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Pascal distribution series and its applications on parabolic starlike functions with positive coefficients

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(Dedicated to My brother Er. G.Malmurugu)

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Abstract

The purpose of this article is to make a connection between the Pascal distribution series and some subclasses of normalized analytic functions whose coefficients are probabilities of the Pascal distribution. To be more precise, we investigate such connections with the classes of parabolic starlike and uniformly convex functions with positive coefficients in the open unit disk $\mathbb{U}$.

Keywords: Starlike functions, Convex functions, Uniformly Starlike functions, Uniformly Convex functions, Hadamard product, Pascal distribution series.
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1 Introduction and Preliminary Results

Let $\mathcal{A}$ be the class of analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}. \tag{1.1}$$

We also let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{V}$ the subclass of $\mathcal{S}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1.2}$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.$$

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We recall the following two subclasses of analytic functions with positive coefficients $\mathcal{V}$, defined and studied by Uralegaddi et al., [24] extensively.

A function $f \in \mathcal{V}$ is said to be starlike of order α $(1 \le \alpha < \frac{4}{3})$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha \quad (z \in \mathbb{U}).$$

This function class is denoted by $\mathcal{VS}^*(\alpha)$. We also write $\mathcal{VS}^*(0) =: \mathcal{VS}^*$, where $\mathcal{VS}^*$ denotes the class of functions $f \in \mathcal{V}$ that $f(\mathbb{U})$ is starlike with respect to the origin.

A function $f \in \mathcal{V}$ is said to be convex of order α $(1 \le \alpha < \frac{4}{3})$ if and only if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha \quad (z \in \mathbb{U}).$$

This class is denoted by $\mathcal{VK}(\alpha)$. Further, $\mathcal{VK} = \mathcal{VK}(0)$, the well-known standard class of convex functions.It is an established fact that $f \in \mathcal{VK}(\alpha) \Longleftrightarrow zf' \in \mathcal{VS}^*(\alpha)$.

Motivated by the earlier works of Ali et al., [1] and Murugusundaramoorthy et al., [11], we define the following two new subclasses $\mathcal{P}_\lambda(\gamma, \beta)$ and $\mathcal{Q}_\lambda(\gamma, \beta)$ of $\mathcal{V}$.

For some γ $(1 \le \gamma < \frac{4}{3})$, λ $(0 \le \lambda \le 1)$, $\beta \ge 0$ and functions of the form (1.1),we let $\mathcal{P}_\lambda(\gamma, \beta)$ be the subclass of $\mathcal{S}$ satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - \gamma\right) < \beta \left|\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right|, \ z \in \mathbb{U}$$

and also let $\mathcal{Q}_\lambda(\gamma, \beta)$, be the subclass of $\mathcal{S}$ satisfying the analytic criteria

$$\Re\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda zf'(z)} - \gamma\right) < \beta \left|\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda zf'(z)} - 1\right|, \ z \in \mathbb{U}.$$

Also let $\mathcal{VP}_\lambda(\gamma, \beta) = \mathcal{P}_\lambda(\gamma, \beta) \cap \mathcal{V}$ and $\mathcal{VQ}_\lambda(\gamma, \beta) = \mathcal{Q}_\lambda(\gamma, \beta) \cap \mathcal{V}$.

By suitably specializing the parameters λ, γ, β one can define the new subclasses as stated in the following Examples:

Example 1.1. For some γ $(1 \le \gamma < \frac{4}{3})$, $\beta \ge 0$ and choosing $\lambda = 1$ and functions of the form (1.2), we let $\mathcal{VP}_1(\gamma, \beta) \equiv \mathcal{VS}_{\mathcal{P}}(\gamma, \beta)$ be the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{f(z)} - \gamma\right) < \beta \left|\frac{zf'(z)}{f(z)} - 1\right| \ z \in \mathbb{U}.$$

and also let $\mathcal{VQ}_1(\gamma, \beta) \equiv \mathcal{VUC}(\gamma, \beta)$ be the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \gamma\right) < \beta \left|\frac{zf''(z)}{f'(z)}\right|, \ z \in \mathbb{U}.$$

Note that

$$\mathcal{VS}_{\mathcal{P}1}(\gamma, 0) \equiv \mathcal{VS}^*(\gamma) \ and \ \mathcal{VQ}_1(\gamma, 0) \equiv \mathcal{VK}(\gamma)$$

the subclasses studied by Uralegaddi et al., [24]and he proved the following necessary and sufficient conditions :

Lemma 1.2. [24] A function $f \in \mathcal{V}$ and of the form (1.2)
(i) belongs to the class $\mathcal{VS}^*(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}(n-\alpha)|a_n| \le \alpha - 1. \tag{1.3}$$

(ii) belongs to the class $\mathcal{VK}(\alpha)$ if and only if

$$\sum_{n=2}^{\infty}n(n-\alpha)|a_n| \le \alpha - 1. \tag{1.4}$$

By taking $\lambda = 0$ we define the following new subclasses of analytic functions with positive coefficients.

Example 1.3. For some $\gamma(1 \leq \gamma < \frac{4}{3})$, $\beta \geq 0, \lambda = 0$ and $f \in \mathcal{V}$ of the form (1.2),we let
(i) $\mathcal{VP}_0(\gamma, \beta) \equiv \mathcal{VSD}(\gamma, \beta)$ the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left(f'(z) - \gamma\right) < \beta\left|f'(z) - 1\right|, \qquad (z \in \mathbb{U})$$

and
(ii) $\mathcal{VQ}_0(\gamma, \beta) \equiv \mathcal{VCD}(\gamma, \beta)$ the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left((zf'(z))' - \gamma\right) < \beta\left|(zf'(z))' - 1\right|, \ z \in \mathbb{U}.$$

Example 1.4. For some $\gamma(1 \leq \gamma < \frac{4}{3})$,$\beta = 0, \lambda = 0$ and $f \in \mathcal{V}$ of the form (1.2),we let
(i) $\mathcal{VP}_0(\gamma, 0) \equiv \mathcal{VR}(\gamma)$ the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left(f'(z)\right) < \gamma, \qquad (z \in \mathbb{U})$$

and
(ii) $\mathcal{VQ}_0(\gamma, 0) \equiv \mathcal{VN}(\gamma)$ the subclass of $\mathcal{V}$ satisfying the analytic criteria

$$\Re\left(zf'(z) + f''\right) < \gamma, \ z \in \mathbb{U}.$$

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions (see[6, 8, 19, 21, 22, 23]). Recently there has been triggering interest to study the geometric properties of analytic functions associating with generalized distributions and Poisons distributions (see [2, 3, 10, 12, 13, 14, 15, 16, 17]). In our present study we establish connections between Pascal distribution series and Geometric Function Theory due to El-Deeb[7] and Bulboaca and Murugusundaramoorthy[5]. A variable x is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities $(1-q)^m$, $\frac{qm(1-q)^m}{1!}$, $\frac{q^2m(m+1)(1-q)^m}{2!}$, $\frac{q^3m(m+1)(m+2)(1-q)^m}{3!}$, $\ldots$ respectively, where$q$ and m are called the parameter,and thus

$$P(x = k) = \binom{k+m-1}{m-1} \cdot q^k(1-q)^m, k = 0, 1, 2, 3, \ldots$$

Very recently, El-Deeb[7] introduce a power series whose coefficients are probabilities of Pascal distribution

$$\Phi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{U} \tag{1.5}$$

where $m \geq 1; 0 \leq q \leq 1$ and we note that, by ratio test the radius of convergence of above series is infinity. Now, we consider the linear operator due to Bulboaca and Murugusundaramoorthy[5]

$$\mathcal{I}_q^m(z) : \mathcal{A} \to \mathcal{A}$$

defined by the convolution or hadamard product

$$\mathcal{I}_q^m f(z) = \Phi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \qquad z \in \mathbb{U} \tag{1.6}$$

Motivated by result on connections between various subclasses of analytic univalent functions associating with generalized distributions (see [2, 3, 10, 12, 13, 14, 15, 16, 17] and also the references cited therein), we establish a number of connections between the classes $\mathcal{VP}_\lambda(\gamma, \beta)$ and $\mathcal{VQ}_\lambda(\gamma, \beta)$ by applying the convolution operator given by (1.6) involving Pascal distribution.

2 Inclusion results

Lemma 2.1. (Characterization Property): A function $f \in \mathcal{V}$ and of the form (1.2)
(i) belongs to the class $\mathcal{VP}_\lambda(\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty}[n(1+\beta)-\lambda(\gamma+\beta)]|a_n| \leq \gamma-1. \tag{2.1}$$

(ii) belongs to the class $\mathcal{VQ}_\lambda(\gamma,\beta)$ if and only if

$$\sum_{n=2}^{\infty}n[n(1+\beta)-\lambda(\gamma+\beta)]|a_n| \leq \gamma-1. \tag{2.2}$$

Proof . Proof of Case (i):
Let f be of the form (1.1) belong to the class $\mathcal{VQ}_\lambda(\gamma,\beta)$. It suffices to show that

$$\Re\left(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\right)-\beta\left|\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\right| \leq \gamma-1.$$

We have

$$\begin{aligned}
&\Re\left(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\right)-\beta\left|\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\right| \\
&\leq \left|(1+\beta)\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\right| \\
&= \frac{(1+\beta)\sum\limits_{n=2}^{\infty}(n-\lambda)|a_n|}{1-\sum\limits_{n=2}^{\infty}\lambda|a_n|}.
\end{aligned}$$

The last expression is bounded above by $\gamma-1$ if

$$\sum_{n=2}^{\infty}[n(1+\beta)-\lambda(\gamma+\beta)]|a_n| \leq \gamma-1$$

Conversely,we need only to prove the if $f \in \mathcal{VP}_\lambda(\gamma,\beta)$ and z is real then

$$\Re\left(\frac{1+\sum\limits_{n=2}^{\infty}n\ a_nz^{n-1}}{1+\sum\limits_{n=2}^{\infty}\lambda\ a_nz^{n-1}}-\gamma\right) < \beta\left|\frac{\sum\limits_{n=2}^{\infty}(n-\lambda)\ a_nz^{n-1}}{1+\sum\limits_{n=2}^{\infty}\lambda\ a_nz^{n-1}}\right|.$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality

$$\sum_{n=2}^{\infty}[n(1+\beta)-\lambda(\gamma+\beta)]\ |a_n| \leq \gamma-1,$$

where $0 \leq \lambda < 1$, $0 \leq \gamma < 1$ and $\beta \geq 0$. This completes the proof of case(i).

Proof of Case (ii)
Let $f \in \mathcal{VQ}_\lambda(\gamma,\beta)$ be of the form (1.1).Then by definition we have

$$f \in \mathcal{VQ}_\lambda(\gamma,\beta) \Longleftrightarrow zf' \in \mathcal{VP}_\lambda(\gamma,\beta),$$

thus we have$f(z) = \left(z+\sum\limits_{n=2}^{\infty}(na_n)z^n\right) \in \mathcal{VP}_\lambda(\gamma,\beta)$. Hence by proceeding on lines similar to case (i), we easily get (2.2). □

By using the result of Uralegaddi et al., [24],we state the following remark:

Remark 2.2. A function $f \in \mathcal{V}$ and of the form (1.2) then $f \in \mathcal{VP}_\lambda(\gamma, \beta)$ if and only if

$$\sum_{n=2}^{\infty} \left[n - \lambda \left(\frac{\gamma+\beta}{1+\beta} \right) \right] |a_n| \leq \left(\frac{\gamma+\beta}{1+\beta} \right) - 1. \tag{2.3}$$

We note that $\mathcal{VS}^* \left(\frac{\gamma+\beta}{1+\beta} \right) \equiv \mathcal{VS}_{\mathcal{P}}(\gamma, \beta)$ $\mathcal{VK} \left(\frac{\gamma+\beta}{1+\beta} \right) \equiv \mathcal{VUC}(\gamma, \beta)$ and $\mathcal{USD}(\alpha) = \mathcal{VSD} \left(\frac{\gamma+\beta}{1+\beta} \right)$ and using the above identities we state (without proof) the following necessary and sufficient conditions for the subclasses defined in the Examples 1.1to 1.4.

Lemma 2.3. A function $f \in \mathcal{V}$ and of the form (1.2),then
(i) $f \in \mathcal{VS}_{\mathcal{P}}(\gamma, \beta)$ if and only if $\sum_{n=2}^{\infty} [n(1+\beta) - (\gamma+\beta)]|a_n| \leq \gamma - 1$
(ii)$f \in \mathcal{VUC}(\gamma, \beta)$ if and only if $\sum_{n=2}^{\infty} n[n(1+\beta) - (\gamma+\beta)]|a_n| \leq \gamma - 1$.

Lemma 2.4. A function $f \in \mathcal{V}$ and of the form (1.2), then
(i) $f \in \mathcal{VSD}(\gamma, \beta)$ if and only if $\sum_{n=2}^{\infty} n(1+\beta)|a_n| \leq \gamma - 1$
(ii) $f \in \mathcal{VCD}(\gamma, \beta)$ if and only if $\sum_{n=2}^{\infty} n^2(1+\beta)|a_n| \leq \gamma - 1$.

Lemma 2.5. A function $f \in \mathcal{V}$ and of the form (1.2)
(i) belongs to the class $\mathcal{VR}(\gamma)$ if and only if $\sum_{n=2}^{\infty} n|a_n| \leq \gamma - 1$
(ii) belongs to the class $\mathcal{VN}(\gamma)$ if and only if $\sum_{n=2}^{\infty} n^2|a_n| \leq \gamma - 1$.

For convenience throughout in the sequel,unless otherwise stated we let
$m \geq 1; 0 \leq q \leq 1$ γ $(1 \leq \gamma < \frac{4}{3})$, λ $(0 \leq \lambda \leq 1)$, $\beta \geq 0$ and we use the following notations:

$$\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} q^n = \frac{1}{(1-q)^m}; \quad \sum_{n=0}^{\infty} \binom{n+m-2}{m-2} q^n = \frac{1}{(1-q)^{m-1}};$$
$$\sum_{n=0}^{\infty} \binom{n+m}{m} q^n = \frac{1}{(1-q)^{m+1}}; \quad \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} q^n = \frac{1}{(1-q)^{m+2}} \tag{2.4}$$

Theorem 2.6. If $m \geq 1$ then $\Phi_q^m(z)$, is in the class $\mathcal{VP}_\lambda(\gamma, \beta)$ if and only if

$$\frac{(1+\beta)qm}{(1-q)} + [(1+\beta) - \lambda(\gamma+\beta)] (1 - (1-q)^m) \leq \gamma - 1. \tag{2.5}$$

Proof . Since $\Phi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m z^n \in \mathcal{VP}_\lambda(\gamma, \beta)$ by virtue of Lemma 2.1 and (2.1) it suffices to show that

$$\sum_{n=2}^{\infty} [n(1+\beta) - \lambda(\gamma+\beta)] \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \leq \gamma - 1.$$

Let $\mathfrak{L}_1(m, \lambda, \beta, \gamma) = \sum_{n=2}^{\infty} [n(1+\beta) - \lambda(\gamma+\beta)] \binom{n+m-2}{m-1} q^{n-1} (1-q)^m$, now by writing $n = (n-1) + 1$ we get

$$\begin{aligned}
\mathfrak{L}_1(m, \lambda, \beta, \gamma) =& (1+\beta) \sum_{n=2}^{\infty} n \binom{n+m-2}{m-1} q^{n-1} (1-q)^m - \lambda(\gamma+\beta) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} (1-q)^m \\
=& (1+\beta)(1-q)^m \sum_{n=2}^{\infty} (n-1) \binom{n+m-2}{m-1} q^{n-1} + (1-q)^m [(1+\beta) - \lambda(\gamma+\beta)] \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1} \\
=& (1+\beta)(1-q)^m \sum_{n=2}^{\infty} qm \binom{n+m-2}{m} q^{n-2} + (1-q)^m [(1+\beta) - \lambda(\gamma+\beta)] \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}
\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_1(m,\lambda,\beta,\gamma) =&(1+\beta)(1-q)^m\sum_{n=0}^{\infty} qm\binom{n+m}{m}q^n + (1-q)^m[(1+\beta)-\lambda(\gamma+\beta)]\left(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1\right)\\ \leq&(1+\beta)(1-q)^m qm\frac{1}{(1-q)^{m+1}} + (1-q)^m[(1+\beta)-\lambda(\gamma+\beta)]\left(\frac{1}{(1-q)^m}-1\right)\\ \leq&\frac{(1+\beta)qm}{(1-q)} + [(1+\beta)-\lambda(\gamma+\beta)]\left(1-(1-q)^m\right).\end{aligned}$$

But this expression is bounded above by $\gamma - 1$ if and only if (2.5) holds. Thus the proof is complete. □

Theorem 2.7. If $m \geq 1$ then $\Phi_q^m(z)$, is in the class $\mathcal{VQ}_\lambda(\gamma,\beta)$ if and only if

$$\frac{(1+\beta)m(m+1)q^2}{(1-q)^2} + \frac{[3(1+\beta)-\lambda(\gamma+\beta)]qm}{1-q} + [(1+\beta)-\lambda(\gamma+\beta)]\left(1-(1-q)^m\right) \leq \gamma - 1.$$

Proof . Since $\Phi_q^m(z) = z + \sum\limits_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n \in \mathcal{VQ}_\lambda(\gamma,\beta)$ by virtue of Lemma 2.1 and (2.2) it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-\lambda(\gamma+\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \gamma - 1.$$

Let

$$\mathfrak{L}_2(m,\lambda,\beta,\gamma) = \sum_{n=2}^{\infty}\left(n^2(1+\beta)-n\lambda(\gamma+\beta)\right)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.$$

Writing $n = (n-1)+1$ and $n^2 = (n-1)(n-2)+3(n-1)+1$, we can rewrite the above term as

$$\begin{aligned}\mathfrak{L}_2(m,\lambda,\beta,\gamma) \;=\;& (1+\beta)(1-q)^m\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}\\ &+[3(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}\\ &+[(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}\end{aligned}$$

$$\begin{aligned}\mathfrak{L}_2(m,\lambda,\beta,\gamma)) \;=\;& (1+\beta)q^2(1-q)^m\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-3}\\ &+[3(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}qm(n-1)\binom{n+m-2}{m}q^{n-2}\\ &+[(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}\\ =\;& (1+\beta)q^2(1-q)^m\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-3}\\ &+[3(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}qm(n-1)\binom{n+m-2}{m}q^{n-2}\\ &+[(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}\end{aligned}$$

$$\begin{aligned}
&= (1+\beta)q^2(1-q)^m\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-3}\\
&\quad+[3(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}qm(n-1)\binom{n+m-2}{m}q^{n-2}\\
&\quad+[(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}\\
&= (1+\beta)m(m+1)q^2(1-q)^m\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n+[3(1+\beta)-\lambda(\gamma+\beta)]qm(1-q)^m\sum_{n=0}^{\infty}\binom{n+m}{m}q^n\\
&\quad+[(1+\beta)-\lambda(\gamma+\beta)](1-q)^m\left(\frac{1}{(1-q)^m}-1\right)\\
&= \frac{(1+\beta)m(m+1)q^2}{(1-q)^2}+\frac{[3(1+\beta)-\lambda(\gamma+\beta)]qm}{1-q}+[(1+\beta)-\lambda(\gamma+\beta)]\left(1-(1-q)^m\right).
\end{aligned}$$

But this expression is bounded above by $\gamma-1$ if and only if (2.6) holds. Thus the proof is complete. □

Corollary 2.8. If $m\geq 1$ then $\Phi_q^m(z)$, is in the class
(i) is in the class $\mathcal{VS_P}(\gamma,\beta)$ if and only if

$$\frac{(1+\beta)qm}{(1-q)(2-(1-q)^m)}\leq\gamma-1.$$

(ii) is in the class $\mathcal{VUC}(\gamma,\beta)$ if and only if

$$\frac{(1+\beta)m(m+1)q^2}{(2-(1-q)^m)(1-q)^2}+\frac{[3+2\beta-\gamma)]qm}{(1-q)(2-(1-q)^m)}\leq\gamma-1.$$

Proof . The proof follows by taking $\lambda=1$ and proceeding as in Theorems 2.6 and 2.7 respectively. □ By taking $\lambda=0$ in Theorem 2.6 and Theorem 2.7 we state the following:

Corollary 2.9. If $m\geq 1$ then $\Phi_q^m(z)$, is in the class
(i) is in the class $\mathcal{VSD}(\gamma,\beta)$ if and only if

$$(1+\beta)\left[\frac{qm}{(1-q)}+1-(1-q)^m\right]\leq\gamma-1$$

(ii) is in the class $\mathcal{VCD}(\gamma,\beta)$ if and only if

$$(1+\beta)\left[\frac{m(m+1)q^2}{(1-q)^2}+\frac{3qm}{1-q}+1-(1-q)^m\right]\leq\gamma-1.$$

3 Image Properties of $\mathcal{I}_q^m$ and $\mathcal{L}(m,z)$ Operators

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(\mu,\delta)$, $(\tau\in\mathbb{C}\backslash\{0\},\ 0<\mu\leq 1;\delta<1)$, if it satisfies the inequality

$$\left|\frac{(1-\mu)\frac{f(z)}{z}+\mu f'(z)-1}{2\tau(1-\delta)+(1-\mu)\frac{f(z)}{z}+\mu f'(z)-1}\right|<1,\quad(z\in\mathbb{U}).$$

The class $\mathcal{R}^\tau(\mu,\delta)$ was introduced earlier by Swaminathan [22](for special cases see the references cited there in) and obtained the following estimate.

Lemma 3.1. [23] If $f\in\mathcal{R}^\tau(\mu,\delta)$ is of form (1.1), then

$$|a_n|\leq\frac{2\,|\tau|\,(1-\delta)}{1+\mu(n-1)},\quad n\in\mathbb{N}\setminus\{1\}. \tag{3.1}$$

The bounds given in (3.1) is sharp.

Making use of the Lemma3.1, we will study the action of the Pascal distribution series on the class $\mathcal{VQ}_\lambda(\alpha,\beta)$ in the following theorem.

Theorem 3.2. If $m \geq 1$ and $f \in \mathcal{R}^\tau(\mu,\delta)$, if the inequality

$$\left[\frac{(1+\beta)qm}{(1-q)} + [(1+\beta)-\lambda(\gamma+\beta)]\left(1-(1-q)^m\right)\right] \leq \frac{\mu(\gamma-1)}{2\left|\tau\right|(1-\delta)} \tag{3.2}$$

is satisfied, then $\mathcal{I}_q^m f(z) \in \mathcal{VQ}_\lambda(\alpha,\beta)$.

Proof . Let f be of the form (1.1) belong to the class $\mathcal{R}^\tau(\mu,\delta)$. By virtue of Lemma 2.1 and (2.2) it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta)-\lambda(\gamma+\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n| \leq \gamma-1$$

Since $f \in \mathcal{R}^\tau(\mu,\delta)$ then by Lemma 3.1 we have

$$|a_n| \leq \frac{2\left|\tau\right|(1-\delta)}{1+\mu(n-1)}, \quad n \in \mathbb{N}\setminus\{1\}.$$

Let $\mathfrak{L}_3(m,\lambda,\beta,\gamma) = \sum_{n=2}^{\infty} n[n(1+\beta)-\lambda(\gamma+\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m|a_n|$

$$\leq \quad 2\left|\tau\right|(1-\delta)\sum_{n=2}^{\infty} n\frac{[n(1+\beta)-\lambda(\gamma+\beta)]}{1+\mu(n-1)}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.$$

Since $1+\mu(n-1) \geq n\mu$, we get

$$\mathfrak{L}_3(m,\lambda,\beta,\gamma) \quad \leq \quad \frac{2\left|\tau\right|(1-\delta)}{\mu}\sum_{n=2}^{\infty}[n(1+\beta)-\lambda(\gamma+\beta)]\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.$$

Proceeding as in Theorem 2.6, we get

$$\mathfrak{L}_3(m,\lambda,\beta,\gamma) \leq \frac{2\left|\tau\right|(1-\delta)}{\mu}\left[\frac{(1+\beta)qm}{(1-q)} + [(1+\beta)-\lambda(\gamma+\beta)]\left(1-(1-q)^m\right)\right].$$

But this expression is bounded above by $\gamma-1$ if and only if (3.2) holds. Thus the proof is complete. □ Putting $\lambda = 1$ and proceeding as in Theorem 3.2 ,we obtain the next special case:

Corollary 3.3. If $m \geq 1$ and $f \in \mathcal{R}^\tau(\mu,\delta)$, if the inequality

$$\left[\frac{(1+\beta)qm}{(1-q)} - (\gamma-1)\left(1-(1-q)^m\right)\right] \leq \frac{\mu(\gamma-1)}{2\left|\tau\right|(1-\delta)} \tag{3.3}$$

is satisfied, then $\mathcal{I}_q^m f(z) \in \mathcal{VUC}(\gamma,\beta)$.

Corollary 3.4. If $m \geq 1$ and $f \in \mathcal{R}^\tau(\mu,\delta)$, if the inequality

$$(1+\beta)\left[\frac{qm}{1-q} + \left(1-(1-q)^m\right)\right] \leq \frac{\mu(\gamma-1)}{2\left|\tau\right|(1-\delta)}$$

is satisfied, then $\mathcal{I}_q^m f(z) \in \mathcal{VCD}(\gamma,\beta)$.

The proof follows by taking $\lambda = 0$ and proceeding as in Theorem 3.2.

Theorem 3.5. Let $m \geq 1$, then $\mathcal{L}(m,z) = \int_0^z \frac{\mathcal{I}_q^m(t)}{t} dt$ is belong to the class $\mathcal{VQ}_\lambda(\gamma,\beta)$ if and only if

$$\frac{(1+\beta)qm}{(1-q)} + [(1+\beta) - \lambda(\gamma+\beta)]\left(1-(1-q)^m\right) \leq \gamma - 1. \tag{3.4}$$

Proof . Since

$$\mathcal{L}(m,z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m \frac{z^n}{n}$$

then by Theorem 2.6 we need only to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - \lambda(\gamma+\beta)]\frac{1}{n}\binom{n+m-2}{m-1} q^{n-1}(1-q)^m \leq \gamma - 1.$$

That is,

$$\sum_{n=2}^{\infty}[n(1+\beta) - \lambda(\gamma+\beta)]\binom{n+m-2}{m-1} q^{n-1}(1-q)^m \quad \leq \quad \gamma - 1.$$

Now by writing $n = (n-1)+1$ and Proceeding as in Theorem 2.6, we get

$$\sum_{n=2}^{\infty}[n(1+\beta) - \lambda(\gamma+\beta)]\binom{n+m-2}{m-1} q^{n-1}(1-q)^m = \left[\frac{(1+\beta)qm}{(1-q)} + [(1+\beta) - \lambda(\gamma+\beta)]\left(1-(1-q)^m\right)\right]$$

which is bounded above by $\gamma - 1$ if and only if (3.4) holds. □

By fixing $\lambda = 1$ and $\lambda = 0$ in above theorem respectively, we state the following corollary:

Corollary 3.6. Let $m \geq 1$, then $\mathcal{L}(m,z) = \int_0^z \frac{\mathcal{I}_q^m(t)}{t} dt$ is belong to the class

1. $\mathcal{VUC}(\gamma,\beta)$ if and only if $\dfrac{(1+\beta)qm}{(1-q)(2-(1-q)^m)} \leq \gamma - 1$
   and
2. $\mathcal{VCD}(\gamma,\beta)$ if and only if $(1+\beta)\left(\frac{qm}{1-q} + 1 - (1-q)^m\right) \leq \gamma - 1$.

Concluding Remark: By specializing $\lambda = 0$ or $\lambda = 1$ analogously one can deduce above results for various subclasses with positive coefficients similar to the classes defined in [4, 20] . Further,by taking $\beta = 0$ and specializing $\lambda = 0$ or $\lambda = 1$ we can deduce above results for the subclasses studied in [24].

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References

[1] R.M. Ali, K.G. Subramanian, V. Ravichandran and O. Ahuja, *Neighborhoods of starlike and convex functions associated with parabola*, J. Inequal. Appl. **2008** (2008), Article ID 346279, 9 pages.

[2] S. Altinkaya and S. Yalcin, *Poisson distribution series for certain subclasses of starlike functions with negative coefficients*, An. Univ. Oradea Fasc. Mat. **24** (2017), no. 2, 5–8.

[3] S. Altinkaya and S. Yalcin, *Poisson distribution series for analytic univalent functions*, Complex Anal. Oper. Theory **12** (2018), no. 5, 1315–1319.

[4] R.Bharati, R.Parvatham and A. Swaminathan, *On subclasses of uniformly convex functions and corresponding class of starlike functions*, Tamkang J. Math. **26** (1997), no. 1, 17–32.

[5] T.Bulboaca and G. Murugusundaramoorthy, *Univalent functions with positive coefficients involving Pascal distribution series*, Commun. Korean Math. Soc. **35** (2020), no. 3, 867–877.

[6] N.E. Cho, S.Y.Woo and S. Owa, *Uniform convexity properties for hypergeometric functions*, Fract. Cal. Appl. Anal. **5** (2002), no. 3, 303–313.

[7] S.M. El-Deeb, T. Bulboaca and J. Dziok, *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J. **59** (2019), 301–314.

[8] E. Merkes and B.T. Scott, *Starlike hypergeometric functions*, Proc. Amer. Math. Soc. **12** (1961) 885–888.

[9] A.O. Mostafa, *A study on starlike and convex properties for hypergeometric functions*, J. Inequal. Pure Appl. Math. **10** (2009), no. 3, 1–16.

[10] G. Murugusundaramoorthy, *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28** (2017), no. 7-8, 1357–1366.

[11] G. Murugusundaramoorthy , K. Vijaya and K. Uma, *Subordination results for a class of analytic functions involving the Hurwitz-Lerch zeta function*, Int. J. Nonlinear Sci. **10** (2010), no. 40, 430–437.

[12] G. Murugusundaramoorthy, K. Vijaya and S. Porwal, *Some inclusion results of certain subclasses of analytic functions associated with Poisson distribution series*, Hacet. J. Math. Stat. **45** (2016), no. 4, 1101–1107.

[13] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. **2014** (2014), 1–3.

[14] S. Porwal, *Mapping properties of certain subclasses of analytic functions associated with generalized distribution series*, Appl. Math. E-Notes, **20** (2020), 39–45.

[15] S. Porwal, *An application of a Poisson distribution series on certain analytic functions*, J. Complex Anal. **2014** (2014), 1-3.

[16] S. Porwal and M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Mat. **27** (2016), 10–21.

[17] S.Porwal and S.Kumar, *Confluent hypergeometric distribution and its applications on certain classes of univalent functions*, Afr. Mat. **28** (2017), 1–8.

[18] T.Rosy, B.A. Stephen, K.G. Subramanian and H.Silverman, *Classes of convex functions*, Internat. J. Math., Math. Sci. **23** (2000), no. 12, 819—825.

[19] H. Silverman, *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl., **172** (1993), 574–581.

[20] K.G. Subramanian, G.Murugusundaramoorthy, P. Balasubrahmanyam and H. Silverman, *Subclasses of uniformly convex and uniformly starlike functions*, Math. Japonica, **42** (1995), no. 3, 517–522.

[21] H.M. Srivastava, G. Murugusundaramoorthy and S. Sivasubramanian, *Hypergeometric functions in the parabolic starlike and uniformly convex domains*, Integral Transform Spec. Funct. **18** (2007), 511–520.

[22] A. Swaminathan, *Certain suffienct conditions on Gaussian hypergeometric functions,*, J. Inequal. Pure Appl. Math. **5** (2004), no. 4, 1–10.

[23] T. Thulasiram, K. Suchithra, T.V. Sudharsan and G. Murugusundaramoorthy, *Some inclusion results associated with certain subclass of analytic functions involving Hohlov operator*, Rev. R. Acad. Cienc.Exactas Fis. Nat. Ser. A Math. **108** (2014), 711–720.

[24] B.A. Uralegaddi, M.D. Ganigi and S.M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25** (1994), no. 3, 225–230.