-open set whenever $\mathfrak{G} = p(\mathcal{Q}) \setminus \{\varnothing\}$.*

Proof . *Let $\mathfrak{G}$ be an α-open set, then there exists $ş \in \tau$ such that $ş \subseteq \mathcal{A} \subseteq \mathrm{Int}\varsigma l\ (ş)$, so $ş$-$\mathcal{A} = \varnothing \notin \mathfrak{G}$ and $\mathcal{A} - \mathrm{Int}\varsigma l(ş) = \varnothing \notin \mathfrak{G}$. Now since $\tau = \tau_{\mathfrak{G}}$, we have $\mathcal{A} - \mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) \notin \mathfrak{G}$.*

Conversely, $ş$-$\mathcal{A} = \varnothing \wedge \mathcal{A} - \mathrm{Int}\varsigma l(\ ş\ ) \notin \mathfrak{G}(\tau) = \tau_{\mathfrak{G}}$. Since $\mathfrak{G} = p(\mathcal{Q}) \setminus \{\varnothing\}$, $u \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathrm{Int}\varsigma l$ $(ş)$. So $ş \subseteq \mathcal{A} \subseteq \mathrm{Int}\varsigma l(ş)$. □

Proposition 2.7. *Let $\mathcal{A}$ be a $\mathfrak{G}^*\alpha$-open set and $\mathcal{B} \subseteq \mathcal{Q}$ such that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathrm{Int}\varsigma l\ _{\mathfrak{G}}(\mathcal{A})$, then $\mathcal{B}$ is a $\mathfrak{G}^*\alpha$-open set.*

Proof . *Suppose that $ş$ is an open set, as a result $ş$-$\mathcal{A} \notin \mathfrak{G} \wedge \mathcal{A} - \mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) \notin \mathfrak{G}$, ($\mathcal{A}$ is $\mathfrak{G}^*\alpha$-open set). Since $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(\mathcal{A})$, $ş$-$\mathrm{Int}\varsigma l_{\mathfrak{G}}(\mathcal{A}) \notin \mathfrak{G} \subseteq ş$-$\mathcal{B} \notin \mathfrak{G} \subseteq ş$-$\mathcal{A} \notin \mathfrak{G}$. Then there exists $ş \in \tau$ such that $ş$-$\mathcal{B} \notin \mathfrak{G}$, and since $\mathcal{A} \subseteq \mathcal{B}$, $Int\varsigma l\ _{\mathfrak{G}}(\mathcal{A}) \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(\mathcal{B})$. So $ş$-$\mathrm{Int}\varsigma l_{\mathfrak{G}}(\mathcal{B}) \notin \mathfrak{G}$, hence $\mathcal{B} \in \mathfrak{G}^*\alpha$-open set.*
□

Remark 2.8. *The two concepts $\mathfrak{G}^*$ $\alpha-$ open set and $\alpha-$ open set are independent. See Examples 2.9 and 2.10.*

Example 2.9. *Let ($\mathcal{Q}$,τ, $\mathfrak{G}$) be any grill topology and $\mathcal{Q} = \{\ q_1,q_2,q_3\ \}$, $\tau = \{\ \mathcal{Q},\phi,\{\ q_1\ \}\ \}$, $\mathfrak{G} = \{\ ş;q_1 \in ş\ \}$, $\mathfrak{G}^*$ $\alpha o(\ \mathcal{Q}\ ) = P(\ \mathcal{Q}\ )$, and $\alpha o(\ \mathcal{Q}\ ) = \{\ \mathcal{Q},\phi,\{\ q_1\ \}\ \} \cup \phi$, hence it is clear that $\{q_2\} \in \mathfrak{G}^*$ $\alpha o(\ \mathcal{Q}\ )$ but $\{q_2\} \notin \alpha o(\mathcal{Q})$.*

Example 2.10. *Let (N,τ,) be any grill topology and $\tau = \{N,\phi,\ \{q_1\}\}$, $\mathfrak{G} = \{ş \subseteq N\ ;ş$ is an infinite set $\}$, it is clear that O (odd number) is an $\alpha-$ open set, but it is not a $\mathfrak{G}^*$ α open set.*

Corollary 2.11. *Suppose that ($\mathcal{Q}$,τ, $\mathfrak{G}$) is a grill topological space and $\mathcal{A}$ subset of $\mathcal{Q}$. If $\mathfrak{G} = p(q) \setminus \{\varnothing\}$. Then $\mathcal{A}$ is $\mathfrak{G}^*\alpha$-open set only if and only if $\mathcal{A}$ is α-open set.*

Proof . *Let $\mathcal{A}$ be $\mathfrak{G}^*$, so there exists $ş \in \tau$; $ş$-$\mathcal{A} \notin \mathfrak{G}$ and $\mathcal{A}$- $Int\varsigma l_{\mathfrak{G}}(\ ş\ ) \notin \mathfrak{G}$, so $ş$-$\mathcal{A} = \varnothing \notin \mathfrak{G}$ $\wedge \mathcal{A}$-$Int\varsigma l_{\mathfrak{G}}(ş\ ) = \varnothing$, $ş \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(\ ş\ )$, so $ş \subseteq \mathcal{A}$ and $\mathcal{A} \subseteq \mathrm{Int}\varsigma l(\ ş\ )$.*

Conversely, it is clear that $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(cs)$, so $u - \mathcal{A} \notin \mathfrak{G}$ and $\mathcal{A} - \mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) = \varnothing \notin \mathfrak{G}$. Then $ş$-$\mathcal{A} \notin \mathfrak{G}$ and $\mathcal{A} - \mathrm{Int}\varsigma l(ş) = \varnothing \notin \mathfrak{G}$. □

Definition 2.12. *Let ($\mathcal{Q}$,τ,$\mathfrak{G}$) be a grill topological space. A subset $\mathcal{A}$ in $\mathcal{Q}$ is called $\mathfrak{G}^*\alpha$-open if $\mathcal{A} \subseteq \Psi(\mathrm{Int}(\mathcal{A}))$.*

Proposition 2.13. *Every $\Psi\alpha$–-open is a $\mathfrak{G}^*\alpha$-open.*

Proof . *Let $\mathcal{A}$ be $\Psi\alpha$-open, so there exists $ş \in \tau$ such that $ş \subseteq \mathcal{A} \subseteq \Psi$ $(ş)$, so $ş$-$\mathcal{A} = \varnothing \wedge \mathcal{A} - \Psi(ş)$ $= \varnothing$ and $ş$-$\mathcal{A} \notin \mathfrak{G} \wedge \mathcal{A} - \Psi\ (\ ş\ ) \notin \mathfrak{G}$, so $ş$-$\mathcal{A} \notin \mathfrak{G} \wedge \mathcal{A} - \mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) \notin \mathfrak{G}$.* □

Proposition 2.14. *Every $\Psi\alpha$-open is an α-open.*

Proof . Since $\Psi\alpha$-open, there exists $ş \in \tau$ such that $ş \subseteq \mathcal{A} \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) \subseteq \mathrm{Int}\varsigma l(ş)$. Thus, $\mathrm{Int}\varsigma l_{\mathfrak{G}}(ş) \subseteq \mathrm{Int}\varsigma l(ş)$ and $ş \subseteq \mathcal{A} \subseteq \mathrm{Int}\varsigma l_{\mathfrak{G}}(\ ş\ ) \subseteq \mathrm{Int}\varsigma l(\ ş\ )$, so $ş \subseteq \mathcal{A} \subseteq \mathrm{Int}\varsigma l(ş)$. □

Theorem 2.15. *A subset $\mathcal{A}$ of a grill ($\mathcal{Q}$,τ, G)) is a $\mathfrak{G}^*\alpha$-open set if and only if there exists $ş \in \tau$ in order for $ş \subseteq \mathcal{A} \subseteq \Psi$ $(ş)$.*

Proof . If $\mathcal{A}$ is a $\mathfrak{G}^*\alpha$-open set, so $\mathcal{A} \subseteq \Psi(\mathrm{Int}(\mathcal{A}))$. Conversely, let $ş \subseteq \mathcal{A} \subseteq \Psi\ (\ ş\ )$ for $ş \in \tau$, therefore $\subseteq \mathcal{A}$. So $ş \subseteq$ $Int(\mathcal{A})$ as well as $\Psi\ (\ ş\ ) \subseteq \Psi\ (\mathrm{Int}(\mathcal{A}))$, as a result, $\mathcal{A} \subseteq \Psi\ (\mathrm{Int}(\mathcal{A}))$. □

Lemma 2.16. $\bigcup_{i\in\Lambda} \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\mathcal{A}_i)) \subseteq \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in\Lambda} \mathcal{A}_i))$.

Proof . We have $\mathcal{A}_i \subseteq \bigcup_{i\in\Lambda} \mathcal{A}_i$. Then $\varsigma l_{\mathfrak{C}}(\mathcal{A}_i) \subseteq \varsigma l_{\mathfrak{C}}(\bigcup_{i\in\Lambda} \mathcal{A}_i)$, $\mathrm{Int}(\varsigma l_{\mathfrak{C}}(\mathcal{A}_i)) \subseteq \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in\Lambda} \mathcal{A}_i)$. Thus,

$$\bigcup_{i\in\Lambda} Int(\varsigma l_{\mathfrak{C}}(\mathcal{A}_i)) \subseteq \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in\Lambda} \mathcal{A}_i).$$

□

Theorem 2.17. *The union of any family of $\mathfrak{C}$*-α open sets is a $\mathfrak{C}$*-α open set.*

Proof . For any $\mathcal{A}_i \in \mathfrak{C}^*$-$\alpha$ open set, we show that $\bigcup_{i\in} \mathcal{A}_i \in$ is a $\mathfrak{C}^*$-α open set. Since, $\mathcal{A}_i \in \mathfrak{C}^*$-$\alpha$ is a open set, there exists $\S_i \in \tau$ such that ($\S_i$-$\mathcal{A}_i$) $\notin \mathfrak{C}$ and $\mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in}\S_i) \notin \mathfrak{C}$ and since $\bigcup_{i\in}\S_i$-$\mathcal{A} \subseteq (\bigcup_{i\in}\S_i$-$\bigcup_{i\in} \mathcal{A}) \notin \mathfrak{C}$ (by Lemma 2.16). But ($\S_i$-$\mathcal{A}_i$) $\subseteq \bigcup_{i\in}\S_i$-$\mathcal{A}_i) \notin \mathfrak{C} \supset (\bigcup_{i\in}\S_i$-$\bigcup_{i\in} \mathcal{A}) \notin \mathfrak{C}$. So there exists an open set w $= \bigcup_{i\in}\S_i$ such that (w-$\bigcup_{i\in} \mathcal{A}) \notin \mathfrak{C}$. Now we prove $(\bigcup_{i\in} \mathcal{A}_i - \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in}\S_i) \notin \mathfrak{C}$. we have $\mathcal{A}_i - \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\S_i) \notin \mathfrak{C}^*$ and $\mathrm{Int}(\varsigma l_{\mathfrak{C}}(\S_i) \subseteq \bigcup_{i\in}(\mathcal{A}_i - \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\bigcup_{i\in}\S_i) \notin \mathfrak{C}$. So $(\bigcup_{i\in} \mathcal{A}_i - \bigcup_{i\in} \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\S_i)) \notin \mathfrak{C}$. Since, $\mathrm{Int}(\varsigma l_{\mathfrak{C}}(\S_i) \subseteq \mathrm{Int}(\varsigma l_{\mathfrak{C}}(\S_i)$, we have $\bigcup_{i\in} \mathcal{A}_i - \mathrm{Int}(\varsigma l)_{\mathfrak{C}}(\bigcup_{i\in}\S_i) \subseteq \bigcup_{i\in} \mathcal{A}_i - \bigcup_{i\in} \mathrm{Int}(\varsigma l)_{\mathfrak{C}}(\S_i)$ this implies $\bigcup_{i\in} \mathcal{A}_i - \mathrm{Int}(\varsigma l)_{\mathfrak{C}}(\bigcup_{i\in})\S_i) \notin \mathfrak{C}$. So $\bigcup_{i\in} \mathcal{A}_i - \mathrm{Int}(\varsigma l_{\mathfrak{C}}(w) \notin \mathfrak{C}$. Thus, $\bigcup_{i\in} \mathcal{A}_i \in \mathfrak{C}^*$-$\alpha$ open set. □

Remark 2.18. *The collection of all $\mathfrak{C}^*\alpha$-open sets is represented supra topology.*

3. Several sorts of open functions

Definition 3.1. *A function $f:(\mathcal{Q},\tau,\mathfrak{C}) \to (Y,\tau',\mathfrak{C})$ is said to be:*

(1) *$\mathfrak{C}^*\alpha$-open function, symbolize by "$\mathfrak{C}^*\alpha$-o function" if $f(\c{s}) \in \mathfrak{C}^*\alpha\ o(y)$, whenever $\c{s} \in \mathfrak{C}^*\alpha\ o(q)$.*

(2) *$\mathfrak{C}^{**}\alpha - o$ function, symbolize by "$\mathfrak{C}^{**}\alpha - o$ function" if $f(\c{s}) \in \mathfrak{C}^*\alpha o(y)$, whenever $\c{s} \in \tau$.*

(3) *$\mathfrak{C}^{***}\alpha - o$ function, symbolize by "$\mathfrak{C}^{***}\alpha - o$ function" if $f(\c{s}) \in \tau'$, when $\c{s} \in \mathfrak{C}^*\alpha\ o(q)$.*

Proposition 3.2. *let $f:(\mathcal{Q},\tau,\mathfrak{C})) \to (Y,\tau',\mathfrak{C})$ be a function*

(i) *If f is a $\mathfrak{C}^{***}\alpha - o$ function, so f is an open function.*

(ii) *If f is a $\mathfrak{C}^{***}\alpha - o$ function so f is $\mathfrak{C}^*\alpha - o$ function.*

(iii) *If f is a $\mathfrak{C}^*\alpha - o$ function so f is $\mathfrak{C}^{**}\alpha - o$ function.*

(iv) *If f is an open function so f is $\mathfrak{C}^{**}\alpha - o$ function.*

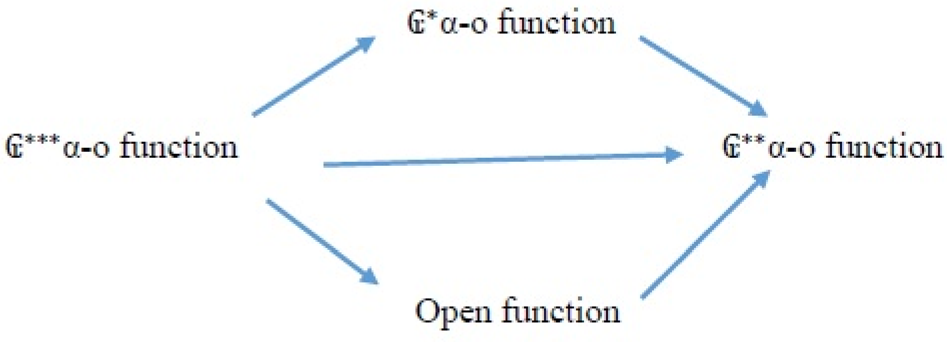
Proof . *(i) Let $\c{s} \in \tau$. By Remark 2.3(i), $\c{s} \in \mathfrak{C}^*\alpha - o(q)$. Since f is a $\mathfrak{C}^{**}\alpha-$ open, $f(\c{s})$ is open in (Y,τ'). Therefore, f is an open function.*

(ii) If $\c{s} \in \mathfrak{C}^*\alpha - o(q)$. Since f is a $\mathfrak{C}^{**}\alpha - o$ function, $f(\c{s}) \in \tau'$. By Remark 2.3(i), $f(\c{s}) \in \mathfrak{C}^*\alpha o(y)$. So f is a $\mathfrak{C}^*$"α" -open function .

(iii) Let $\c{s} \in \tau$. By Remark 2.3(i), $\c{s} \in \mathfrak{C}^*\alpha - o(q)$. because f is a $\mathfrak{C}^*\alpha - o$ function , so $f(\c{s}) \in \mathfrak{C}^*\alpha o(y)$. So f is a $\mathfrak{C}^{**}\alpha o$ function.

(iv) Suppose that $\c{s} \in \tau$. Since f is an open function, $f(\c{s}) \in \tau'$. By Remark 2.3(i), $\c{s} \in \mathfrak{C}^*\alpha o(y)$. So f is a $\mathfrak{C}^{**}\alpha o$ function. □

The following diagram, created to explain the connections that exist between numerous nations were presented in Definition 3.1.



functions via 𝔊*α-o function

Diagram 1

Example 3.3. *Let* $\mathcal{Q} = \{ q_1,q_2,q_3 \}$, $\tau = \{ \mathcal{Q},\phi,\{ q_1 \} \}$, $\mathfrak{G} = P(\mathcal{Q}) \setminus \{\phi\}$, $f :(\mathcal{Q},\tau,\mathfrak{G}) \to (\mathcal{Q},\tau,\mathfrak{G})$, $f (q) = q$, *It is clear that* f *is an open function,* $\mathfrak{G}^*\alpha o(\mathcal{Q}) = \{ \varsigma \subseteq \mathcal{Q} ; q_1 \in \varsigma \} \cup \{\phi\}$. *So there exists* $\{ q_1,q_2 \} \in \mathfrak{G}^*\alpha o(\mathcal{Q})$ *such that* $f(\{q_1,q_2\}) \notin \tau$,. *Then we observe that* f *is not* $\mathfrak{G}^{***}\alpha o$ *function.*

Example 3.4. *Let* $\mathcal{Q} = \{q_1,q_2,q_3\}$, $\tau = \{\mathcal{Q},\phi,\{q_1\},\{ q_1,q_2 \}\}$, $\mathfrak{G} = \{\varsigma \subseteq \mathcal{Q};\ q_3 \in \varsigma\}$, $\tau_{\mathfrak{G}} = P(\mathcal{Q})$, $\mathfrak{G}^*\alpha o(\mathcal{Q}) = P(\mathcal{Q})$, $f : (\mathcal{Q},\tau,\mathfrak{G}) \to (\mathcal{Q},\tau,\mathfrak{G})$, $f(q_1) = \{q_2\}$, $f(q_2) = \{q_1\}$, $f(q_3) = \{q_3\}$, *we observe that the function is a* $\mathfrak{G}^{**}\alpha - o$ *function and* $\mathfrak{G}^*\alpha$*-o function but is not open and* $\mathfrak{G}^{***}\alpha - o$ *function, because* $f(q_1) = \{q_2\} \notin \tau$.

Example 3.5. *Let* $\mathcal{Q} = \{q_1,q_2,q_3 \}$, $\tau = \{\mathcal{Q},\phi,\{q_1,q_2\}\}$, $\mathfrak{G} = P(\mathcal{Q}) \setminus \{\phi\} \cup \{q_1\}$. *Define* $f :(\mathcal{Q},\tau,\mathfrak{G}) \to (\mathcal{Q},\tau,\mathfrak{G})$ *such that* $f(q_1) = \{q_2\}$, $f(q_2) = \{q_3\}$, $f(q_3) = \{q_1\}$. *Then* $\tau_{\mathfrak{G}} = \{\mathcal{Q},\phi,\{q_2,q_1 \},\{q_3,q_2\},\{q_2\}\}$, *therefore* $\mathfrak{G}^*\alpha o\ (q) = \{\mathcal{Q},\phi,\{q_2\},\ \{q_3,q_2\},\{q_2,\ q_1,\{q_1\}\}$. *So,* f *is a* $\mathfrak{G}^{**}\alpha - o$ *function, but* f *is not a* $\mathfrak{G}^{**}\alpha - o$ *function, because there exist* $\{q_3,q_2\} \in \mathfrak{G}^{**}\alpha - o(q)$, *but* $f(\{q_3,q_2\}) = \{q_1,q_3\} \notin \mathfrak{G}^{**}\alpha - o(\mathcal{Q})$.

Definition 3.6. *The function* $f :(\mathcal{Q},\tau,\mathfrak{G}) \to (Y,\tau',\mathfrak{G})$ *is called*

(i) $\mathfrak{G}^*\alpha$*-closed function, symbolize by "*$\mathfrak{G}^*\alpha$*-c function" if* $f(\varsigma) \in \mathfrak{G}^*\alpha\ c(y)$, *whenever* $\varsigma \in \mathfrak{G}^*\alpha c$.

(ii) $\mathfrak{G}^{**}\alpha$*-closed, symbolize by "*$\mathfrak{G}^*\alpha$*-c function" if* $f(\varsigma) \in \mathfrak{G}^*\alpha\ c(y)$, *whenever* ς *is a closed set in* $(\mathcal{Q},\tau)$.

(iii) $\mathfrak{G}^{***}\alpha$*-closed function, symbolize by "*$\mathfrak{G}^{***}\alpha$*-c" if* $f(\varsigma)$ *is closed set in* $(Y,\tau',)$ *whenever* $\varsigma \in \mathfrak{G}^*\alpha c(q)$.

Proposition 3.7. *Let* $f :(\mathcal{Q},\tau,\ \mathfrak{G}) \to (Y,\tau',\mathfrak{G})$ *be a function, then*

(i) f *is an closed function, when* f *is a* $\mathfrak{G}^{***}\alpha - c$ *function.*
(ii) f *is* $\mathfrak{G}^*\alpha - c$ *function, when* f *is a* $\mathfrak{G}^{***}\alpha - c$ *function.*
(iii) f *is* $\mathfrak{G}^{***}\alpha - c$ *function when* f *is a* $\mathfrak{G}^*\alpha - c$ *function.*
(iv) f *is* $\mathfrak{G}^{**}\alpha - c$ *function when* f *is a closed function.*

Proof . By Remark 2.3(i) and Definition 3.6 the prove holds. □ The inverse of Proposition 3.7 is not true. See Examples 3.3 and 3.4.

Remark 3.8. *When f is onto so:*

(i) $\mathfrak{C}^*\alpha$*-open and* $\mathfrak{C}^*\alpha$*-closed functions are identical.*

(ii) $\mathfrak{C}^*\alpha$*-open and* $\mathfrak{C}^{**}\alpha$*-closed functions are identical.*

(iii) $\mathfrak{C}^{***}\alpha$*-open and* $\mathfrak{C}^{***}\alpha$*-closed functions are identical.*

Proof . *Considering that f is an onto function together with Definitions 3.1 3.6) prove the above statments.* □

The following diagram explain the ties that bind these two types of closed functions

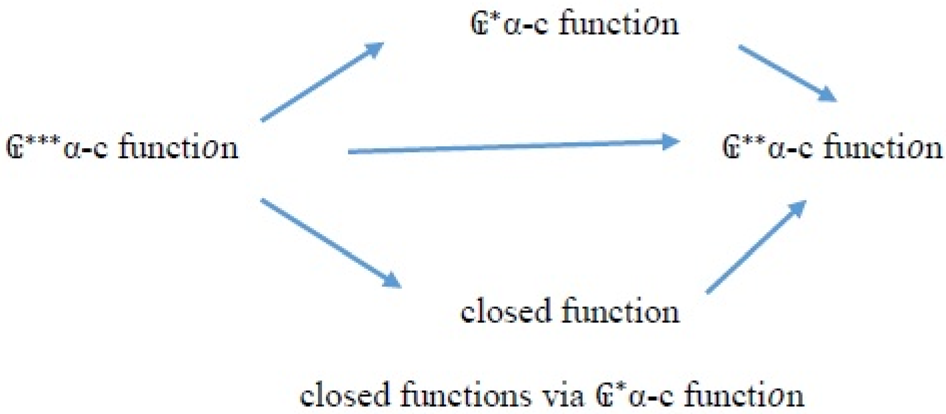


Diagram 2

4. Some types of continuous functions

In the following, new type of continuous functions will present their definitions and the relationships between those functions will explain.

Definition 4.1. *The function* $f :(\mathcal{Q},\tau,\mathfrak{C}) \to (\mathcal{Q},\tau,\mathfrak{C})$ *is called*

(1) $\mathfrak{C}^*\alpha$*-continues function, shortly "*$\mathfrak{C}^*\alpha$*-continues function, if* $f^{-1}(ş) \in \mathfrak{C}^*\alpha o(\mathcal{Q})$*, for all* $ş \in \tau$.

(2) strongly $\mathfrak{C}^*\alpha$*-continues function shortly strongly* $\mathfrak{C}^*\alpha$*-continuous function, if* $f^{-1}(ş) \in \tau$*, for every* $ş \in \mathfrak{C}^*\alpha o(Y)$.

(3) $\mathfrak{C}^*\alpha$*-irresolute function, shortly* $\mathfrak{C}^*\alpha$*-irresolute function, if* $f^{-1}(ş) \in \mathfrak{C}^*\alpha o(\mathcal{Q})$*, for every* $ş \in \mathfrak{C}^*\alpha o(Y)$.

Proposition 4.2. *Let* $f :(\mathcal{Q},\tau,\mathfrak{C}) \to (\mathcal{Q},\tau,\mathfrak{C})$ *be a function. Then*

(1) f α *is* α $\mathfrak{C}^*(\mathcal{Q})\alpha-$*irresolute function, when* f *is strongly-*$\mathfrak{C}^*(\mathcal{Q})\alpha-$*continuous function.*

(2) f *is continuous function, when* f *is strongly-*$\mathfrak{C}^*(\mathcal{Q})\alpha-$*continuous function.*

(3) f *is* $\mathfrak{G}^*(\mathcal{Q})\alpha-$*continuous function, when* f *is continuous function.*

(4) if f *is a* $\mathfrak{G}^*(\mathcal{Q})\alpha-$*irresolute function, then* f *is* $\mathfrak{G}^*(\mathcal{Q})\alpha-$*continuous function.*

Proof . *(1) Let* ş $\in \mathfrak{G}^*\alpha o(Y)$*, since* f *is a strongly* $\mathfrak{G}^*(\mathcal{Q})\alpha$*-continuous function,* $f^{-1}($ ş $) \in \tau$*. By Remark 3.8,* $f^{-1}($ ş $) \in \mathfrak{G}^*\alpha o(\mathcal{Q})$*, this implies that* f *is* $\mathfrak{G}^*(\mathcal{Q})\alpha-$*irresolute function.*

(2) Let ş $\in \tau$*. By Remark 2.3(i),* ş $\in \mathfrak{G}^*\alpha o(Y)$*. Since* f *is strongly-*$\mathfrak{G}^*(\mathcal{Q})\alpha-$*continuous function,* f^{-1}(ş) *is an open set in* $(\mathcal{Q},\tau)$*. This implies that* f *is a continues function.*

(3)Let ş $\in \tau$*. Since* f *is a continues function,* f^{-1}(ş) *is an open set in* $(\mathcal{Q},\tau)$*. By Remark 2.3(i)* f^{-1}(ş) $\in \mathfrak{G}^*\alpha o(\mathcal{Q})$*, so,* f *iα* $\mathfrak{G}^*(\mathcal{Q})\alpha-$*continuous function.*

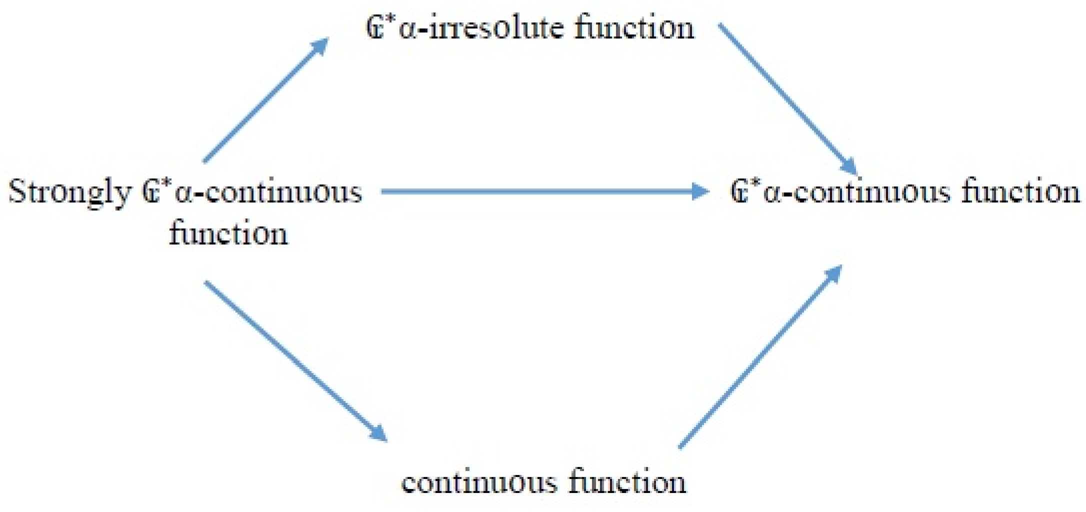
(4) Let ş $\in \tau$*. By Remark 2.3(i),* ş $\in \mathfrak{G}^*\alpha o(Y)$*. Since* f *is* $\mathfrak{G}^*\alpha-$*irresolute function,* f^{-1}(ş)$\in \mathfrak{G}^*\alpha o(\mathcal{Q})$*, αo* f *iα* $\mathfrak{G}^*(\mathcal{Q})\alpha--$*continuous function.* □

The inverse of Proposition 4.2 is not true in general.

Example 4.3. *The function* $f:(\mathcal{Q},\tau,\mathfrak{G}) \to (\mathcal{Q},\tau,\mathfrak{G}'')$ *such that* $f(q) = q$*, for each* $q \in \mathcal{Q}$*, where* $\mathcal{Q} = \{q_1,q_2,q_3\}$*,* $\tau = \{\mathcal{Q},\phi,\{q_1\}\}$*,* $\mathfrak{G} = p(\mathcal{Q})\backslash\{\phi\}$*,* $\mathfrak{G}'' = \{$ş$;q_1\in$ş$\}$*,* $\mathfrak{G}^*\alpha O(\mathcal{Q}) = \{$ş$:\ q_1 \in$ ş$\} \cup\{\phi\}$*,* $\mathfrak{G}''^*\alpha O(\mathcal{Q}) = p(\mathcal{Q})$*, so that,* f *is* $\mathfrak{G}^*\alpha O(\mathcal{Q})$ *continuous function and continuous function but it is not irresolute and not strongly because, there exists* $\{q_2,q_3\}\in\mathfrak{G}''^*\alpha O(\mathcal{Q})$*,* $f^{-1}\{q_2,q_3\} = \{q_2,q_3\}\notin \mathfrak{G}''$*.*

Example 4.4. *Consider the function* $f:(\mathcal{Q},\tau,\mathfrak{G}) \to (\mathcal{Q},\tau,\mathfrak{G}'')$ *such that* $f(\{q_1\}) = \{q_2\}$ $f(\{q_2\}) = \{q_1\}$*,* $f(\{q_3\}) = \{q_3\}$*, where* $\mathcal{Q} = \{q_3,q_2,q_1\}$*,* $\tau = \{\mathcal{Q},\phi,\{q_1\}\}$*,* $\mathfrak{G}''= P(\mathcal{Q})\backslash\{\phi\}$*,* $\mathfrak{G} =\{$ ş$;\ q_1\in$ş$\}$*,* $\mathfrak{G}^*\alpha O(\mathcal{Q}) = P(q)$*,* $\mathfrak{G}''^*\alpha O(\mathcal{Q}) = \{$ş$;q_1\in$ ş$\} \cup\{\phi\}$*. Then* f *is* $\mathfrak{G}^*\alpha O(\mathcal{Q})$ *continuous function and irresolute function but it is not continues and not strongly function since* $f^{-1}(q_1) = \{q_2\}\notin \tau$*.*

The following diagram, explains the relations between the concept in Definition 4.1.



Continuity via 𝔊*α-open set

Diagram 3

References

[1] G. Choquet, *Sur les notions de filtre et grille*, Compt. Rend. Acad. Sci. Paris 224 (1947) 171–173.

[2] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer Math. Month. 70 (1963) 36–41.

[3] M.O. Mustafa and R.B. Esmaeel, *Separation axioms with grill-toplogical open set*, J. Phys. Conf. Ser. 1879(2) (2021) 022107.

[4] M.O. Mustafa and R.B. Esmaeel, *Some properties in grill-topological open and closed sets*, J. Phys. Conf. Ser. 1897(1) (2021), 012038.

[5] A.I. Nasir and R.B. Esmaeel, *On α-J-space On -J-space*, JARDCS 11(1) (2019) 1379–1382.

[6] B. Roy and M.N. Mukherjee, *On a typical topology induced by a grill*, Soochow J. Math. 33(4) (2007) 771–786.

[7] S.G. Saeed and R.B. Esmaeel, *Separation axioms via αg-open set*, J. Phys. Conf. Ser. 1591(1) (2020) 012099.

[8] D. Saravanakumar and N. Kalaivani, *On grill Sp-open set in grill topological spaces*, J. New Theory 23 (2018) 85–92.