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A quasi-static contact problem with friction in electro viscoelasticity with long-term memory body with damage and thermal effects

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Abstract

In this paper, we consider a mathematical model that describes the quasi-static process of contact between a piezoelectric body and a deformable foundation. A nonlinear thermo-electro-viscoelastic constitutive law with long term memory and damage is used and the contact is described with the normal compliance condition and a version of Coulomb's law of friction. We derive variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field, the temperature field and the damage field, existence and uniqueness of a weak solution of the problem is proved. The proof is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed points.

Keywords: Electro-viscoelasticity, quasistatic, Damage, temperature, differential equations, fixed point.
2020 MSC: 74C10, 49J40, 74M10, 74M15, 47H10

1 Introduction

Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or complex metal forming processes are just a few examples. Because of the importance of contact processes in structural and mechanical systems, considerable progress has been achieved recently in modeling and mathematical analysis and numerical simulations and so, the engineering literature concerning this topic is rather extensive [3, 16, 9].

The piezoelectricity lie between the coupling of the mechanical and electrical material properties, This coupling, leads to the appearance of electric field in the presence of a mechanical stress, and conversely in the presence of the electric potential the mechanical stress is generated. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. General models for elastic materials with piezoelectric effects can be found in [2, 19, 20, 21].

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In order to model the effect of temperature in the behaviour of some real bodies like metals, magmas, polymers and so on, thermo-elastic and thermo-viscoplastic constitutive laws has been studied by mathematicians, physicists and engineers, see for examples and details [13, 17, 18, 5, 1].

In this paper, we also consider the damage of the material. The effect due to the damage leads to the decrease in the load carrying capacity of the body, is also included. The effective functioning and safety of a mechanical system may be deteriorated by this decrease as the material undergoes damage. Because of the importance of this topic, General novel models for damage were derived in [6, 7] from the virtual power principle. The mathematical analysis of one-dimensional problems can be found in [8]. Contact problems with damage have been investigated in [25, 11, 12, 23].

Contact problems involving viscoelastic materials with long memory have been studied in [10, 24, 26].

Here we continue this line of research and study a quasistatic contact problem with coulomb friction in thermo-electro-viscoelasticity with long-term memory body. when the foundation is deformable and conductive.

In Section 2 we present contact model and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. We prove in Section 4 the existence and uniqueness of the solution.

2 Problem statement

The physical setting is the following, A body occupies the domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with outer Lipschitz surface which is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $meas(\Gamma_1) > 0$ and $meas(\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0, T)$ and the displacement vanishes there. A volume force of density f_0 acts in $\Omega \times (0, T)$ and surface tractions of density f_2 act on $\Gamma_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_2 is prescribed on $\Gamma_b \times (0, T)$. The body may come in contact with a conductive obstacle over the part Γ_3, the potential contact surface. A gap g may exist between the contact surface Γ_3 and the foundation, measured along the outward normal vector ν over the potential contact surface Γ_3. We admit a possible external heat source applied in $\Omega \times (0, T)$, given by the function ρ.

The classical formulation of the mechanical problem of electro viscoelasticity with long-term memory body with damage and thermal effects, be stated as follows.

Problem *P*

Find a displacement field $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times (0, T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0, T) \to \mathbb{R}$, an electric displacement field $\boldsymbol{D} : \Omega \times (0, T) \to \mathbb{R}^d$, and a damage field $\alpha : \Omega \times (0, T) \to \mathbb{R}$ such that

$$\begin{aligned}&\boldsymbol{\sigma}(t) = \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\left(\varepsilon\left(\boldsymbol{u}(t)\right), \theta(t), \alpha(t)\right) + \\ &\int_0^t \mathcal{M}\left(t-s, \varepsilon\left(\boldsymbol{u}(s)\right), \theta(s), \alpha(s)\right) ds - (\mathcal{E})^* E\left(\varphi(t)\right)\end{aligned} \quad \text{in } \Omega \times (0, T), \tag{2.1}$$

$$\boldsymbol{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \boldsymbol{B}\nabla(\varphi) \quad \text{in } \Omega \times (0, T), \tag{2.2}$$

$$\dot{\theta} - \kappa_0 \Delta\theta = \Theta\left(\boldsymbol{\sigma}, \varepsilon\left(\boldsymbol{u}\right), \theta, \alpha\right) + \rho \quad \text{in } \Omega \times (0, T), \tag{2.3}$$

$$\dot{\alpha} - k\Delta\alpha + \partial\varphi_K(\alpha) \ni S\left(\varepsilon(\boldsymbol{u}), \alpha\right), \quad \text{in } \Omega \times (0, T), \tag{2.4}$$

$$\operatorname{Div} \boldsymbol{\sigma} + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.5}$$

$$\operatorname{div} \boldsymbol{D} - q_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{2.6}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_1 \times (0, T), \tag{2.7}$$

$$\boldsymbol{\sigma\nu} = f_2 \quad \text{on } \Gamma_2 \times (0, T), \tag{2.8}$$

$$-\sigma_\nu = p_\nu\left(u_\nu - g\right) \quad \text{on } \Gamma_3 \times (0, T) \tag{2.9}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \leq \mu p_\nu\left(u_\nu - g\right) \\ \|\boldsymbol{\sigma}_\tau\| < \mu p_\nu\left(u_\nu - g\right) \Rightarrow \dot{\boldsymbol{u}}_\tau = 0 \\ \|\boldsymbol{\sigma}_\tau\| = \mu p_\nu\left(u_\nu - g\right) \Rightarrow \text{there exists } \lambda \geq 0 \\ \qquad \text{such that } \boldsymbol{\sigma}_\tau = -\lambda\dot{\boldsymbol{u}}_\tau, \end{cases} \quad \text{on } \Gamma_3 \times (0, T) \tag{2.10}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \qquad \text{on } \Gamma \times (0,T), \tag{2.11}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0,T), \tag{2.12}$$

$$\boldsymbol{D}.\nu = q_2 \qquad \text{on } \Gamma_b \times (0,T), \tag{2.13}$$

$$\boldsymbol{D}\cdot\boldsymbol{\nu} = \psi\left(u_\nu - g\right)\phi_l\left(\varphi - \varphi_0\right) \qquad \text{on } \Gamma_3 \times (0,T), \tag{2.14}$$

$$k_0\frac{\partial\theta}{\partial\nu} + B\theta = 0 \qquad \text{on } \Gamma \times (0,T), \tag{2.15}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0, \qquad \text{in } \Omega. \tag{2.16}$$

First, equations (2.1) and (2.2) represent the thermo-electro-viscoelastic constitutive law with long term-memory and damage, were $\mathcal{A}$ and $\mathcal{B}$ are the viscosity and elasticity operators, respectively, and $\mathcal{M}$ is the relaxation operator, where θ represents the absolute temperature and α is the damage field. $E(\varphi) = -\nabla\varphi$ is the electric field, $\mathcal{E} = (e_{ijk})$ represent the third order piesoelectric tensor, $\mathcal{E}^*$ is its transposition.

Equation (2.3) represents the energy conservation where Θ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and ρ is a given volume heat source

The evolution of the damage field is governed by the inclusion of parabolic type given by the relation (2.4) where $\varphi_K(\alpha)$ denotes the subdifferential of the indicator function of the set K of admissible damage functions defined by $K = \left\{\alpha \in H^1(\Omega) \mid 0 \le \alpha \le 1 \text{ a.e. in } \Omega\right\}$, and S is the mechanical source of the damage

Equations (2.5) and (2.6) represent the equilibrium equations for the stress and electric displacement fields. Equations (2.7)-(2.8) are the displacement-traction conditions.

Frictional contact conditions of the form (2.9) and (2.10) have been used in various papers [12, 23]. and which describe the contact on the surface Γ_3 , described by the normal compliance function p, such that $p(u) = 0$ when $u \le 0$, g is the initial gap and the condition, $u_\nu - g \ge 0$ represents the penetration of body in the foundation.

In (2.10) the tangential stress cannot exceed the friction threshold $\mu p_\nu(u_\nu - g)$. In addition, when the threshold is not reached, there is no slip (the tangential velocity vanishes). When this threshold is reached, the body begins to slide and the tangential stress tends to oppose the tangential movement

The relation (2.11) describes a homogeneous Neumann boundary condition. (2.12) and (2.13) represent the electric boundary conditions.

Equality (2.14) represents the electrical condition on the potential contact surface, where ϕ_l is the potential of the electric foundation. The function ϕ_l is given by

$$\phi_l(s) = \begin{cases} -l & \text{if } s < -l, \\ s & \text{if } -l \le s \le l, \\ l & \text{if } s > l. \end{cases} \tag{2.17}$$

here l is a large positive constant, it may be arbitrarily large, higher than any possible peak voltage in the system. The function ψ is given bellow. For more details see [15]. (2.15) represent a Fourier boundary condition for the temperature. Finally, The functions $\boldsymbol{u}_0$, θ_0 and α_0 in (2.16) are the initial data.

3 Variational formulation and preliminaries

For a weak formulation of the problem, first we introduce some notation. Let d be a positive integer. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i, \quad \forall \boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^d \quad \text{and} \quad \boldsymbol{\sigma}\cdot\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathbb{S}^d,$$
$$\|\boldsymbol{u}\| = (\boldsymbol{u}\cdot\boldsymbol{u})^{\frac{1}{2}}, \quad \forall \boldsymbol{u} \in \mathbb{R}^d \quad \text{and} \quad \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma}\cdot\boldsymbol{\sigma})^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma} \in \mathbb{S}^d.$$

Here and everywhere in this paper the indices i, j, k run from 1 to d. The convention of summation over repeated indices is used and the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a regular boundary Γ. we introduce the spaces

$$H = L^2(\Omega;\mathbb{R}^d), \quad H_1 = \{\boldsymbol{u} \in H \mid \varepsilon(\boldsymbol{u}) \in \mathcal{H}\} = H^1(\Omega;\mathbb{R}^d),$$
$$\mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij}) \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\} = L^2(\Omega;\mathbb{S}^d), \quad \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H} \mid \operatorname{Div}\boldsymbol{\sigma} \in H\}.$$

where $\varepsilon : H^1\left(\Omega;\mathbb{R}^d\right) \to L^2\left(\Omega;\mathbb{S}^d\right)$ and Div : $\mathcal{H}_1 \to L^2\left(\Omega;\mathbb{R}^d\right)$ denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u})), \quad \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}\right), \quad \mathrm{Div}(\boldsymbol{\sigma}) = \sigma_{ij,j}.$$

The spaces H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are Hilbert spaces equipped with the inner products

$$(\boldsymbol{u},\boldsymbol{v})_H = \int_\Omega u_i v_i dx \quad \forall \mathbf{u},\mathbf{v} \in H, \quad (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H},$$

$$(\mathbf{u},\boldsymbol{v})_{H_1} = (\boldsymbol{u},\boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \quad \forall \boldsymbol{u},\boldsymbol{v} \in H_1,$$
$$(\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma},\boldsymbol{\tau})_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma}, \mathrm{Div}\,\boldsymbol{\tau})_H, \quad \boldsymbol{\sigma},\boldsymbol{\tau} \in \mathcal{H}_1.$$

The associated norms in H, H_1, $\mathcal{H}$ and $\mathcal{H}_1$ are denoted by $\|.\|_H$, $\|.\|_{H_1}$, $\|.\|_{\mathcal{H}}$ and $\|.\|_{\mathcal{H}_1}$ respectively.

Given $\boldsymbol{u} \in H^{\frac{1}{2}}(\Gamma;\mathbb{R}^d)$ we denote by u_ν and $\boldsymbol{u}_\tau$ the normal and the tangential components of $\boldsymbol{u}$ on the boundary, i.e. $u_\nu = \boldsymbol{u}.\boldsymbol{\nu}$ and $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu\boldsymbol{\nu}$ Similarly, for a regular tensor field $\boldsymbol{\sigma} : \Gamma \to \mathbb{S}^d$ we define its normal and tangential components by $\sigma_\nu = \sigma\nu \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, and we recall that the following Green formula holds

$$(\boldsymbol{\sigma},\varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathrm{Div}\,\boldsymbol{\sigma},\boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu.\boldsymbol{v} da, \quad \forall \boldsymbol{v} \in H_1.$$

and for the displacement field we need the closed subspace of $H^1(\Omega)^d$ defined by

$$V = \left\{\boldsymbol{v} \in H^1(\Omega)^d \mid \boldsymbol{v} = 0 \text{ on } \Gamma_1\right\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds and there exists a constant $C_0 > 0$, that depends only on Ω and Γ_1 such that

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}} \geq C_0\|\boldsymbol{v}\|_{H^1(\Omega)^d}, \quad \forall \boldsymbol{v} \in V.$$

A proof of Korn's inequality may be found in ([22] ,p.79). On V, we consider the inner product and the associated norm given by

$$(\boldsymbol{u},\boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \ \|\boldsymbol{v}\|_V = \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \ \boldsymbol{u},\boldsymbol{v} \in V. \tag{3.1}$$

It follows that $\|.\|_{H^1(\Omega)^d}$ and $\|.\|_V$ are equivalent norms on V and therefore $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem and (3.1) there exists a constant $c_0 > 0$ depending only on Ω, Γ_1 and Γ_3 such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_3)^d} \leq c_0\|\boldsymbol{v}\|_V, \quad \forall \boldsymbol{v} \in V. \tag{3.2}$$

For the electric displacement field we use the Hilbert space

$$\mathcal{W} = \left\{\mathbf{D} = (D_i) \mid D_i \in L^2(\Omega), \mathrm{div}\,\mathbf{D} \in L^2(\Omega)\right\},$$

endowed with the inner product

$$(\mathbf{D},\mathbf{E})_{\mathcal{W}} = (\mathbf{D},\mathbf{E})_H + (\mathrm{div}\,\mathbf{D}, \mathrm{div}\,\mathbf{E})_{L^2(\Omega)},$$

and the associated norm $\|.\|_{\mathcal{W}}$. The electric potential field is to be found in

$$W = \left\{\xi \in H^1(\Omega), \xi = 0 \text{ on } \Gamma_a\right\}.$$

Since $meas(\Gamma_a) > 0$, the Friedrichs-Poincaré inequality holds

$$\|\nabla\zeta\|_H \geq c_F\|\zeta\|_{H^1(\Omega)}, \quad \forall \zeta \in W, \tag{3.3}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W we use the inner product

$$(\varphi,\xi)_W = \int_\Omega \nabla\varphi.\nabla\xi dx, \tag{3.4}$$

and $\|.\|_W$ the associated norm. It follows from (3.3) that $\|.\|_{H^1(\Omega)}$ and $\|.\|_W$ are equivalent norms on W and therefore $(W, \|.\|_W)$ is a real Hilbert space.

Moreover, by the Sobolev trace theorem, there exist a constant $\tilde{c}_0$ such that

$$\|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0\|\psi\|_W, \quad \forall\psi \in W. \tag{3.5}$$

Moreover, when $\boldsymbol{D} \in \mathcal{W}$ is a regular function, the following Green's type formula holds

$$(\boldsymbol{D}, \nabla\zeta)_H + (\operatorname{div}\boldsymbol{D}, \zeta)_{L^2(\Omega)} = \int_\Gamma \boldsymbol{D}\cdot\boldsymbol{\nu}\zeta da, \quad \forall\zeta \in H^1(\Omega). \tag{3.6}$$

For any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$, where $1 \leq p \leq \infty$ and $k \geq 1$. For $T > 0$ we denote by $C(0,T;X)$ and $C^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\boldsymbol{f}\|_{C(0,T;X)} = \max_{t\in[0,T]}\|\boldsymbol{f}(t)\|_X.$$

$$\|\boldsymbol{f}\|_{C^1(0,T;X)} = \max_{t\in[0,T]}\|\boldsymbol{f}(t)\|_X + \max_{t\in[0,T]}\|\dot{\boldsymbol{f}}(t)\|_X.$$

In the study of the problem P, we consider the following assumptions

The viscosity operator $\mathcal{A} : \Omega\times\mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} (a)\text{ There exists } L_{\mathcal{A}} > 0 \quad \text{such that} \\ \|\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leq L_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|, \\ \quad\text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x}\in\Omega. \\ (b)\text{ There exists } m_{\mathcal{A}} > 0 \quad \text{such that} \\ (\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}_2)).(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geq m_{\mathcal{A}}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2, \\ \quad\text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e } \boldsymbol{x}\in\Omega. \\ (c)\text{ The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega, \\ \quad\text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d. \\ (d)\text{ The mapping } \boldsymbol{x}\mapsto\mathcal{A}(\boldsymbol{x},\boldsymbol{0}) \in \mathcal{H}. \end{cases} \tag{3.7}$$

The elasticity operator $\mathcal{B} : \Omega\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{S}^d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{B}} > 0 \quad \text{such that} \\ \|\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_1,\theta_1,\alpha_1) - \mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon}_2,\theta_2,\alpha_2)\| \leq L_{\mathcal{B}}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| \\ + \|\alpha_1 - \alpha_2\|), \text{ for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{\varepsilon},\theta,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad\text{for all } \boldsymbol{\varepsilon}\in\mathbb{S}^d, \theta,\alpha\in\mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{B}(\boldsymbol{x},\boldsymbol{0},0,0) \in \mathcal{H}. \end{cases} \tag{3.8}$$

The relaxation function $\mathcal{M} : \Omega\times(0,T)\times\mathbb{S}^d\times\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_{\mathcal{M}} > 0 \quad \text{such that} \\ \|\mathcal{M}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1,\theta_1,\alpha_1) - \mathcal{M}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2,\theta_2,\alpha_2)\| \leq L_{\mathcal{M}}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\theta_1 - \theta_2\| \\ + \|\alpha_1 - \alpha_2\|), \text{ for all } t\in(0,T), \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \theta_1,\theta_2,\alpha_1,\alpha_2\in\mathbb{R}, \text{ a.e. } \boldsymbol{x}\in\Omega. \\ \text{(b) The mapping } \boldsymbol{x}\mapsto\mathcal{M}(\boldsymbol{x},t,\boldsymbol{\varepsilon},\theta,\alpha) \text{ is Lebesgue measurable on } \Omega, \\ \text{for all } \boldsymbol{\varepsilon},\in\mathbb{S}^d, t\in(0,T), \text{ for all } \theta,\alpha\in\mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x}\mapsto\mathcal{M}(\boldsymbol{x},t,\boldsymbol{\varepsilon},\theta,\alpha) \text{ is continuous in } \Omega, \\ \text{for all } \boldsymbol{\varepsilon},\in\mathbb{S}^d, t\in(0,T), \text{ for all } \theta,\alpha\in\mathbb{R}. \\ \text{(d) The mapping } \boldsymbol{x}\mapsto\mathcal{M}(\boldsymbol{x},t,\boldsymbol{0},0,0) \in \mathcal{H}. \end{cases} \tag{3.9}$$

The function $\Theta : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} (a) \text{ There exists a constant } L_\Theta > 0 \quad \text{such that} \\ \|\Theta(\boldsymbol{x}, \boldsymbol{\sigma}_1, \boldsymbol{\varepsilon}_1, \theta_1, \alpha_1) - \Theta(\boldsymbol{x}, \boldsymbol{\sigma}_2, \boldsymbol{\varepsilon}_2, \theta_2, \alpha_2)\| \leq L_\Theta(\|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\| + \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ + \|\theta_1 - \theta_2\| + \|\alpha_1 - \alpha_2\|), \\ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in \mathbb{S}^d, \text{ and } \theta_1, \theta_2, \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e } \boldsymbol{x} \in \Omega. \\ (b) \text{ For any } \boldsymbol{\sigma}, \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ and } \theta, \alpha \in \mathbb{R}, \boldsymbol{x} \mapsto \Theta(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \theta, \alpha) \text{ is Lebesgue} \\ \quad \text{measurable on } \Omega. \\ (c) \text{ The mapping } \boldsymbol{x} \mapsto \Theta(\boldsymbol{x}, \mathbf{0}, \mathbf{0}, 0, 0) \in L^2(\Omega). \end{cases} \tag{3.10}$$

The function $S : \Omega \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_S > 0 \quad \text{such that} \\ \|S(\boldsymbol{x}, \boldsymbol{\varepsilon}_1, \alpha_1) - S(\boldsymbol{x}, \boldsymbol{\varepsilon}_2, \alpha_2)\| \leq L_S(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| + \|\alpha_1 - \alpha_2\|), \\ \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ for all } \alpha_1, \alpha_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, \boldsymbol{\varepsilon}, \alpha) \text{ is Lebesgue measurable on } \Omega, \\ \qquad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d, \text{ for all } \alpha \in \mathbb{R}. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto S(\boldsymbol{x}, \mathbf{0}, 0) \in L^2(\Omega). \end{cases} \tag{3.11}$$

The surface electrical conductivity function $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\psi > 0 \text{ such that} \\ \|\psi(., u_1) - \psi(., u_2)\| \leq L_\psi \|u_1 - u_2\|, \text{ for all } u_1, u_2 \in \mathbb{R}. \\ \text{(b) There exists } M_\psi > 0 \text{ such that } \|\psi(\boldsymbol{x}, u)\| \leq M_\psi, \\ \qquad \text{for all } u \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(c)} \boldsymbol{x} \mapsto \psi(\boldsymbol{x},) \text{ is measurable on } \Gamma_3, \text{ for all } u \in \mathbb{R}, \\ \text{(d)} \boldsymbol{x} \mapsto \psi(\boldsymbol{x}, u) = 0, \quad \text{for all } u \leq 0. \end{cases} \tag{3.12}$$

Electric permittivity operator $\boldsymbol{B} = (b_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a) } \boldsymbol{B}(\boldsymbol{x}, E) = (b_{ij}(\boldsymbol{x})E_j) \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) } b_{ij} = b_{ji} \in L^\infty(\Omega), 1 \leq i, j \leq d. \\ \text{(c) There exists a constant } m_{\boldsymbol{B}} > 0 \text{ such that} \\ \quad \boldsymbol{B}E.E \geq m_{\boldsymbol{B}}\|E\|^2, \text{ for all } E = (E_i) \in \mathbb{R}^d, \text{ a.e. in } \Omega. \end{cases} \tag{3.13}$$

The piezoelectric operator $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies

$$\begin{cases} (a) \quad \mathcal{E} = (e_{ijk}), e_{ijk} \in L^\infty(\Omega), 1 \leq i, j, k \leq d. \\ (b) \quad \mathcal{E}(\mathrm{x})\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{\tau}, \text{ for all } \boldsymbol{\sigma} \in \mathbb{S}^d, \text{ and all } \boldsymbol{\tau} \in \mathbb{R}^d. \end{cases} \tag{3.14}$$

The normal compliance function $p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+$ satisfies

$$\begin{cases} \text{(a) There exists } L_\nu > 0 \text{ such that} \\ \quad \|p_\nu(\boldsymbol{x}, u_1) - p_\nu(\boldsymbol{x}, u_2)\| \leq L_\nu \|u_1 - u_2\| \\ \forall u_1, u_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_3 \\ \text{(d) For any } u \in \mathbb{R}, \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, u) \text{ is measurable on } \Gamma_3 \\ \text{(e) } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x}, u) = 0, \text{ for all } u \leq 0. \end{cases} \tag{3.15}$$

We assume that the gap function g the initial potential φ_0 the friction coefficient μ, the volume heat source ρ, the initial data α_0, $\boldsymbol{u}_0$ and θ_0 the volume of forces f_0 and f_2 and the charges densities q_0, q_2, the energy coefficient k_0, the microcrack diffusion coefficient k_1 satisfy

$$g \in L^2(\Gamma_3), \; g \geq 0 \text{ a.e. on } \Gamma_3, \; \varphi_0 \in L^2(\Gamma_3), \tag{3.16}$$
$$\mu \in L^\infty(\Gamma_3), \; \mu \geq 0 \text{ a.e. on } \Gamma_3, \tag{3.17}$$
$$\boldsymbol{u}_0 \in V, \quad \alpha_0 \in K, \quad \theta_0 \in H^1(\Omega), \tag{3.18}$$
$$f_0 \in C\left(0,T;L^2(\Omega)^d\right), \; f_2 \in C\left(0,T;L^2(\Gamma_2)^d\right), \tag{3.19}$$
$$q_0 \in C\left(0,T;L^2(\Omega)\right), \; q_2 \in C\left(0,T;L^2(\Gamma_b)\right), \tag{3.20}$$
$$B > 0, \quad k_i > 0, \quad i = 0,1, \quad \rho \in C\left(0,T;L^2(\Omega)\right). \tag{3.21}$$

We introduce the following bilinear forms $a_i : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, $i = 0,1$ by

$$a_0(\zeta,\xi) = k_0 \int_\Omega \nabla\zeta \cdot \nabla\xi dx + B \int_\Gamma \zeta\xi d\gamma, \tag{3.22}$$
$$a_1(\zeta,\xi) = k_1 \int_\Omega \nabla\zeta \cdot \nabla\xi dx. \tag{3.23}$$

Next. We define four mappings $j : V \times V \to \mathbb{R}$, $h : V \times W \to W$, $f : [0,T] \to V$ and $q : [0,T] \to W$, respectively, by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_1} p_\nu\left(\boldsymbol{u}_\nu - g\right)\boldsymbol{v}_\nu da + \int_{\Gamma_3} \mu p_\nu\left(\boldsymbol{u}_\nu - g\right)\|\boldsymbol{v}_\tau\| da, \tag{3.24}$$
$$(h(\mathbf{u},\varphi),\zeta)_W = \int_{\Gamma_3} \psi\left(u_\nu - g\right)\phi_l\left(\varphi - \varphi_0\right)\zeta da, \tag{3.25}$$
$$(\boldsymbol{f}(t),\boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v} dx + \int_{\Gamma_2} \boldsymbol{f}_2(t)\cdot\boldsymbol{v} da, \tag{3.26}$$
$$(q(t),\zeta)_W = \int_\Omega q_0(t)\zeta dx - \int_{\Gamma_b} q_2(t)\zeta da, \tag{3.27}$$

for all $\boldsymbol{u},\boldsymbol{v} \in V$, $\varphi,\zeta \in W$ and $t \in [0,T]$. Note that

$$\boldsymbol{f} \in C\left(0,T;V\right), \quad q \in C(0,T;W). \tag{3.28}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)-(2.16).

problem $\boldsymbol{PV}$

Find a displacement field $\boldsymbol{u} : (0,T) \to V$, a stress field $\boldsymbol{\sigma} : (0,T) \to \mathcal{H}$, an electric potential $\varphi : (0,T) \to W$, a damage field $\alpha : (0,T) \to H^1(\Omega)$, and a temperature $\theta : (0,T) \to H^1(\Omega)$ such that

$$\begin{aligned}\boldsymbol{\sigma}(t) = {} & \mathcal{A}\varepsilon\left(\dot{\boldsymbol{u}}(t)\right) + \mathcal{B}\left(\varepsilon\left(\boldsymbol{u}(t)\right),\theta(t),\alpha(t)\right) + \\ & \int_0^t \mathcal{M}\left(t-s,\varepsilon\left(\boldsymbol{u}(s)\right),\theta(s),\alpha(s)\right)ds - (\mathcal{E})^* E\left(\varphi(t)\right)\end{aligned} \tag{3.29}$$
$$(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v}) - \varepsilon(\dot{\boldsymbol{u}}(t))_{\mathcal{H}} + j(\boldsymbol{u}(t),\boldsymbol{v}) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \geq (\mathbf{f}(t),\boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \tag{3.30}$$
$$(\boldsymbol{B}\nabla\varphi(t),\nabla\zeta)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}(t)),\nabla\zeta)_H + (h(\boldsymbol{u}(t),\varphi(t)),\zeta)_W = (q(t),\zeta)_W, \tag{3.31}$$
$$\begin{aligned}&(\dot{\theta}(t),\mathbf{v})_{L^2(\Omega)} + a_0(\theta(t),\mathbf{v}) = (\psi(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{u}(t)),\theta(t)),\mathbf{v})_{L^2(\Omega)} \\ &\quad + (\rho(t),\mathbf{v})_{L^2(\Omega)}, \quad \forall \mathbf{v} \in H^1(\Omega), \text{ a.e. } t \in (0,T),\end{aligned} \tag{3.32}$$
$$\begin{aligned}&\alpha(t) \in K, (\dot{\alpha}(t),\xi - \alpha(t))_{L^2(\Omega)} + a(\alpha(t),\xi - \alpha(t)) \\ &\geq \left(S\left(\varepsilon(\mathbf{u}(t)),\alpha(t)\right),\xi - \alpha(t)\right)_{L^2(\Omega)}, \forall \xi \in K, t \in (0,T),\end{aligned} \tag{3.33}$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \theta(0) = \theta_0, \quad \alpha(0) = \alpha_0. \tag{3.34}$$

Our main existence and uniqueness result for Problem PV is in the following section.

4 Existence and uniqueness

Theorem 4.1. Assume that (3.7)-(3.21) hold, Then there exists a unique solution $(\boldsymbol{u}, \boldsymbol{\sigma}, \varphi, \theta, \alpha, \boldsymbol{D})$ to problem PV. Moreover, the solution has the regularity

$$\boldsymbol{u} \in C^1(0,T;V), \tag{4.1}$$
$$\varphi \in C(0,T;W), \tag{4.2}$$
$$\boldsymbol{\sigma} \in C(0,T;\mathcal{H}), \tag{4.3}$$
$$\theta \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{4.4}$$
$$\alpha \in W^{1,2}\left(0,T;L^2(\Omega)\right) \cap L^2\left(0,T;H^1(\Omega)\right), \tag{4.5}$$
$$\boldsymbol{D} \in C(0,T;\mathcal{W}). \tag{4.6}$$

The functions $\boldsymbol{u}$, $\boldsymbol{\sigma}$, φ, θ, α, and $\boldsymbol{D}$ which satisfy (3.29)-(3.34) are called a weak solution of the contact problem P. We conclude that, under the assumptions (3.7)-(3.21), the mechanical problem (2.1)-(2.16) has a unique weak solution satisfying (4.1)-(4.6).

It follows from (3.31) that $\operatorname{div} \boldsymbol{D} - q_0 = 0$ for all $t \in (0,T)$, and therefore the regularity (4.2) of φ, combined with (3.13),(3.14), and (3.20) implies (4.6).

The proof of theorem 4.1, is carried out is several steps and is based on arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and fixed points.

We denote by C a constant whose value may change from line to line when no confusing can arise.

Let $\boldsymbol{\eta} \in C(0,T;\mathcal{H})$ and $\boldsymbol{g} \in C(0,T;V)$ we consider the following variational problem.

Problem $\mathcal{P}_{\boldsymbol{\eta}}^{\mathbf{1}}$

Find a displacement field $\boldsymbol{u}_\eta : [0,T] \to V$ and a stress field $\boldsymbol{\sigma}_\eta : [0,T] \to \mathcal{H}$ such that for all $t \in [0,T]$

$$\boldsymbol{\sigma}_\eta = \mathcal{A}\left(\varepsilon\left(\dot{\boldsymbol{u}}_\eta\right)\right) + \boldsymbol{\eta}, \tag{4.7}$$
$$\left(\boldsymbol{\sigma}_\eta(t), \varepsilon\left(\boldsymbol{w} - \dot{\boldsymbol{u}}_\eta\right)\right)_{\mathcal{H}} + j(\boldsymbol{u}_\eta, \boldsymbol{w}) - j\left(\boldsymbol{u}_\eta, \dot{\boldsymbol{u}}_\eta\right) \geq \left(\mathbf{f}(t), \boldsymbol{w} - \dot{\boldsymbol{u}}_\eta\right)_V, \quad \forall \boldsymbol{w} \in V, \tag{4.8}$$
$$\boldsymbol{u}_\eta(0) = \boldsymbol{u}_0. \tag{4.9}$$

We consider the following variational inequality

Problem $\mathcal{P}_{\boldsymbol{\eta}}^{\mathbf{2}}$

Find $\boldsymbol{v}_{\eta \boldsymbol{g}} : \Omega \times (0,T) \to V$ such that

$$\begin{aligned}\left(\mathcal{A}\varepsilon\left(\boldsymbol{v}_{\eta \boldsymbol{g}}\right), \varepsilon\left(\boldsymbol{w} - \boldsymbol{v}_{\eta \boldsymbol{g}}\right)\right)_{\mathcal{H}} &+ j(\boldsymbol{g}, \boldsymbol{w}) - j\left(\boldsymbol{g}, \boldsymbol{v}_{\eta \boldsymbol{g}}\right) \\ &\geq \left(\mathbf{f}, \boldsymbol{w} - \boldsymbol{v}_{\eta \boldsymbol{g}}\right)_V - \left(\boldsymbol{\eta}, \varepsilon\left(\boldsymbol{w} - \boldsymbol{v}_{\eta \boldsymbol{g}}\right)\right)_{\mathcal{H}}, \quad \forall \boldsymbol{w} \in V.\end{aligned} \tag{4.10}$$

Lemma 4.2. For all $\boldsymbol{g} \in C(0,T;V)$, $\boldsymbol{\eta} \in C(0,T;V)$, $\mathcal{P}_\eta^2$ has a unique solution with the regularity $\boldsymbol{v}_{\eta \boldsymbol{g}} \in C(0,T;V)$.

Proof . Using Riesz Representation Theorem, we may define an element $\mathbf{F} \in C(0,T;V)$ by

$$(\mathbf{F}(t), \boldsymbol{v})_V = (\mathbf{f}(t), \boldsymbol{v})_V - (\eta(t), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \ \forall \boldsymbol{v} \in V.$$

We define the operator $A : V \to V$ such that

$$(A\boldsymbol{u}, \boldsymbol{v}) = (\mathcal{A}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}}, \forall \boldsymbol{u}, \boldsymbol{v} \in V. \tag{4.11}$$

It follows from (4.11) and (3.7)(a) that for all $\boldsymbol{u}_1, \boldsymbol{u}_2 \in V$ and $\boldsymbol{v} \in V$ we have

$$\begin{aligned}\left\|(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{v})_V\right\| &= \left\|\left(\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_1\right)\right) - \mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_2\right)\right), \varepsilon(\boldsymbol{v})\right)\right\|_{\mathcal{H}}, \\ &\leq \left\|\mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_1\right)\right) - \mathcal{A}\left(\varepsilon\left(\boldsymbol{u}_2\right)\right)\right\|_{\mathcal{H}} \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \left\|\varepsilon\left(\boldsymbol{u}_1\right) - \varepsilon\left(\boldsymbol{u}_2\right)\right\|_{\mathcal{H}} \|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}}, \\ &\leq L_{\mathcal{A}} \left\|\boldsymbol{u}_1 - \boldsymbol{u}_2\right\|_V \|\boldsymbol{v}\|_V,\end{aligned}$$

which shows that $A : V \to V$ is Lipschitz continuous. Now, by (4.11) and (3.7)(b) we find

$$\begin{aligned}(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V &= (\mathcal{A}(\varepsilon(\boldsymbol{u}_1)) - \mathcal{A}(\varepsilon(\boldsymbol{u}_2)), \varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2))_{\mathcal{H}} \\ &\geq m_{\mathcal{A}} \|\varepsilon(\boldsymbol{u}_1) - \varepsilon(\boldsymbol{u}_2)\|^2_{\mathcal{H}} \geq C \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_V .\end{aligned}$$

And according to Korn's inequality, it comes

$$(A\boldsymbol{u}_1 - A\boldsymbol{u}_2, \boldsymbol{u}_1 - \boldsymbol{u}_2)_V \geq C \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_V ,$$

i.e., that $A : V \to V$ is a strongly monotone operator on V. And we can easily check that $\boldsymbol{v} \mapsto j(\boldsymbol{g}(t), \boldsymbol{v})$ is convex lower semicontinuous and proper. It follows from classical results for elliptic variational inequalities (see [4]) that there exists a unique $\boldsymbol{v}_{\eta \boldsymbol{g}} \in V$, which is a solution of (4.10). To establish its regularity by showing that $\boldsymbol{v}_{\eta\boldsymbol{g}} \in C(0, T; V)$. We let $t_1, t_2 \in [0, T]$ and denote $\boldsymbol{\eta}_i = \boldsymbol{\eta}(t_i)$, $\boldsymbol{g}_i = \boldsymbol{g}(t_i)$, $f_i = f(t_i)$ and $\boldsymbol{v}_i = \boldsymbol{v}_{\eta\boldsymbol{g}}(t_i)$. Using the relation (4.10) we find that

$$\begin{aligned}&(\mathcal{A}\varepsilon(\boldsymbol{v}_1) - \mathcal{A}\varepsilon(\boldsymbol{v}_2), \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \leq (f_1 - f_2, \boldsymbol{v}_1 - \boldsymbol{v}_2)_V \\ &+ (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2, \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} + j(\boldsymbol{g}_1, \boldsymbol{v}_2) - j(\boldsymbol{g}_1, \boldsymbol{v}_1) + j(\boldsymbol{g}_2, \boldsymbol{v}_1) - j(\boldsymbol{g}_2, \boldsymbol{v}_2) .\end{aligned} \tag{4.12}$$

Moreover, we have

$$(\mathcal{A}\varepsilon(\boldsymbol{v}_1) - \mathcal{A}\varepsilon(\boldsymbol{v}_2), \varepsilon(\boldsymbol{v}_1 - \boldsymbol{v}_2))_{\mathcal{H}} \geq m_{\mathcal{A}} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|^2_V . \tag{4.13}$$

From the definition of the functional j given by (3.24), we have

$$\begin{aligned}&j(\boldsymbol{g}_1, \boldsymbol{v}_1) - j(\boldsymbol{g}_1, \boldsymbol{v}_2) + j(\boldsymbol{g}_2, \boldsymbol{v}_1) - j(\boldsymbol{g}_2, \boldsymbol{v}_2) \\ &\quad \leq c_0^2 L_\nu \|\mu\|_{L^\infty(\Gamma_3)} \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V .\end{aligned} \tag{4.14}$$

Using these bounds in (4.12), we obtain

$$\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \leq C (\|f_1 - f_2\|_V + \|\eta_1 - \eta_2\|_{\mathcal{H}} + \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V) , \tag{4.15}$$

then the conclusion that $\boldsymbol{v}_{\eta\boldsymbol{g}} \in C([0, T]; V)$ follows from the continuity of f, $\boldsymbol{\sigma}$ and $\boldsymbol{g}$ in their respective spaces V and $\mathcal{H}$. □

With the help of Lemma 4.2, we are in a position to show the following existence and uniqueness result for Problem $\mathcal{P}^1_\eta$.

Lemma 4.3. There exists a unique solution to Problem $\mathcal{P}^1_\eta$ such that $\boldsymbol{u}_\eta \in C^1(0, T; V)$ and $\boldsymbol{\sigma}_\eta \in C(0, T; \mathcal{H})$.

Proof . We consider an operator $\Lambda_\eta : C(0, T; V) \to C(0, T; V)$ defined by

$$\Lambda_\eta \boldsymbol{g} = \boldsymbol{g}_\eta,\ \boldsymbol{g} \in C(0, T; V), \tag{4.16}$$

where

$$\boldsymbol{g}_\eta(t) = u_0 + \int_0^t \boldsymbol{v}_{\eta\boldsymbol{g}}(s) ds,\ t \in (0, T). \tag{4.17}$$

and $\boldsymbol{v}_{\eta\boldsymbol{g}}$ is the solution of (4.10). We will show that this operator has a unique fixed point $\boldsymbol{g}_\eta \in C([0, T]; V)$. To this end, let $\boldsymbol{g}_1, \boldsymbol{g}_2 \in C([0, T]; V)$ and denote by $\boldsymbol{v}_i = \boldsymbol{v}_{\eta\boldsymbol{g}_i}$, $i = 1, 2$, the corresponding solutions of (4.10). Using (4.16) and (4.17) we have

$$\|\Lambda_\eta \boldsymbol{g}_1(t) - \Lambda_\eta \boldsymbol{g}_2(t)\|_V \leq \int_0^t \|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V ds, \quad \forall t \in [0, T]. \tag{4.18}$$

Using estimates similar to those in the proof of Lemma 4.2 (see (4.12)-(4.15)) we see that

$$\|\boldsymbol{v}_1(s) - \boldsymbol{v}_2(s)\|_V \leq C \|\boldsymbol{g}_1(s) - \boldsymbol{g}_2(s)\|_V , \quad s \in [0, T]. \tag{4.19}$$

Taking into account (4.18) we obtain

$$\|\Lambda_\eta \boldsymbol{g}_1(t) - \Lambda_\eta \boldsymbol{g}_2(t)\|_V \leq C \int_0^t \|\boldsymbol{g}_1(s) - \boldsymbol{g}_2(s)\|_V \, ds, \quad \forall t \in [0, T]. \tag{4.20}$$

Let is introduce the following notations

$$\begin{cases} I_1 = \displaystyle\int_0^t \|\boldsymbol{g}_1(s) - \boldsymbol{g}_2(s)\|_V \, ds, \\ \vdots \\ I_k = \displaystyle\int_0^t \int_0^{s_{k-1}} \cdots \int_0^{s_1} \|\boldsymbol{g}_1(r) - \boldsymbol{g}_2(r)\|_V \,, \end{cases}$$

and by induction, by denoting by Λ_η^m the m power of the operator Λ_η, we obtain

$$\begin{aligned} &\left\|\Lambda_\eta^m \boldsymbol{g}_1(t) - \Lambda_\eta^m \boldsymbol{g}_2(t)\right\|_V \\ &\qquad \leq C^m \left(\sum_{k=1}^m C_m^k I^{m-k} \|\boldsymbol{g}_1(t) - \boldsymbol{g}_2(t)\|_V \right), \end{aligned}$$

for all $t \in (0, T)$ and $m \in \mathbb{N}$,

$$\begin{aligned} I^{m-k}\|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V &= \int_{(m-k)\text{ fois}} \cdot \int \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_V \\ &\leq \int_0^s \int \cdots \int_{(m-k)\text{ fois}} \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{C(0,T;V)} \\ &\leq \frac{t^{m-k}}{k!} \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{C(0,T;V)} \\ &\leq \frac{T^{m-k}}{k!} \|\boldsymbol{g}_1 - \boldsymbol{g}_2\|_{C(0,T;V)}, \end{aligned}$$

$$\begin{aligned} &\|\Lambda_\eta^m \boldsymbol{g}_1(t) - \Lambda_\eta^m \boldsymbol{g}_2(t)\|_{C(0,T;V)} \\ &\qquad \leq C^m \left(\sum_{k=1}^m C_m^k \frac{T^{m-k}}{k!} \|\boldsymbol{g}_1(t) - \boldsymbol{g}_2(t)\|_{C(0,T;V)} \right) \\ &\qquad \leq \frac{(CT)^m}{m!} \|\boldsymbol{g}_1(t) - \boldsymbol{g}_2(t)\|_{C(0,T;V)}, \end{aligned} \tag{4.21}$$

which implies that for a sufficiently large m the operator Λ_η^m is a contraction on $C(0,T;V)$, Thus Λ_η has a unique fixed point $\boldsymbol{g}_\eta^* \in C([0,T];V)$. Next, let $\boldsymbol{v}_\eta \in C([0,T];V)$ and $\boldsymbol{\sigma}_\eta \in C([0,T];\mathcal{H})$ be given by

$$\boldsymbol{v}_\eta = \boldsymbol{v}_{\eta \boldsymbol{g}_\eta^*}, \ \boldsymbol{\sigma}_\eta = \boldsymbol{\sigma}_{\eta \boldsymbol{g}_\eta^*} = \mathcal{A}\varepsilon\left(\boldsymbol{v}_{\eta \boldsymbol{g}_\eta^*}\right) + \boldsymbol{\eta}. \tag{4.22}$$

Moreover, using (4.17) and (4.22), we let $\boldsymbol{u}_\eta : [0,T] \to V$ be the function

$$\boldsymbol{u}_\eta(t) = \boldsymbol{u}_0 + \int_0^t \boldsymbol{v}_\eta(s) ds, \quad \forall t \in [0, T]. \tag{4.23}$$

Clearly, (4.7) and (4.9) are satisfied. Moreover, by (4.22), (4.23) and (4.16), (4.17), it follows that $\boldsymbol{u}_\eta = \boldsymbol{g}_\eta^*$ and $\dot{\boldsymbol{u}}_\eta = \boldsymbol{v}_\eta$. Therefore, if $\boldsymbol{g} = \boldsymbol{g}_\eta^*$ in (4.10), then we obtain (4.8).

To prove the regularity of $\boldsymbol{\sigma}_\eta$, we choose $\boldsymbol{w} = \boldsymbol{v}_{\eta \boldsymbol{g}}(t) \pm \varphi$ in (4.8) where $\varphi \in C_0^\infty(\Omega)^d$, $t \in (0, t)$ we find

$$(\boldsymbol{\sigma}_{\eta \boldsymbol{g}}, \varepsilon(\varphi))_{\mathcal{H}} = (\mathbf{F}, \varphi)_V, \quad \forall \varphi \in C_0^\infty(\Omega)^d, \text{ on } [0, T]. \tag{4.24}$$

Using (3.26) we deduce

$$\mathrm{Div}\,\sigma_\eta + \mathbf{f}_0 = 0, \quad \text{on } [0,T], \tag{4.25}$$

and then, assumption (3.19) and equation (4.25) imply that $\boldsymbol{\sigma}_\eta \in C([0,T];\mathcal{H})$. This establishes the existence part in Lemma 4.3. The uniqueness is a consequence of the uniqueness of the fixed point of the operator Λ_η defined by (4.16), (4.17) and the unique solvability of the Problem $\mathcal{P}_\eta^2$ and relations (3.7) and (4.7). □

In the second step we use the displacement field $\boldsymbol{u}_\eta$ obtained in Lemma 4.3, to construct the following variational problem for the an electrical potential.

Problem $\mathcal{P}_\eta^3$

Find an electrical potential $\varphi_\eta : (0,T) \to W$ such that

$$\begin{aligned}(B\nabla\varphi_\eta(t), \nabla\zeta)_H - (\mathcal{E}\varepsilon(\boldsymbol{u}_\eta(t)), \nabla\zeta)_H + (h(\boldsymbol{u}_\eta(t), \varphi_\eta(t)), \zeta)_W \\ = (q(t), \zeta)_W, \text{ for all } \zeta \in W, t \in (0,T).\end{aligned} \tag{4.26}$$

We have the following result for problem $\mathcal{P}_\eta^3$

Lemma 4.4. Problem (4.26) has unique solution φ_η which satisfies the regularity (4.2). Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (4.26) corresponding to $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C([0,T];\mathcal{H})$, then there exists $C > 0$ such that

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \le C\,\|\boldsymbol{u}_{\eta_1}(t) - \boldsymbol{u}_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \tag{4.27}$$

To prove the above lemma, we use an abstract existence and unique result which may be found in [15] .

For $\lambda \in C(0,T;L^2(\Omega))$, we consider the following variational problem.

Problem $\mathcal{P}_\lambda$

Find the temperature field $\theta_\lambda : (0,T) \to L^2(\Omega)$

$$\begin{aligned}\left(\dot\theta_\lambda(t), \mathbf{v}\right)_{L^2(\Omega)} + a_0(\theta_\lambda(t), \mathbf{v}) = (\lambda(t) + \rho(t), \mathbf{v})_{L^2(\Omega)}, \\ \forall \mathbf{v} \in L^2(\Omega), \text{ a.e. } t \in (0,T),\end{aligned} \tag{4.28}$$

$$\theta_\lambda(0) = \theta_0, \text{ in } \Omega. \tag{4.29}$$

Lemma 4.5. There exists a unique solution θ_λ to the auxiliary problem $\mathcal{P}_\lambda$ satisfying (4.4).

Proof .

By an application of the Poincaré-Friedrichs inequality, we can find a constant $B' > 0$ such that

$$\int_\Omega \|\nabla\zeta\|^2 dx + \frac{B}{k_0}\int_\Gamma \|\zeta\|^2 d\gamma \ge B' \int_\Omega \|\zeta\|^2 dx, \quad \forall \zeta \in V.$$

Thus, we obtain

$$\mathfrak{a}_0(\zeta,\zeta) \ge c_1\|\xi\|_V^2, \quad \forall \zeta \in V,$$

where $c_1 = k_0 min(1, B')/2$, which implies that a_0 is V-elliptic. Consequently, based on classical arguments of functional analysis concerning parabolic equations [14], the variational equation (4.28) has a unique solution θ_λ satisfying (4.4). □

In the third step we let $\chi \in L^2(0,T;L^2(\Omega))$

Problem $\mathcal{P}_\chi$

Find the damage field $\alpha_\chi : (0,T) \to L^2(\Omega)$ such that $\alpha_\chi(t) \in K$ and

$$\begin{aligned}&(\dot{\alpha}_\chi(t), \xi - \alpha_\chi)_{L^2(\Omega)} + a_1(\alpha_\chi(t), \xi - \alpha_\chi(t)) \\ &\qquad \geq (\chi(t), \xi - \alpha_\chi(t))_{L^2(\Omega)} \quad \forall \xi \in K, \text{ a.e. } t \in (0,T),\end{aligned} \tag{4.30}$$

$$\alpha_\chi(0) = \alpha_0. \tag{4.31}$$

For the study of problem $\mathcal{P}_\chi$, we have the following result.

Lemma 4.6. There exists a unique solution α_χ to the auxiliary problem $\mathcal{P}_\chi$ satisfying (4.5).

The above lemma follows from a standard result for parabolic variational inequalities, (see [25]).

Finally, as a consequence of these results and using the properties of the operator $\mathcal{G}$ the operator $\mathcal{E}$, the functions Θ and S for $t \in (0,T)$, we consider the element

$$\Lambda(\boldsymbol{\eta}, \lambda, \chi)(t) = \big(\Lambda^1(\boldsymbol{\eta}, \lambda, \chi)(t), \Lambda^2(\boldsymbol{\eta}, \lambda, \chi)(t), \Lambda^3(\boldsymbol{\eta}, \lambda, \chi)(t)\big) \in \mathcal{H} \times L^2(\Omega) \times L^2(\Omega), \tag{4.32}$$

defined by

$$\begin{aligned}\big(\Lambda^1(\boldsymbol{\eta}, \lambda, \chi)(t), \boldsymbol{v}\big)_{\mathcal{H}\times V} &= (\mathcal{B}(\varepsilon(\boldsymbol{u}(t), \theta_\lambda(t), \alpha_\chi(t)), \varepsilon(\boldsymbol{v}))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta(t), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} \\ &+ \left(\int_0^t \mathcal{M}(t-s, \boldsymbol{\varepsilon}(\boldsymbol{u}_\eta(s)), \theta_\lambda(s), \alpha_\chi(s))\, ds, \boldsymbol{\varepsilon}(\boldsymbol{v})\right)_{\mathcal{H}}, \forall \boldsymbol{v} \in V,\end{aligned} \tag{4.33}$$

$$\Lambda^2(\boldsymbol{\eta}, \lambda, \chi)(t) = \Theta(\boldsymbol{\sigma}_\eta, \varepsilon(\boldsymbol{u}_\eta(t)), \theta_\lambda(t), \alpha_\chi(t)). \tag{4.34}$$

$$\Lambda^3(\boldsymbol{\eta}, \lambda, \chi)(t) = S(\varepsilon(\boldsymbol{u}_\eta(t)), \alpha_\chi(t)). \tag{4.35}$$

Here, for every $(\boldsymbol{\eta}, \lambda, \chi) \in C(0,T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$. $\boldsymbol{u}_\eta$, φ_η, θ_λ, α_χ and $\boldsymbol{\sigma}_\eta$ represent the displacement field, the electric potential field, the temperature field, the damage field and the stress field , obtained in Lemmas 4.2, 4.3, 4.4,4.5 and 4.6 respectively. We have the following result.

Lemma 4.7. The mapping Λ has a fixed point $(\boldsymbol{\eta}, \lambda, \chi) \in C(0,T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega))$, such that $\Lambda(\boldsymbol{\eta}^*, \lambda^*, \chi^*) = (\boldsymbol{\eta}^*, \lambda^*, \chi^*)$.

Proof . Let $t \in (0,T)$ and $(\boldsymbol{\eta}_1, \lambda_1, \chi_1), (\boldsymbol{\eta}_2, \lambda_2, \chi_2) \in C\big(0,T; \mathcal{H} \times L^2(\Omega) \times L^2(\Omega)\big)$. We use the notation that $\boldsymbol{u}_{\eta_i} = \boldsymbol{u}_i$, $\dot{\boldsymbol{u}}_{\eta_i} = \boldsymbol{v}_{\eta_i} = \boldsymbol{v}_i$, $\theta_{\lambda_i} = \theta_i$, $\varphi_{\eta_i} = \varphi_i$, $\alpha_{\boldsymbol{\eta}} = \alpha_i$ and $\boldsymbol{\sigma}_{\eta_i,\theta_i} = \boldsymbol{\sigma}_i$ for $i = 1,2$.

Let us start by using (3.8), (3.9) and (3.14), we have

$$\begin{aligned}&\|\Lambda^1(\boldsymbol{\eta}_1, \lambda_1, \chi_1)(t) - \Lambda^1(\boldsymbol{\eta}_2, \lambda_2, \chi_2)(t)\|^2_{\mathcal{H}} \\ &\quad \leq \|\mathcal{B}(\varepsilon(\mathbf{u}_1(t)), \theta_1(t), \alpha_1(t)) - \mathcal{B}(\varepsilon(\mathbf{u}_2(t)), \theta_2(t), \alpha_2(t))\|^2_{\mathcal{H}} \\ &\quad + \|\mathcal{E}^*\nabla\varphi_1(t) - \mathcal{E}^*\nabla\varphi_2(t)\|^2_{\mathcal{H}} \\ &\quad + \int_0^t \|\mathcal{M}(t-s, \varepsilon(u_1(s)), \theta_1(s), \alpha_1(s)) - \mathcal{M}(t-s, \varepsilon(u_2(s)), \theta_2(s), \alpha_2(s))\|^2_{\mathcal{H}}\, ds, \\ &\quad \leq C\Big(\|\varphi_1(t) - \varphi_2(t)\|^2_W + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V + \|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \\ &\quad + \|\alpha_1(s) - \alpha_2(s)\|^2_{L^2(\Omega)} + \int_0^t \|\boldsymbol{u}_1(s) - \boldsymbol{u}_2(s)\|^2_V\, ds \\ &\quad + \int_0^t \|\theta_1(s) - \theta_2(s)\|^2_{L^2(\Omega)}\, ds + \int_0^t \|\alpha_1(s) - \alpha_2(s)\|^2_{L^2(\Omega)}\, ds\Big).\end{aligned} \tag{4.36}$$

By similar arguments, from (4.34), (3.10) we obtain

$$\begin{aligned}&\big\|\Lambda^2(\boldsymbol{\eta}_1, \lambda_1, \chi_1)(t) - \Lambda^2(\boldsymbol{\eta}_2, \lambda_2, \chi_2)(t)\big\|^2_{\mathcal{H}} \\ &\leq C\Big(\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|^2_V + \|\boldsymbol{u}_1(t) - \boldsymbol{u}_2(t)\|^2_V \\ &\quad + \|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} + \|\alpha_1(t) - \alpha_2(t)\|^2_{L^2(\Omega)}\Big), \quad \text{a.e. } t \in (0,T).\end{aligned} \tag{4.37}$$

Similarly, using (4.35), (3.11) implies

$$\left\|\Lambda^3\left(\boldsymbol{\eta}_1, \lambda_1, \chi_1\right)(t)-\Lambda^3\left(\boldsymbol{\eta}_2, \lambda_2, \chi_2\right)(t)\right\|_{\mathcal{H}}^2 \leq C\left(\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\|\alpha_1(t)-\alpha_2(t)\|_{L^2(\Omega)}^2\right). \tag{4.38}$$

It follows now from (4.36), (4.37) and (4.38) that

$$\begin{aligned}
&\|\Lambda\left(\boldsymbol{\eta}_1, \lambda_1, \chi_1\right)(t)-\Lambda\left(\boldsymbol{\eta}_2, \lambda_2, \chi_2\right)(t)\|_{\mathcal{H}}^2 \\
&\quad\leq C\Big(\|\varphi_1(t)-\varphi_2(t)\|_W^2+\|\boldsymbol{\sigma}_1(t)-\boldsymbol{\sigma}_2(t)\|_{\mathcal{H}}^2 \\
&\quad+\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V^2+\int_0^t\|\boldsymbol{u}_1(s)-\boldsymbol{u}_2(s)\|_V^2\, ds \\
&\quad+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2+\int_0^t\|\theta_1(s)-\theta_2(s)\|_{L^2(\Omega)}^2\, ds \\
&\quad+\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|_{L^2(\Omega)}^2\, ds\Big).
\end{aligned} \tag{4.39}$$

Taking into account that

$$\boldsymbol{\sigma}_i(t)=\mathcal{A}\left(\varepsilon\left(\dot{\boldsymbol{u}}_i(t)\right)\right)+\boldsymbol{\eta}_i(t), \quad \forall t \in[0, T], \tag{4.40}$$

it follows that

$$\begin{aligned}
&\left(\mathcal{A}\left(\varepsilon\left(\boldsymbol{v}_1(s)\right)\right)-\mathcal{A}\left(\varepsilon\left(\boldsymbol{v}_2(s)\right)\right), \varepsilon\left(\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\right)\right)_{\mathcal{H}} \\
&\quad\leq j\left(\boldsymbol{v}_1(s), \boldsymbol{v}_2(s)\right)+j\left(\boldsymbol{v}_2(s), \boldsymbol{v}_1(s)\right)-j\left(\boldsymbol{v}_1(s), \boldsymbol{v}_1(s)\right)-j\left(\boldsymbol{v}_2(s), \boldsymbol{v}_2(s)\right) \\
&\qquad-\left(\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s), \varepsilon(\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s))\right)_{\mathcal{H}}.
\end{aligned}$$

So, by using (3.7), (3.2) and (3.24), we deduce that

$$\begin{aligned}
m_{\mathcal{A}}\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2 \leq & c_0^2 L_\nu\|\mu\|_{L^\infty(\Gamma_3)}\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2 \\
&+\|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V^2,
\end{aligned}$$

which, implies

$$\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V \leq C\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\|_{\mathcal{H}}. \tag{4.41}$$

Moreover, from (4.23), we obtain

$$\|\boldsymbol{u}_1(t)-\boldsymbol{u}_2(t)\|_V \leq \int_0^t\|\boldsymbol{v}_1(s)-\boldsymbol{v}_2(s)\|_V\, ds. \tag{4.42}$$

So, using the inequality above, we find

$$\begin{aligned}
\int_0^t\|\mathbf{u}_1(s)-\mathbf{u}_2(s)\|_V\, ds &\leq C \int_0^t \int_0^s\|\eta_1(r)-\eta_2(r)\|_{\mathcal{H}}\, dr ds \\
&\leq \int_0^T\|\eta_1(s)-\eta_2(s)\|_{\mathcal{H}}\, ds.
\end{aligned} \tag{4.43}$$

From (4.28) we deduce that

$$\left(\dot{\theta}_1-\dot{\theta}_2, \theta_1-\theta_2\right)_{L^2(\Omega)}+a_0\left(\theta_1-\theta_2, \theta_1-\theta_2\right)+\left(\lambda_1-\lambda_2, \theta_1-\theta_2\right)_{L^2(\Omega)}=0.$$

We integrate this equality with respect to time, using the initial conditions $\theta_1(0)=\theta_2(0)=\theta_0$ and inequality $a_0\left(\theta_1-\theta_2, \theta_1-\theta_2\right) \geq 0$ to find

$$\frac{1}{2}\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2 \leq \int_0^t\left(\lambda_1(s)-\lambda_2(s), \theta_1(s)-\theta_2(s)\right)_{L^2(\Omega)} ds,$$

which implies that

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)} \le \int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\theta_1(s)-\theta_2(s)\|^2_{L^2(\Omega)}\,ds.$$

This inequality combined with Gronwall's inequality leads to

$$\|\theta_1(t)-\theta_2(t)\|^2_{L^2(\Omega)} \le C\int_0^t \|\lambda_1(s)-\lambda_2(s)\|^2_{L^2(\Omega)}\,ds. \tag{4.44}$$

Form (4.30), deduced that

$$\begin{aligned}(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{L^2(\Omega)} &+ a_1(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\\ &\le (\chi_1-\chi_2,\alpha_1-\alpha_2)_{L^2(\Omega)}, \ \text{a.e. } t\in(0,T).\end{aligned}$$

integrate inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$, and inequality

$$a_1(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\ge 0$$

we find

$$\tfrac{1}{2}\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)} \le C\int_0^t (\chi_1(s)-\chi_2(s),\alpha_1(s)-\alpha_2(s))_{L^2(\Omega)}ds,$$

which implies

$$\begin{aligned}&\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)}\\ &\quad\le C\left(\int_0^t \|\chi_1(s)-\chi_2(s)\|^2_{L^2(\Omega)}\,ds + \int_0^t \|\alpha_1(s)-\alpha_2(s)\|^2_{L^2(\Omega)}\,ds\right).\end{aligned}$$

This inequality combined with the Gronwall inequality leads to

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{L^2(\Omega)} \le C\int_0^t \|\chi_1(s)-\chi_2(s)\|^2_{L^2(\Omega)}\,ds \ ,\forall t\in[0,T]. \tag{4.45}$$

Form the previous inequality and estimates (4.45), (4.44), (4.43) and (4.39) it follows now that

$$\begin{aligned}&\|\Lambda(\boldsymbol{\eta}_1,\lambda_1,\chi_1)(t)-\Lambda(\boldsymbol{\eta}_2,\lambda_2,\chi_2)(t)\|^2_{\mathcal{H}\times L^2(\Omega)\times L^2(\Omega)}\\ &\quad\le C\int_0^T \|(\boldsymbol{\eta}_1,\lambda_1,\chi_1)(s)-(\boldsymbol{\eta}_2,\lambda_2,\chi_2)(s)\|^2_{\mathcal{H}\times L^2(\Omega)\times L^2(\Omega)}\,ds.\end{aligned} \tag{4.46}$$

Reiterating this inequality m times we obtain

$$\begin{aligned}&\|\Lambda^m(\boldsymbol{\eta}_1,\lambda_1,\chi_1)-\Lambda^m(\boldsymbol{\eta}_2,\lambda_2,\chi_2)\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))}\\ &\quad\le \frac{C^mT^m}{m!}\|(\boldsymbol{\eta}_1,\lambda_1,\chi_1)-(\boldsymbol{\eta}_2,\lambda_2,\chi_2)\|^2_{C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))}.\end{aligned}$$

Thus, for m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$, and so Λ has a unique fixed point. □

Now we have every thing that is required to prove Theorem 4.1.

Existence

Let $(\boldsymbol{\eta}^*,\lambda^*,\chi^*)\in C(0,T;\mathcal{H}\times L^2(\Omega)\times L^2(\Omega))$ be the fixed point of Λ defined by (4.32)-(4.35) and let $(\boldsymbol{u}_\eta,\boldsymbol{\sigma}_\eta)$ be the solution of problem $\mathcal{P}^1_\eta$.Let φ_η be the solution of problem $\mathcal{P}^3_\eta$ for $\eta=\eta^*$, let θ_{λ^*} be the solution of problem $\mathcal{P}_\lambda$ for $\lambda=\lambda^*$ and let α_{χ^*} be the solution of problem $\mathcal{P}_\chi$ for $\chi=\chi^*$. The equalities $\Lambda^1(\eta^*,\lambda^*,\chi^*)=\eta^*$, $\Lambda^2(\eta^*,\lambda^*,\chi^*)=\lambda^*$ and $\Lambda^3(\eta^*,\lambda^*,\chi^*)=\chi^*$ combined with (4.33)-(4.35) show that (3.29)-(3.33) are satisfied. Next (3.34) and the regularity (4.1)-(4.5) follow from Lemmas 4.3, 4.4, 4.5 and 4.6. Which concludes the existence part of the theorem.

Uniqueness

The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator Λ. and the unique solvability of the Problems $\mathcal{P}_\eta^1$, $\mathcal{P}_\eta^3$, $\mathcal{P}_\lambda$ and $\mathcal{P}_\chi$ which completes the proof.

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