Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 221–225
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.19508.2083

Maps preserving strong Jordan multiple $*$-product on $*$-algebras

Ali Taghavi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, P. O. Box 47416-1468, Babolsar, Iran

(Communicated by Ali Jabbari)

Abstract

Let $\mathcal{A}$ be an arbitrary $*$-algebra with unit I over the real or complex field $\mathbb{F}$ that contains a nontrivial idempotent P_1 and $n \geq 1$ a natural number and $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ be a surjective map on $\mathcal{A}$ such that φ satisfies condition

$$\varphi(P) \bullet_{n-1} \varphi(P) \bullet \varphi(A) = P \bullet_{n-1} P \bullet A,$$

for every $A \in \mathcal{A}$ and projection $P \in \{P_1,\ I - P_1\}$, where $A \bullet_{n-1} A$ with repeat $n-1$ times A is the Jordan multiple $*$-product. Then $\varphi(A) = \varphi(I)A$ for all $A \in \mathcal{A}$ and $\varphi(I)^2 = I$.

Keywords: Maps Preserving, Strong Jordan Multiple, $*$-Product
2020 MSC: 46J10, 47B48

1 Introduction and Preliminaries

Let $\mathcal{A}$ be a $*$-algebra. For $A, B \in \mathcal{A}$, we define Jordan $*$-product and Lie $*$-product of A, B respectively by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. The products are found playing a more and more important role in some recent researches and studying the new products was the main focus of many mathematicians over the past years (see [1, 2, 3, 8, 10, 12, 13, 14, 15]). Continuing it in [2], let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras. The authors studied nonlinear bijective map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfying $\Phi([A, B]_*) = [\Phi(A), \Phi(B)]_*$ for all $A, B \in \mathcal{A}$ if and only if Φ is a $*$-ring isomorphism. In [1] which $\mathcal{M}$ and $\mathcal{N}$ are two von Neumann algebras, it is proved that a not necessarily linear bijective map $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ satisfies $\varphi([S, T]_*) = [\varphi(T), \varphi(S)]_*$ for all $T, S \in \mathcal{M}$ if and only if φ is the direct sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. Also in [8] where $\mathcal{A}$ and $\mathcal{B}$ are two factor von Neumann algebras, it is characterized that a not necessarily linear bijective map $\Phi : \mathcal{A} \longrightarrow \mathcal{B}$ satisfies $\Phi(A \bullet B) = \Phi(A) \bullet \Phi(B)$ for all $A, B \in \mathcal{A}$ if and only if Φ is a $*$-ring isomorphism.

Let $\mathcal{R}$ be an associative ring (or an associative algebra over a field $\mathbb{F}$). Then recall a map $\varphi : \mathcal{R} \longrightarrow \mathcal{R}$ preserves strong commutativity or strong Lie Product if $[\varphi(A), \varphi(B)] = [A, B]$, for each $A, B \in \mathcal{A}$ that $[A, B]$ is Lie product i.e. $[A, B] = AB - BA$. Similarly φ preserves strong Jordan product if $\varphi(A) \circ \varphi(B) = A \circ B$, for each $A, B \in \mathcal{A}$ that $A \circ B$ is Jordan product i.e. $A \circ B = AB + BA$. The structure of linear (or nonlinear) maps that preserve strong commutativity and strong Jordan product have been investigated in [6, 12, 14]. Gonga et al [6] proved that every

Email address: taghavi@umz.ac.ir (Ali Taghavi)

Received: January 2020 *Accepted:* August 2021

nonlinear map φ that preserves strong Jordan product on any algebra $\mathcal{R}$ with unit I over a field $\mathbb{F}$, has the form of $\varphi(A) = \varphi(I)A$, for all $A \in \mathcal{R}$, where $\varphi(I) \in \mathcal{R}$ and $\varphi(I)^2 = I$.

For a ring $\mathcal{R}$ and a positive integer k, recall that the k-commutator of elements $A, B \in \mathcal{R}$ is defined by $[A, B]_k = [[A, B]_{k-1}, B]$ with $[A, B]_0 = A$ and $[A, B]_1 = [A, B] = AB - BA$; similarly we define $A \circ_k B = (A \circ_{k-1} B) \circ B$ with $A \circ_0 B = A$ and $A \circ_1 B = A \circ B = AB + BA$. A map $\varphi : \mathcal{R} \longrightarrow \mathcal{R}$ is called strong k-commutativity preserver if $[\varphi(A), \varphi(B)]_k = [A, B]_k$ for all $A, B \in \mathcal{R}$ and φ is called strong k-Jordan product if $\varphi(A) \circ_k \varphi(B) = A \circ_k B$ for each $A, B \in \mathcal{R}$. Qi [5], characterizes the structure of a strong 2-commutativity preserving map on prime algebra. Also Lin and Hou [9] characterized the structure of a map that preserves Strong 3-commutativity on standard algebras. Moreover recently in [15] authors proved the concrete form of a map that preserves strong 2-Jordan product on standard operator algebras, properly infinite von Neumann algebras and nest algebras.

The aim of this paper is to extend this work by studying surjective maps that preserves strong skew Jordan multiple $*$-product on general $*$-algebras. We prove that if $\mathcal{A}$ be an arbitrary $*$-algebra (with identity I) over the real or complex field $\mathbb{F}$ that contains a nontrivial idempotent P_1 and $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ satisfies below condition

$$\varphi(P) \bullet_{n-1} \varphi(P) \bullet \varphi(A) = P \bullet_{n-1} P \bullet A,$$

for every $A \in \mathcal{A}$ and projection $P \in \{P_1,\ I - P_1\}$, then $\varphi(A) = \varphi(I)A$ for all $A \in \mathcal{A}$ and $\varphi(I)^2 = I$, where, $n \geq 1$ is a natural number and $A \bullet_{n-1} A$ with repeat $n-1$ times A is the Jordan multiple $*$-product.

Now we are ready to state the main results of the present paper.

2 The Main Results

We begin by showing a preliminary lemma.

Lemma 2.1. Let $\mathcal{A}$ be an arbitrary $*$-algebra over the real or complex field $\mathbb{F}$ that contains a nontrivial idempotent P and $n \geq 1$ a natural number. If $P \bullet_{n-1} P \bullet A = 0$, then $PA = 0 = AP$.

Proof . Since $P \bullet_{n-1} P \bullet A = 0$ holds for $A \in \mathcal{A}$, by applying mathematical induction we conclude that $P \bullet_{n-1} P \bullet A = 2^{n-1} P \bullet A = 2^{n-1}(PA + AP) = 0$. Thus

$$AP + PA = 0. \tag{2.1}$$

Now by multiplying P from the left side and the right side of the Equation 2.1 we obtain $PAP + PA = 0$ and $AP + PAP = 0$. Again by using 2.1 we have $PAP = PA = AP = 0$. □ Following, we will state the main results and proofs.

Theorem 2.2. Let $\mathcal{A}$ be an arbitrary $*$-algebra with unit I over the real or complex field $\mathbb{F}$ that contains a nontrivial idempotent P_1 and $n \geq 1$ a natural number. Assume that $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ is a surjective map satisfying the condition

$$\varphi(P) \bullet_{n-1} \varphi(P) \bullet \varphi(A) = P \bullet_{n-1} P \bullet A, \tag{2.2}$$

for all $A \in \mathcal{A}$ and projection $P \in \{P_1,\ I - P_1\}$. Then $\varphi(A) = \varphi(I)A$ for all $A \in \mathcal{A}$ and $\varphi(I)^2 = I$.

Proof . We assume $P_2 = I - P_1$ and we organize the proof into several steps.

Step 1. φ is injective.

Let $A_1, A_2 \in \mathcal{A}$ such that $\varphi(A_1) = \varphi(A_2)$. By applying the Equation 2.2 we have

$$\begin{aligned} 0 &= \varphi(P) \bullet_{n-1} \varphi(P) \bullet (\varphi(A_1) - \varphi(A_2)) \\ &= P \bullet_{n-1} P \bullet (A_1 - A_2). \end{aligned}$$

From the Lemma 2.1, it follows that $PA_1 = PA_2$. In fact, $P_iA_1 = P_iA_2,\quad i = 1, 2$ which implies that $A_1 = A_2$. Consequently, φ is a injective map.

Step 2. i) $\varphi(A^*) = \varphi(A)^*$ for all $A \in \mathcal{A}$.

ii) $\varphi(P)^{n+1} = P$ for every $P \in \{P_1,\ P_2\}$.

i) Let $B \in \mathcal{A}$, $P \in \{P_1,\ P_2\}$ and $\varphi(P) \bullet_{n-1} \varphi(P) = C$. Then using Equation 2.2 and by applying mathematical induction, we obtain

$$\begin{aligned} CB + BC^* &= \varphi(P) \bullet_{n-1} \varphi(P) \bullet \varphi(\varphi^{-1}(B)) \\ &= P \bullet_{n-1} P \bullet \varphi^{-1}(B) \\ &= 2^{n-1} P \bullet \varphi^{-1}(B). \end{aligned}$$

for all $B \in \mathcal{A}$. That is

$$2^{n-1} P \bullet \varphi^{-1}(B) = CB + BC^*, \tag{2.3}$$

for all $B \in \mathcal{A}$. Similarly, for every $B^* \in \mathcal{A}$ we have

$$2^{n-1} P \bullet \varphi^{-1}(B^*) = CB^* + B^*C^*. \tag{2.4}$$

Now, by adjionting from the two sides of the Equation 2.3 we have

$$2^{n-1} P \bullet \varphi^{-1}(B)^* = B^*C^* + CB^*. \tag{2.5}$$

Thus with comparison the Equation 2.4 and 2.5 we can conclude that $P \bullet \varphi^{-1}(B^*) = P \bullet \varphi^{-1}(B)^*$ and so $P \bullet (\varphi^{-1}(B^*) - \varphi^{-1}(B)^*) = 0$ for all $B \in \mathcal{A}$. Then applying the Lemma 2.1 we obtain $P\varphi^{-1}(B^*) = P\varphi^{-1}(B)^*$ for all $B \in \mathcal{A}$ and $P \in \{P_1,\ P_2\}$. In fact, $P_i\varphi^{-1}(B^*) = P_i\varphi^{-1}(B)^*,\ i = 1,\ 2$ which implies that $\varphi^{-1}(B^*) = \varphi^{-1}(B)^*$ for all $B \in \mathcal{A}$.

Now, let $\varphi(A) = B$ so $\varphi^{-1}(B) = A$, hence we can conclude that $\varphi(A^*) = \varphi(\varphi^{-1}(B)^*) = \varphi(\varphi^{-1}(B^*)) = B^* = \varphi(A)^*$ for all $A \in \mathcal{A}$.

ii) From (i) we have $\varphi(P^*) = \varphi(P)^*$. Hence by choosing $A = P$ in the Equation 2.2 and by applying mathematical induction we obtain

$$2^n \varphi(P)^{n+1} = \varphi(P) \bullet_n \varphi(P) = P \bullet_n P = 2^n P,$$

which follows that $\varphi(P)^{n+1} = P$.

Step 3. For every $A \in \mathcal{A}$ and $P \in \{P_1,\ P_2\}$, we have

$$\varphi(P)\varphi(A) + \varphi(A)\varphi(P) = PA + AP. \tag{2.6}$$

It is easy to conclude from step 2 that $P_i\varphi(P_j) = \varphi(P_j)P_i,\ i, j = 1, 2$. Again, using the Step 2 we can compute

$$\begin{aligned} 2^{n-1}(\varphi(p)^n\varphi(A) + \varphi(A)\varphi(P)^n) &= (\varphi(P) \bullet_n \varphi(P) \bullet \varphi(A)) \\ &= (P \bullet_n P \bullet A) \\ &= 2^{n-1}(PA + AP) \end{aligned}$$

for every $A \in \mathcal{A}$. That is

$$\varphi(P)^n\varphi(A) + \varphi(A)\varphi(P)^n = PA + AP. \tag{2.7}$$

We prove the result in two cases.

Case 1. Let $n = 2k - 1$ and $k \in \mathbb{N}$. By the Step 2 we have $(\varphi(P)^2)^k = P$. Since $\varphi(P)$ is selfadjoint, $\varphi(P)^2$ is k^{th} root of P and so $\varphi(P)^2 = P$. Thus 2.7 turns to Equation 2.6.

Case 2. Let $n = 2k$ and $k \in \mathbb{N}$. By replacing A by P_2 in Equation 2.7 and choosing $P = P_1$ we have

$$\varphi(P_1)^n\varphi(P_2) + \varphi(P_2)\varphi(P_1)^n = 0. \tag{2.8}$$

By multiplying $\varphi(P_1)$ from the right side and the left side of the Equation 2.8 we obtain

$$P_1\varphi(P_2)\varphi(P_1) + \varphi(P_1)P_1\varphi(P_2) = 0, \tag{2.9}$$

Multiplying the sides of the Equation 2.9 by $\varphi(P_1)$ from the left and once from the right respectively, we compute

$$\varphi(P_1)\varphi(P_2)P_1\varphi(P_1) + P_1\varphi(P_1)^2\varphi(P_2) = 0,$$

and

$$\varphi(P_2)P_1\varphi(P_1)^2 + P_1\varphi(P_1)\varphi(P_2)\varphi(P_1) = 0.$$

These follows that

$$\varphi(P_2)P_1\varphi(P_1)^2 = P_1\varphi(P_1)^2\varphi(P_2). \tag{2.10}$$

Now, by applying the Step 2 we have $P\varphi(P)^{n+2} = P\varphi(P) = \varphi(P)P$. Since $n+2$ is an even number, $P\varphi(P)^{n+2} = P\varphi(P)$ is a positive element and so 2.10 implies that

$$\varphi(P_2)P_1\varphi(P_1) = P_1\varphi(P_1)\varphi(P_2).$$

Using Equation 2.9 and this Equation we get

$$P_1\varphi(P_1)\varphi(P_2) = 0.$$

Hence, from the Step 2 we obtain $P_1\varphi(P_2) = 0$ and so $\varphi(P_1) = (P_1 + P_2)\varphi(P_1) = P_1\varphi(P_1)$. This means which $\varphi(P) = P\varphi(P)$ for every $P \in \{P_1,\ P_2\}$. Hence, positivity of $P\varphi(P)$ follows that $\varphi(P)$ is a positive element. Then $\varphi(P)^{n+1} = P$ implies that $\varphi(P)$ is $(n+1)^{th}$ root P and so $\varphi(P) = P$. It can be shown easily that 2.7 turns to Equation 2.6. So in both cases the result is proved.

Step 4. $PA\varphi(P) = \varphi(P)AP$ for all $A \in \mathcal{A}$ and $P \in \{P_1,\ P_2\}$.

Let $A \in \mathcal{A}$ and $P \in \{P_1,\ P_2\}$. By multiplying $\varphi(P)$ from the left side and P from the right side of the Equation 2.6 and by applying the proof of the Step 3 we have

$$P\varphi(A)P + \varphi(P)\varphi(A)\varphi(P) = 2\varphi(P)AP. \tag{2.11}$$

In a similar way multiplying $\varphi(P)$ from the right side and P from the left side of the Equation 2.6 and applying the proof of the Step 3 we have

$$P\varphi(A)P + \varphi(P)\varphi(A)\varphi(P) = 2PA\varphi(P). \tag{2.12}$$

Then with comparison of two Equations 2.11 and 2.12, we conclude that $PA\varphi(P) = \varphi(P)AP$ for all $A \in \mathcal{A}$ and $P \in \{P_1,\ P_2\}$.

Step 5. $\varphi(A) = \varphi(I)A$ for all $A \in \mathcal{A}$.

By Step 4 $\varphi(P)$, commutes with $P\phi(A)P$ and so from Equation 2.11 we have

$$P\varphi(A)P = \varphi(P)AP. \tag{2.13}$$

On the other hand, replacing P in 2.6 by P_1 and by multiplying P_2 from the right side and $\phi(P)$ from the left side we have

$$P_1\varphi(A)P_2 = \varphi(P_1)AP_2. \tag{2.14}$$

Taking P in 2.13 by P_1 and then by adding the sides of this equation with the Equation 2.14, we obtain

$$P_1\varphi(A) = \varphi(P_1)A, \tag{2.15}$$

and similarly,

$$P_2\varphi(A) = \varphi(P_2)A. \tag{2.16}$$

In a similar way from the two Equations 2.15 and 2.16, we can compute

$$\varphi(A) = (\varphi(P_1) + \varphi(P_2))A. \tag{2.17}$$

Substituting $A = I$ in 2.17, we obtain $\varphi(I) = \varphi(P_1) + \varphi(P_2)$ and hence

$$\varphi(A) = \varphi(I)A, \qquad \text{for all } A \in \mathcal{A}, \qquad \text{and} \qquad \varphi(I)^2 = I.$$

This completes the proof. $\square$

Definition 2.3. Let $\mathcal{A}$ be an arbitrary algebra. An additive mapping $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ is called a left (resp.right) multiplier if $\varphi(xy) = \varphi(x)y$ (resp.$\varphi(xy) = x\varphi(y)$), holds for all $x, y \in \mathcal{A}$. A multiplier is an additive mapping which is both right as well as left multiplier.

The following corollary follows directly from Theorem 2.2.

Corollary 2.4. Let $\mathcal{A}$ be an arbitrary $*$-algebra with unit I over the real or complex field $\mathbb{F}$ that contains a nontrivial idempotent and $n \geq 1$ a natural number. Assume that $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$ is a surjective map satisfying in the conditione of Theorem 2.2, then φ is a multiplier.

Acknowledgments

The authors would like to thank anonymous referee for a thorough and detailed report with many helpful comments and suggestions.

References

[1] Z. Bai and S. Dou, *Maps preserving product $XY - YX^*$ on von Neuman algebras*, J. Math. Anal. Appl. **386** (2012), 103–109.

[2] J. Cui and C.K. Li *Maps preserving product $XY - YX^*$ on factor von Neuman algebras*, Linear Algebra Appl. **431** (2009), 833–842.

[3] L. Dai andF, Lu. *Nonlinear maps preserving Jordan $*$-products*, J. Math. Anal. Appl. **409** (2014), 180–188.

[4] L. Fang, *Linear maps preserving the idempotency of Jordan products of operators*, Linear Algebra Appl. **22** (2011), 767–779.

[5] X. Qi, *Strong 2-commutativity preserving maps on prime rings*, Publ. Math. Debrecen **88** (2016), no. 1-2, 119–129.

[6] L. Gonga, X. Qi, J. Shao and F. Zhang, *Strong (skew) ξ-Lie commutativity preserving maps on algebras*, Cogent. Math. Statist. **2** (2015), no. 1, 1003175.

[7] H. Gao, *$*$-Jordan-triple muitiplicative surjective maps on B(H)*, J. Math. Anal. Appl. **401** (2013), 397–403.

[8] C. Li, F. Lu and X. Fang, *Nonlinear mappings preseving product $XY + YX^*$ on factor von Neuman algebra*, Linear Algebra Appl. **438** (2013), no. 5, 2339–2345.

[9] M.Y. Liu and J.C. Hou, *Strong 3-commutativity preserving maps on standard algebras*, Acta Math. Sinica English Ser. **33** (2017), no. 12, 1659–1670.

[10] L. Molnar, *Multiplicative Jordan triple isomorphisms on the self-adjoint elements of von Neuman algebras*, Linear Algebra Appl. **419** (2006), 586–600.

[11] X. Qi and J. Hou, *Additivity of Lie multiplictive maps on triangular algebras*, Linear Multilinear Algebra **59** (2011), 391–397.

[12] X. Qi, *Strong 3-commutativity preserving maps on prime rings*, Publ. Math. Debrecen **88** (2016), no. 2, 119–129.

[13] A. Taghavi, F. Kolivand and H. Rohi, *A note on η-Lie products preserving maps on some algebra*, Mediterr. J. Math. **14** (2017), no. 1, 1–10.

[14] A. Taghavi and F. Kolivand, *A note on strong skew Jordan product preserving maps on von Neuman algebras*, Period. Math. Hungar. **75** (2017), no. 2, 330–335.

[15] A. Taghavi and F. Kolivand, *Maps preserving strong 2-Jordan product on some algebras*, Asian-Eur. J. Math. **10** (2017), no. 3, 1750044.