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Multi-wavelet Bessel sequences in Sobolev spaces in $L^2(\mathbb{K})$

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Abstract

This paper is devoted to the study of some properties of multiwavelet Bessel sequences in Sobolev spaces over local fields of positive characteristics.

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1 Introduction

The wavelet transform is a simple mathematical tool that cuts up data or functions into different frequency components, and then studies each components with a resolution matched to its scale. The main feature of the wavelet transform is to hierarchically decompose general functions, as a signal or a process, into a set of approximation functions with different scales. One of the important factor behind the stable decomposition of a signal for analysis or transmission is related to the type of representation used for its spanning set (representation system). A careful choice of the spanning set enables us to solve a variety of analysis tasks. During the last two decades, many researchers have contributed in the designing and time-frequency analysis of these representation systems for the various spaces, namely, finite and infinite abelian groups, Euclidean spaces, locally compact abelian groups.

On the other hand, there is a substantial body of work that has been concerned with the construction of wavelets and frames on local fields. For example, R. L. Benedetto and J. J. Benedetto [5] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets and only allows the construction of wavelet functions whose Fourier transforms are characteristic functions of some sets. Ahmad and his collaborators in the series of papers investigated frame theory on local fields and obtained various interesting results [4, 3, 2, 11, 12, 13, 14].

The paper is structured as follows. In section 2, we discuss the preliminaries on local fields, definition of Sobolov spaces and also discuss some auxiliary results about Bessel sequences. In Section 3, we provide the complete characterization of multiwavelet Bessel wavelet sequences in Sobolev spaces over local fields.

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2 Fourier Analysis on Non-Archemedian Fields

A local field $\mathbb{K}$ is a locally compact, non-discrete and totally disconnected field. If it is of characteristic zero, then it is a field of p-adic numbers $\mathbb{Q}_p$ or its finite extension. If $\mathbb{K}$ is of positive characteristic, then $\mathbb{K}$ is a field of formal Laurent series over a finite field $GF(p^c)$. If $c = 1$, it is a p-series field, while for $c \neq 1$, it is an algebraic extension of degree c of a p-series field. Let $\mathbb{K}$ be a fixed local field with the ring of integers $\mathfrak{D} = \{x \in \mathbb{K} : |x| \leq 1\}$. Since $\mathbb{K}^+$ is a locally compact abelian group, we choose a Haar measure dx for $\mathbb{K}^+$. The field $\mathbb{K}$ is locally compact, non-trivial, totally disconnected and complete topological field endowed with non–Archimedean norm $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ satisfying

(a) $|x| = 0$ if and only if $x = 0$;

(b) $|x\,y| = |x||y|$ for all $x, y \in \mathbb{K}$;

(c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in \mathbb{K}$.

Property (c) is called the ultrametric inequality. Let $\mathfrak{B} = \{x \in \mathbb{K} : |x| < 1\}$ be the prime ideal of the ring of integers $\mathfrak{D}$ in $\mathbb{K}$. Then, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $GF(q)$, where $q = p^c$ for some prime p and $c \in \mathbb{N}$. Since $\mathbb{K}$ is totally disconnected and $\mathfrak{B}$ is both prime and principal ideal, so there exist a prime element $\mathfrak{p}$ of $\mathbb{K}$ such that $\mathfrak{B} = \langle \mathfrak{p} \rangle = \mathfrak{p}\mathfrak{D}$. Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in \mathbb{K} : |x| = 1\}$. Clearly, $\mathfrak{D}^*$ is a group of units in $\mathbb{K}^*$ and if $x \neq 0$, then can write $x = \mathfrak{p}^n y, y \in \mathfrak{D}^*$. Moreover, if $\mathcal{U} = \{a_m : m = 0, 1, \ldots, q-1\}$ denotes the fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$, then every element $x \in \mathbb{K}$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell\, \mathfrak{p}^\ell$ with $c_\ell \in \mathcal{U}$. Recall that $\mathfrak{B}$ is compact and open, so each fractional ideal $\mathfrak{B}^k = \mathfrak{p}^k\mathfrak{D} = \left\{x \in \mathbb{K} : |x| < q^{-k}\right\}$ is also compact and open and is a subgroup of K^+. We use the notation in Taibleson's book [15]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural, non-negative integers and integers, respectively.

Let χ be a fixed character on $\mathbb{K}^+$ that is trivial on $\mathfrak{D}$ but non-trivial on $\mathfrak{B}^{-1}$. Therefore, χ is constant on cosets of $\mathfrak{D}$ so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(y, x), x \in \mathbb{K}$. Suppose that χ_u is any character on $\mathbb{K}^+$, then the restriction $\chi_u|\mathfrak{D}$ is a character on $\mathfrak{D}$. Moreover, as characters on $\mathfrak{D}, \chi_u = \chi_v$ if and only if $u - v \in \mathfrak{D}$. Hence, if $\{u(n) : n \in \mathbb{N}_0\}$ is a complete list of distinct coset representative of $\mathfrak{D}$ in $\mathbb{K}^+$, then, as it was proved in [15], the set $\left\{\chi_{u(n)} : n \in \mathbb{N}_0\right\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong GF(q)$ where $GF(q)$ is a c-dimensional vector space over the field $GF(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that $\text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q)$. For $n \in \mathbb{N}_0$ satisfying

$$0 \leq n < q, \quad n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \quad 0 \leq a_k < p, \quad \text{and } k = 0, 1, \ldots, c-1,$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\,\mathfrak{p}^{-1}. \tag{2.1}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,\ n \in \mathbb{N}_0,\ 0 \leq b_k < q, k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{p}^{-1} + \cdots + u(b_s)\mathfrak{p}^{-s}. \tag{2.2}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. But, if $r, k \in \mathbb{N}_0$ and $0 \leq s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{p}^{-k} + u(s)$. Further, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k) : k \in \mathbb{N}_0\} = \{u(k) : k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter we use the notation $\chi_n = \chi_{u(n)}$, $n \geq 0$.

Let the local field K be of characteristic $p > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi(\zeta_\mu \mathfrak{p}^{-j}) = \begin{cases} exp(2\pi i/p), & \mu = 0 \ \ and \ \ j = 1 \\ 1 \ \ \mu = 1 \cdots c-1 \ \ or \ j \neq 1 \end{cases} \tag{2.3}$$

2.1 Fourier Transforms on Local Fields

The Fourier transform of $f \in L^1(K)$ is denoted by $\hat{f}(\xi)$ and defined by

$$\mathcal{F}\big\{f(x)\big\} = \hat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\,dx. \tag{2.4}$$

It is noted that

$$\hat{f}(\xi) = \int_K f(x)\,\overline{\chi_\xi(x)}dx = \int_K f(x)\chi(-\xi x)\,dx.$$

The properties of Fourier transforms on local field K are much similar to those of on the classical field $\mathbb{R}$. In fact, the Fourier transform on local fields of positive characteristic have the following properties:

- The map $f \to \hat{f}$ is a bounded linear transformation of $L^1(K)$ into $L^\infty(K)$, and $\|\hat{f}\|_\infty \leq \|f\|_1$.
- If $f \in L^1(K)$, then $\hat{f}$ is uniformly continuous.
- If $f \in L^1(K) \cap L^2(\mathbb{K})$, then $\|\hat{f}\|_2 = \|f\|_2$.

The Fourier transform of a function $f \in L^2(\mathbb{K})$ is defined by

$$\hat{f}(\xi) = \lim_{k\to\infty} \hat{f}_k(\xi) = \lim_{k\to\infty} \int_{|x|\leq q^k} f(x)\overline{\chi_\xi(x)}\, dx, \tag{2.5}$$

where $f_k = f\,\mathbf{\Phi}_{-k}$ and $\mathbf{\Phi}_k$ is the characteristic function of $\mathfrak{B}^k$. Furthermore, if $f \in L^2(\mathfrak{D})$, then we define the Fourier coefficients of f as

$$\hat{f}\big(u(n)\big) = \int_{\mathfrak{D}} f(x)\overline{\chi_{u(n)}(x)}\, dx. \tag{2.6}$$

The series $\sum_{n\in\mathbb{N}_0} \hat{f}\big(u(n)\big)\chi_{u(n)}(x)$ is called the Fourier series of f. From the standard L^2-theory for compact abelian groups, we conclude that the Fourier series of f converges to f in $L^2(\mathfrak{D})$ and Parseval's identity holds:

$$\|f\|_2^2 = \int_{\mathfrak{D}} \big|f(x)\big|^2 dx = \sum_{n\in\mathbb{N}_0} \Big|\hat{f}\big(u(n)\big)\Big|^2. \tag{2.7}$$

For $s \in \mathbb{K}$, the Sobolev space $\mathbb{H}^s(\mathbb{K})$ consists of all distributions f such that

$$\|f\|^2_{\mathbb{H}^s(\mathbb{K})} = \int_{\mathbb{K}} |\widehat{f}(\zeta)|^2(1+\|\zeta\|_2^2)^s d\zeta < \infty,$$

where $\|.\|_2$ denotes the Euclidean norm on $\mathbb{K}$. It is noted that, $H^s(\mathbb{K})$ is a separable Hilbert space under the definition of the inner product

$$\langle f, g\rangle_{\mathbb{H}^s(\mathbb{K})} = \int_{\mathbb{K}} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)}(1+\|\zeta\|_2^2)d\zeta, \quad f,\ g \in \mathbb{H}^s(\mathbb{K}).$$

Obviously, $\mathbb{H}(\mathbb{K}) = L^2(\mathbb{K})$ and $\mathbb{H}^{s_1} \subseteq \mathbb{H}^{s_2}(\mathbb{K})$ iff $s_1 \geq s_2$. Furthermore, for every $g \in \mathbb{H}^{-s}(\mathbb{K})$,

$$\langle f, g\rangle = \int_{\mathbb{K}} \widehat{f}(\zeta)\overline{\widehat{g}(\zeta)}d\zeta, \quad f \in \mathbb{H}^s(\mathbb{K})$$

gives a continuous functional on $\mathbb{H}^s(\mathbb{K})$.

For $f, g : \mathbb{K} \to \mathbb{C}$, we define

$$[f, g]_t(\xi) = \sum_{k\in\mathbb{N}_0} f(\xi + u(k))\overline{g(\xi + u(k))}(1+\|.+u(k)\|_2^2)^t, \quad t \in \mathbb{K}.$$

By $\Gamma_{\mathfrak{p}}$ a full set of $q\mathbb{N}_0/\mathbb{N}_0$, i.e a set of representatives of distinct cosets of $q\mathbb{N}_0/\mathbb{N}_0$. We write

$$f_{j,k}(\xi) = q^{\frac{jd}{2}} f(\mathfrak{p}^j\xi - u(k)) \quad \text{and} \quad f^s_{j,k}(\xi) = q^{-js} f_{j,k}(\xi) = q^{j(\frac{d}{2}-s)} f(\mathfrak{p}^j\xi - u(k))$$

for a distribution $f, j \in \mathbb{Z}$, $\ k \in \mathbb{N}_0$ and $s \in \mathbb{K}$.

Given $r \in \mathbb{N}$, let $\varphi = (\varphi_1, \varphi_2, ..., \varphi_r)^T \in (\mathbb{H}^s(\mathbb{K}))^r$ be an M-refinable function vector satisfying the refinement equation, i.e., there exists an $r \times r$ order matrix $\widehat{a}$, called refinement mask symbol such that

$$\widehat{\varphi}(\mathfrak{p}\xi) = \widehat{a}^l(\xi)\widehat{\varphi}(\xi) \quad a.e\ \mathbb{K}. \tag{2.8}$$

Given $L \in \mathbb{N}$, wavelet function vectors $\psi_l = (\psi_1^l, \psi_2^l, ..., \psi_r^l)^T$ with $l = 1, 2, 3, ..., L$ are defined by

$$\widehat{\psi}(\mathfrak{p}\xi) = \widehat{b}^l(\xi)\widehat{\varphi}(\xi) \quad l = 1, 2, ..., L \tag{2.9}$$

where $\widehat{b}^l(\xi) = (\widehat{b}^l_{n,m}(\xi))^r_{n,m}$ with $l = 1, 2, ..., L$ being a sequence of $r \times r$ order matrices of $\mathbb{N}_0$-periodic measurable functions on $\mathbb{K}$ called wavelet masks symbol. Define a multi-wavelet system

$$\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_l) = \{\varphi_{n;0,k} : n = 1, 2, ..., r; k \in \mathbb{N}_0\} \cup$$

$$\{\psi^{l,s}_{n;j,k} : n = 1, 2, ..., r; j \in \mathbb{N}_0,\ k \in \mathbb{N}_0,\quad l = 1, 2, ..., L\}. \tag{2.10}$$

$\mathcal{W}^s(\varphi; \psi_1, \psi_2, ...\psi_L)$ is called a multi-wavelet Bessel sequence (MWBS) in $\mathbb{H}^s(\mathbb{K})$ if there exists $B > 0$ such that

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{n;0,k}\rangle_{\mathbb{H}^s\mathbb{K}}|^2 + \sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle f, \psi^{l,s}_{n;j,k}\rangle_{\mathbb{H}^s(\mathbb{K})}|^2 \leq B\|f\|^2_{\mathbb{H}^s(\mathbb{K})}, \quad \forall \quad f \in \mathbb{H}^s(\mathbb{K}),$$

where B is called a bessel bound; it is called a multi-wavelet frame (MWF) in $\mathbb{H}^s(\mathbb{K})$ if there exist $0 < A \leq B < \infty$ such that

$$A\|f\|_{\mathbb{H}^s(\mathbb{K})} \leq \sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{n;0,k}\rangle_{\mathbb{H}^s\mathbb{K}}|^2 + \sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle f, \psi^{l,s}_{n;j,k}\rangle_{\mathbb{H}^s(\mathbb{K})}|^2$$
$$\leq B\|f\|^2_{\mathbb{H}^s(\mathbb{K})}, \quad \forall \quad f \in \mathbb{H}^s(\mathbb{K}),$$

where A and B are called frame bounds.

3 Multi Wavelet Bessel Sequences in Sobolev Spaces over Local Fields

In this section, we provide some necessary lemmas which are used for later. By a standard argument, we have

Lemma 3.1. Let $s \in \mathbb{K}$, define λ by

$$\widehat{\lambda f}(\xi) = (1 + \|.\|_2^2)^{s/2}\widehat{f}(\xi)$$

for $f \in \mathbb{H}^s(\mathbb{K})$ or $L^2(\mathbb{K})$. Then λ is a unitary operator both from $\mathbb{H}^s(\mathbb{K})$ onto $L^2(\mathbb{K})$ and $L^2(\mathbb{K})$ onto $\mathbb{H}^{-s}(\mathbb{K})$

Lemma 3.2. Let $s \in \mathbb{K}$ and $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_L)$ is a $MWBS$ in $\mathbb{H}^s(\mathbb{K})$ with Bessel bound B if and only if

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle f, \varphi_{n;0,k}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle f, \psi^{\ell,s}_{n;j,k}\rangle|^2 \leq B\|f\|^2_{\mathbb{H}^{-s}(\mathbb{K})} \text{ for } \ f \in \mathbb{H}^{-s}(\mathbb{K}). \tag{3.1}$$

Proof. By Lemma 3.1, we know that $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_L)$ is a MWBS in $\mathbb{H}^s(\mathbb{K})$ with Bessel bound B if and only if

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle f, \lambda\varphi_{n;0,k}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle f, \lambda\psi^{\ell,s}_{n;j,k}\rangle|^2 \leq B\|f\|^2_{\mathbb{H}^{-s}(\mathbb{K})} \quad \text{for} \quad f \in L^2(\mathbb{K}) \tag{3.2}$$

Since λ is a unitary operator, we have

$$\langle f, \lambda\varphi_{n;0,k}\rangle = \langle \lambda f, \varphi_{n;0,k} \quad \text{and} \quad \langle f, \lambda\psi^{\ell,s}_{n;j,k} = \langle \lambda f, \psi^{\ell,s}_{n;j,k}\rangle,$$

and

$$\|f\|^2_{L^2(\mathbb{K})} = \|\lambda f\|_{\mathbb{H}^{-s}(\mathbb{K})}.$$

It follows that (12) is equivalent to

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle f, \lambda\varphi^n_{0,k}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle f, \lambda\psi^{\ell,s}_{n;j,k}\rangle|^2 \leq B\|\lambda f\|^2_{\mathbb{H}^{-s}(\mathbb{K})} \quad \text{for} \quad f \in L^2(\mathbb{K}). \tag{3.3}$$

This leads to the Lemma since λ is a unitary operator from $L^2(\mathbb{K})$ to $\mathbb{H}^{-s}(\mathbb{K})$ by Lemma 3.1. □

Lemma 3.3. Let $0 \neq s \in \mathbb{K}$ and $\varphi = (\varphi_1, \varphi_2, ..., \varphi_r)^T \in (H^s(\mathbb{K}))^r$. If $[\widehat{\varphi}_n, \widehat{\varphi}_n]_t \in L^\infty(\mathbb{K})$ for some $t > s$ with $n = 1, 2, ..., r$, then

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{n;0,k}\rangle|^2 \leq \sum_{n=1}^{r} \|[\widehat{\varphi}_n, \widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})} \tag{3.4}$$

for $g \in H^{-s}(\mathbb{K})$.

Proof . Since for any $n \in \{1,2,...,r\}$ $\varphi_n \in H^s(\mathbb{K})$ and $g \in H^{-s}(\mathbb{K})$, we have $\widehat{g}\overline{\widehat{\varphi_n}} \in L^2(\mathbb{K})$. Applying the Plancheral theorem and the Parseval identity, by a simple computation we have

$$\begin{aligned}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_n(\xi-u(k))\rangle|^2 &= \sum_{k\in\mathbb{N}_0}\left|\int_{\mathbb{K}} \widehat{g}(\xi)\overline{\widehat{\varphi}(\xi)}\chi_k(\xi)d\xi\right|^2\\ &= \sum_{k\in\mathbb{N}_0}\left|\sum_{k'\in\mathbb{N}_0}\int_{\mathfrak{D}} \widehat{g}(\xi+u(k'))\overline{\widehat{\varphi}(\xi+u(k'))}\chi_k(\xi)d\xi\right|^2\\ &= \int_{\mathfrak{D}}\left|\sum_{k\in\mathbb{N}_0}\widehat{g}(\xi+u(k'))\overline{\widehat{\varphi}(\xi+u(k'))}d\xi\right|^2\\ &= \int_{\mathfrak{D}} |[\widehat{g},\widehat{\varphi}_n]_o(\xi)|^2 d\xi \end{aligned} \tag{3.5}$$

By the Cauchy Schwarz's inequality, we have $|[\widehat{g},\widehat{\varphi}_n]_0(\xi)|^2 \leq [\widehat{g},\widehat{g}]_{-s}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s(\xi)$ for almost every $\xi \in \mathbb{K}$. Since $t > s$ and $[\widehat{\varphi},\widehat{\varphi}]_t \in L^\infty(\mathbb{K})$, it follows that

$$[\widehat{\varphi},\widehat{\varphi}]_s(\xi) \leq [\widehat{\varphi},\widehat{\varphi}]_t(\xi).$$

Therefore, $[\widehat{\varphi},\widehat{\varphi}]_s \in L^\infty(\mathbb{K})$ and thus we deduce from (3.5) that

$$\begin{aligned}\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_n(\xi-u(k))\rangle|^2 &\leq \sum_{n=1}^{r}\int_{\mathfrak{D}} [\widehat{g},\widehat{g}]_{-s}(\xi)[\widehat{\varphi},\widehat{\varphi}]_s(\xi)d\xi\\ &\leq \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\int_{\mathfrak{D}} [\widehat{g},\widehat{g}]_{-s}(\xi)d\xi\\ &= \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}} |\widehat{g}(\xi)|^2(1+\|\xi\|_2^2)^{-s}d\xi\\ &= \sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}\|g\|^2_{H^{-s}(\mathbb{K})}. \end{aligned} \tag{3.6}$$

□

Lemma 3.4. Let $0 \neq s < t$, and $\widehat{b}^\ell(\xi) = \left(\widehat{b}^\ell_{n,m}(\xi)\right)^r_{n,m=1}$, $\ell = 1,2,...,L$ be a sequence of $r\times r$ order matrices of $\mathbb{N}_0$-periodic measurable functions on $\mathbb{K}$, define

$$\triangle_{s,t}(\xi) = \sum_{j=0}^{\infty} q^{-2js}(1+\|\xi\|_2^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^\ell_{n,m}(\mathfrak{p}^{-j-1}\xi)|^2(1+\|\mathfrak{p}^{-j-1}\|_2^2)^{-t}, \quad \xi \in \mathbb{K}.$$

If there exists a non-negative number $\alpha > -s$ and a positive constant C such that

$$\sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^\ell_{n,m}(.)|^2 \leq C min(1, \|.\|_2^{2\alpha}), \quad a.e\ on\ \mathbb{K} \tag{3.7}$$

then $\triangle_{s,t} \in L^\infty(\mathbb{K})$.

Proof . Let us consider the two cases $s > 0$ and $s < 0$ separately. Suppose $s > 0$. Since $t > s$, by Lemma 3.3, we have

$$\triangle_{s,t}(\xi) \leq \sum_{j=0}^{\infty} q^{-2js}(1+o_1^2\|\xi\|_2^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^\ell_{n,m}(\mathfrak{p}^{-j-1})|^2(1+q^{-2j-2}o_2^2\|\xi\|^2)^{-t}. \tag{3.8}$$

By lemma 3.3, there exists a positive constant C' such that

$$B_{s,t}(\xi) = \sum_{j=0}^{\infty} q^{-2js}(1+o_1^2\|\xi\|^2)^s(1+q^{-2j-2}o_2^2\|\xi\|^2)^{-t} \leq C^t, \quad \forall\ \ \xi\ \ \mathbb{K}. \tag{3.9}$$

This implies that $\triangle_{s,t}(\xi) \le C'C, \ \ \forall \xi \in \mathbb{K}, \ \ i.e., \ \ \triangle_{s,t} \in L^{\infty}(\mathbb{K})$. Suppose $s < 0$. without loss of generality, we assume that $s < t < 0$. By Lemma 3.3, we have

$$\triangle_{s,t}(\xi) \le \sum_{j=0}^{\infty} q^{-2js}(1+o_2^2\|\xi\|_2^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-j-1})|^2(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t}.$$

$$=: \ominus_{s,t}(\xi) \tag{3.10}$$

For $o_1\|\xi\| \le 1$ and $j \ge 0$, we have

$$(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t} \le 2^{-t} \quad \text{and} \quad (1+o_2^2\|\xi\|^2)^s \le 1.$$

Since $\alpha \ge 0, \ \ \alpha + s > 0$, by Lemma 3.3. and equation (3.7), we have the following estimate

$$\begin{aligned}\ominus_{s,t}(\xi) &\le 2^{-t}\sum_{j=0}^{\infty} q^{-2js}\sum_{j=0}^{\infty} q^{-2js}\sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-j-1}\xi)^2\\
&\le 2^{-t}C\sum_{j=0}^{\infty} q^{-2js}\|\mathfrak{p}^{-j-1}\xi\|_2^{2\alpha}\\
&\le 2^{-t}Cq^{-2\alpha}\sum_{j=0}^{\infty} m^{-2j(\alpha+s)}(o_1\|\xi\|)^{2\alpha}\\
&\le 2^{-t}Cq^{-2\alpha}\sum_{j=0}^{\infty} q^{-2j(\alpha+s)} = \frac{2^{-t}Cq^{-2\alpha}}{1-q^{-2(\alpha+s)}} < \infty.\end{aligned} \tag{3.11}$$

For $o_1\|\xi\| > 1$, there exists $J \in \mathbb{N}_0$ such that $q^J \le o_1\|\xi\| < q^{J+1}$. Then for $j = 0, 1, ..., J$, we have

$$(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t} \le (1+q^{2(J-j)})^{-t} = q^{-2(J-j)t}(q^{-2(J-j)}+1)^{-t} \le 2^{-t}q^{-2(J-j)t}$$

and

$$(1+o_2^2\|\xi\|^2)^s \le (1+o_2^2o_1^{-2}q^{2J})^s \le o_2^{2s}o_1^{-2s}q^{2Js}.$$

Write $\ominus_{s,t}(\xi) = \ominus^1_{s,t}(\xi) + \ominus^2_{s,t}(\xi)$, where

$$\ominus^1_{s,t}(\xi) = \sum_{j=0}^{J} q^{-2js}(1+o_2^2\|\xi\|^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-j-1}\xi)|^2(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t},$$

$$\ominus^2_{s,t}(\xi) = \sum_{j=J+1}^{\infty} q^{-2js}(1+o_2^2\|\xi\|^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-j-1}\xi)|^2(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t}.$$

Then by $q^J \le o_1\|\zeta\| < q^{J+1}$ and $J \in \mathbb{N}_0$, it follows from $s < t < 0$ that

$$\begin{aligned}\ominus^1_{s,t}(\xi) &= \sum_{j=0}^{J} q^{-2js}(1+o_2^2\|\xi\|^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}^{\ell}_{n,m}(\mathfrak{p}^{-j-1}\xi)|^2(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t},\\
&\le Co_2^{2s}o_1^{-2s}2^{-t}\sum_{j=0}^{J} q^{-2(J-j)(t-s)} \le Co_2^{2s}o_1^{-2s}2^{-t}\sum_{j=0}^{\infty} q^{-2j(t-s)}\\
&= o_2^{2s}o_1^{-2s}2^{-t}\frac{1}{1-q^{-2(t-s)}} < \infty.\end{aligned} \tag{3.12}$$

Since $q^j \le o_1\|\xi\| < q^{J+1}$, we have for $j \ge J+1$

$$(1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t} \le (1+q^{2(J-j)})^{-t} \le 2^{-t}$$

and

$$(1+o_2^2\|\xi\|^2)^s \leq (1+o_2^2 o_1^{-2} q^{2J})^s \leq o_2^{2s} o_1^{-2s} q^{2Js}.$$

Since $\alpha \geq 0,\ \alpha+s>0$, by Lemma 3.3 and equation (3.7), we have

$$\ominus_{s,t}^2(\xi) = \sum_{j=J+1}^{\infty} q^{-2js}(1+o_2^2\|\xi\|^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-j-1}\xi)|^2 (1+q^{-2j-2}o_1^2\|\xi\|^2)^{-t}.$$

$$\leq q^t o_2^{2s} o_1^{-2s} C \sum_{j=J+1}^{\infty} q^{-2(j-J)s} \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-j-1}\xi)|^2$$

$$\leq q^t o_2^{2s} o_1^{-2s} C \sum_{j=J+1}^{\infty} q^{-2(j-J)s} \|\mathfrak{p}^{-j-1}\xi\|_2^{2\alpha}$$

$$\leq q^t o_2^{2s} o_1^{-2s} C \sum_{j=J+1}^{\infty} q^{-2(j-J)s} q^{-2\alpha(j+1)} (o_1\|\xi\|)^{2\alpha}$$

$$\leq q^t o_2^{2s} o_1^{-2s} C \sum_{j=J+1}^{\infty} q^{-2(j-J)(\alpha+s)}$$

$$= q^t o_2^{2s} o_1^{-2s} C \sum_{j=1}^{\infty} q^{-2j(\alpha+s)} = q^t o_2^{2s} o_1^{-2s} C \sum_{j=1}^{\infty} q^{-2j(\alpha+s)}$$

$$= q^t o_2^{2s} o_1^{-2(\alpha+s)} C \frac{q^{-2}(\alpha+s)}{1-q^{-2s}} < \infty. \tag{3.13}$$

Therefore, for the case $s<0$, we conclude that $\triangle_{s,t} \in L^\infty(\mathbb{K})$. $\square$

Now we proceed to prove the main result of this paper.

Theorem 3.1. Given $s \in \mathbb{K}$, let $\varphi = (\varphi_1, \varphi_2, ..., \varphi_r)^T \in (H^s(\mathbb{K}))^r$ be a $\mathfrak{p}$-refinable function vector satisfying the refinable equation , and let $\widehat{b}^{\ell}(\xi) = (\widehat{b}_{n,m}^{\ell}(\xi))_{n,m}^{r}$, $\ell = 1,2,...,L$ be a sequence of $r \times r$ order matrices of $\mathbb{N}_0$-periodic measurables functions on $\mathbb{K}$, $\psi_\ell = (\psi_1^\ell, \psi_2^\ell, ..., \psi_r^\ell)^T$, $\ell = 1,2,...,L$, be the wavelet function vectors defined by (2.9), and $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_L)$ be the multi-wavelet system defined by (2.10). Assume that

(i) $[\widehat{\varphi}_n, \widehat{\varphi}_n]_t \in L^\infty(\mathbb{K})$ for some $t>s$ with $n = 1,2,...,r$
(ii) There exists a non-negative number $\alpha > -s$ and a positive constant C such that

$$\sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\xi)|^2 \leq C min(1, \|\xi\|_2^{2\alpha}), \quad a.e \quad on \quad \mathbb{K}.$$

Then $\mathcal{W}^s(\varphi; \psi_1, \psi_2, ..., \psi_L)$ is a MWBS in $H^s(\mathbb{K})$.

Proof. For the case $s=0$, we take $0 < s_0 < min\{t, \alpha\}$, then the conditions (i) and (ii) hold for $s = s_0$. Therefore, the conclusion holds for s=0 if it holds for $s = s_0$. So, in order to finish the proof, we need to prove the conclusion holds for $s \neq 0$.
By Lemma 3.2, it is enough to prove that there exists a positive constant B such that

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{n;0,k}\rangle|^2 + \sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{n;j,k}^{\ell,s}\rangle|^2 \leq B\|g\|_{H^{-s}(\mathbb{K})} \quad for \quad g \in H^{-s}(\mathbb{K}). \tag{3.14}$$

For the first part, by Lemma 3.3, we have

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{N}_0} |\langle g, \varphi_{n;0,k}^{\ell,s}\rangle|^2 \leq \sum_{n=1}^{r} \|[\widehat{\varphi}_n, \widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})} \|g\|_{H^{-s}(\mathbb{K})}^2 \quad for \quad g \in H^{-s}(\mathbb{K}). \tag{3.15}$$

Next, we check the second part. For $g \in H^{-s}(\mathbb{K})$, compute

$$\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{n;j,k}^{\ell,s}\rangle|^2 = q^{-j(1+2s)} \sum_{k\in\mathbb{N}_0} \left| \int_{\mathbb{K}} \widehat{g}(\xi) \overline{\widehat{\psi}_n^{\ell}(\mathfrak{p}^{-1}\xi)} \chi_k(\mathfrak{p}^{-j}\xi) d\xi \right|$$

$$= q^{-j(1+2s)} \sum_{k\in\mathbb{N}_0} \left| \sum_{k'\in\mathbb{N}_0} \int_{\mathfrak{D}} \widehat{g}(\mathfrak{p}^j(\xi+u(k'))) \overline{\widehat{\psi}_n^{\ell}(\xi+u(k'))} \chi_k(\xi) d\xi \right|^2$$

$$= q^{-j(1+2s)} \sum_{k\in\mathbb{N}_0} \left| \sum_{k'\in\mathbb{N}_0} \int_{\mathfrak{D}} \widehat{g}(\mathfrak{p}^j(\xi+u(k'))) \overline{\widehat{\psi}_n^{\ell}(\xi+u(k'))} d\xi \right|^2$$

$$= q^{j(1-2s)} \int_{\mathfrak{D}} |[\widehat{g}(\mathfrak{p}^j\xi), \widehat{\psi}_n^{\ell}(\xi)]_0(\xi)|^2 d\xi \tag{3.16}$$

By definition in (2.9), we can get each component of ψ_ℓ

$$\psi_n^{\ell}(.) = \sum_{m=1}^{r} \widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-1}\xi)\widehat{\varphi}_n(\mathfrak{p}^{-1}\xi) \quad for \quad n=1,2,...,r \quad and \quad \ell=1,2,...,L,$$

and it follows from (3.16) that

$$\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{n;j,k}^{\ell,s}\rangle|^2$$

$$= q^{J(d-2s)} \int_{\mathfrak{D}} \left| \sum_{k\in\mathbb{N}_0} \sum_{m=1}^{r} \widehat{g}(\mathfrak{p}^j(\xi+u(k))) \overline{\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-1}(\zeta+u(k)))\widehat{\varphi}_n(\mathfrak{p}^{-1}(\xi+u(k)))} \right|^2 d\xi$$

$$= q^{j(d-2s)} \int_{\mathfrak{D}} \left| \sum_{\gamma\in\Gamma_{\mathfrak{p}}} \sum_{m=1}^{r} \overline{\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-1}\xi+\gamma)} [\widehat{g}(\mathfrak{p}^{j+1}\xi), \widehat{\varphi}_n]_0(\mathfrak{p}^{-1}\xi+\gamma) \right|^2 d\xi$$

$$\leq q^{(j+1)d-2s} \sum_{\gamma\in\Gamma_{\mathfrak{p}}} \int_{\mathfrak{D}} \left| \sum_{m=1}^{r} \overline{\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-1}\xi+\gamma)} [\widehat{g}(\mathfrak{p}^{j+1}.), \widehat{\varphi}_n]_0(\mathfrak{p}^{-1}\xi+\gamma) \right|^2 d\xi$$

$$\leq q^{(j+2)d-2js} \int_{\mathfrak{D}} \sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\xi)|^2 [\widehat{g}(\mathfrak{p}^{j-1}.\widehat{g}(\mathfrak{p}^{j+1})]_{-t}(\xi)[\widehat{\varphi}_n, \widehat{\varphi}_n]_t(\xi) d\xi$$

$$\leq q^{(j+2)d-2js} \max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \int_{\mathfrak{D}} \sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\xi)|^2 [\widehat{g}(\mathfrak{p}^{j+1}(\xi)), \widehat{g}(\mathfrak{p}^{j+1}\xi)]_{-t}(\xi) d\xi$$

$$\leq q^{(j+2)d-2js} \max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \int_{\mathbb{K}} \sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\xi)|^2 |\widehat{g}(\mathfrak{p}^{j+1}(\xi))|^2 (1+\|\xi\|_2^2)^{-t} d\xi$$

$$= q^{d-2js} \max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \times$$

$$\sum_{\mathbb{K}} \sum_{m=1}^{r} |\widehat{b}_{n,m}(\mathfrak{p}^{-j-1}\xi)|^2 |\widehat{g}(\xi)|^2 |\widehat{g}(\xi)|^2 (1+\|\mathfrak{p}^{-j-1}\xi\|_2^2)^{-t} d\xi \tag{3.17}$$

Hence, we conclude that

$$\sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0} |\langle g, \psi_{n;j,k}^{\ell,s}\rangle|^2 \leq q^d \max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n, \widehat{\varphi}_n]_t\|_{L^\infty(\mathbb{K})}\} \int_{\mathbb{K}} |\widehat{g}(\xi)|^2 (1+\|\xi\|_2^2)^{-s} \quad \times$$

$$\sum_{j=0}^{\infty} q^{-2js}(1+\|\xi\|_2^2)^s \sum_{\ell=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\widehat{b}_{n,m}^{\ell}(\mathfrak{p}^{-j-1}\xi)|^2 (1+\|\mathfrak{p}^{-j-1}\xi\|_2^2)^{-t} d\xi \tag{3.18}$$

By Lemma 3.4., we get from (3.18) that

$$\begin{aligned}&\sum_{n=1}^{r}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{N}_0}|\langle g,\psi_{n;j,k}\rangle|^2\\&\leq q^d\max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\}\|\triangle_{s,t}\|_{L^\infty(\mathbb{K})}\int_{\mathbb{K}}|\widehat{g}(\xi)|^2(1+\|\xi\|_2^2)^{-s}\\&=q^d\max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\}\|\triangle_{s,t}\|_{L^\infty(\mathbb{K})}\|g\|_{H^{-s}(\mathbb{K})}.\end{aligned}$$

Consequently, (3.14) holds with

$$B=\sum_{n=1}^{r}\|[\widehat{\varphi}_n,\widehat{\varphi}_n]_s\|_{L^\infty(\mathbb{K})}+q^d\max_{1\leq n\leq r}\{\|[\widehat{\varphi}_n,\widehat{\varphi}_n]\}\|\triangle_{s,t}\|_{L^\infty(\mathbb{K})}\}\|\triangle_{s,t}\|_{L^\infty(\mathbb{K})}. \tag{3.19}$$

The proof is completed. □

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