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Some types of Smarandache filters of a Smarandache BH-algebra

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Abstract

In this paper, the notions of a Smarandache p-filter, a Smarandache n-fold p-filter, Smarandache q-filter, a Smarandache-n-fold q-filter of a Smarandache BH-Algebra are introduced. Some properties of them with some theorems, proportions and examples are given.

Keywords: BCK-algebra, BH-algebra, Smarandache filter.
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1. Introduction

The idea of BCK-algebras was formulated first in [4, 5]. In the same year another algebraic structure called BCI-algebra which was a popularization of a BCK-algebra was given by K. Iséki [6]. In 1983, Hu and Li introduced the notion of a BCH-algebra which was a popularization of BCK/BCI-algebras [8, 11]. Hoo show that the notions of an ideal and a filter in a BCI-algebra [7]. A BH-algebra is an algebraic structure introduced by Jun et al in [10] which was a popularization of BCH/BCI/BCK-algebras. The notions of a Smarandache BCI-algebra, Smarandache ideal of a Smarandache BCI-algebra are given by Jun in [9]. Abbass and Dahham introduced the concept of completely closed filter of a BH-algebra in [1]. Abbass and Luhaib introduced the idea of Smarandache filter of a Smarandache BH-Algebra in [3]. In this paper, the notions of a Smarandache-p-filter, a Smarandache n-fold p-filter, Smarandache q-filter, a Smarandache-n-fold q-filter and of a Smarandache BH-Algebra are given.

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2. Preliminaries

In this section, several basic connotations about a BCI-algebra, a BCK-algebra, a Smarandache BH-algebra, and a Smarandache filter of a Smarandache are reviewed.

Definition 2.1. *[9] A BCI-algebra is an algebra* $(Ӿ, \square, 0)$ *, where* X *is a nonempty set,* $\square$ *is a binary operation and 0 is a constant, for all* $x, y, z \in Ӿ$*, satisfying the following axioms:*
i. $((x \square y) \square (x \square z)) \square (z \square y) = 0$,
ii. $(x \square (x \square y)) \square y = 0$,
iii. $x \square x = 0$,
iv. $x \square y = 0$ *and* $y \square x = 0$ *imply* $x = y$.

Definition 2.2. *[8] BCK-algebra is a BCI-algebra satisfying the axiom:* $0 \square x = 0$*, for all* $x \in Ӿ$.

Definition 2.3. *[10] A BH-algebra is a nonempty set* $Ӿ$ *with a constant 0 and a binary operation* $\square$ *satisfying the following conditions:*
i. $x \square x = 0$*, for all* $x \in Ӿ$.
ii. $x \square y = 0$ *and* $y \square x = 0$ *imply* $x = y$*, for all* $x, y \in Ӿ$.
iii. $x \square 0 = x$*, for all* $x \in Ӿ$.

Definition 2.4. *[10] A nonempty subset* S *of a BH-algebra* $Ӿ$ *is called a subalgebra of* $Ӿ$ *if* $x \square y \in S$*, for all* $x, y \in S$.

Definition 2.5. *[1] A filter of a BH-algebra* $Ӿ$ *is a non-empty subset* F *of* $Ӿ$ *such that:*
(F_1 **)** *if* $x \in F$ *and* $y \in F$*, then* $y \square (y \square x) \in F$ *and* $x \square (x \square y) \in F$.
(F_2 **)** *If* $x \in F$ *and* $x \square y = 0$ *then* $y \in F$ *for all* $y \in Ӿ$.
Further F *is a closed filter if* $0 \square x \in F$*, for all* $x \in F$.

Definition 2.6. *[2] Let* $Ӿ$ *be a BH-algebra and* F *be a filter of* $Ӿ$*. Then* F *is called a p-**filter** denoted by* $p - f$ *if it satisfies:*

$$\text{if } x, y \in F \text{ imply } (x \square z) \square (y \square z) \in F \text{ for all } y, z \in Ӿ.$$

Definition 2.7. *[2] Let* F *be a filter of a BH-algebra* $Ӿ$*. If* $x, y \in F$ *and there exists a fixed* $n \in N$ *such that* $z^n \in Ӿ$ *imply* $(x \square z^n) \square (y \square z^n) \in F$*, for all* $z \in Ӿ$*. Then* F *is said to be a **n-fold p-filter** of* $Ӿ$.

Definition 2.8. *[2] Let* $Ӿ$ *be a BH-algebra and* F *be a filter of* $Ӿ$*. Then* F *is called a **q-filter** denoted by **q-f** if it satisfies:*

$$\text{If } x \square z \in F, y \in F \text{ imply } x \square (y \square z) \in F, \text{ for all } x, z \in Ӿ.$$

Definition 2.9. *[2] Let* $Ӿ$ *be a BH-algebra,* F *be a filter of* $Ӿ$*, and there exists a fixed* $n \in N$ *such that* $x \square z^n \in F, y \in F$*, for all* $x, z \in \in$ *imply* $x \square (y \square z^n) \in F$*. Then* F *is called a n-fold q-filter of* $Ӿ$.

Definition 2.10. *[3] A Smarandache BH-algebra is defined to be a BH-algebra* $Ӿ$ *in which there exists a proper subset* Q *of* $Ӿ$ *denoted by **S. BH-algebra** such that*
i. $0 \in Q$ *and* $|Q| \geq 2$.
ii. Q *is a BCK-algebra under the operation of* $Ӿ$.

Definition 2.11. *[3] A non-empty subset F of a S. BH-algebra X is called a* ***Smarandache filter*** *of X denoted by* ***S.f****, if it satisfies (F_1) and*

$$(F_3)\ \textit{If } x\in F \textit{ and } x\,\square\, y=0 \textit{ then } y\in F\ ,\ \forall y\in Q.$$

Proposition 2.12. *[3] Let X be a S. BH-algebra and let $\{F_\beta, \beta\in\Omega\}$ be a family of S.f of X. Then $\bigcap_{\beta\in\Omega} F_\beta$ is an S.f of X.*

Proposition 2.13. *[3] Let X be a S.f and let $\{F_i, i\in\lambda\}$ be a chain of S.f of X. Then $\bigcup_{\beta\in\Omega} F_\beta$ is a S.f of X.*

Theorem 2.14. *[3] Let X be a S. BH-algebra, and F be a S.f of X such that $x\,\square\, y\neq 0$, for all $y\notin F$ and $x\in F$. Then F is a filter of X.*

3. Main Results

In this section, the notions of a Smarandache-p-filter, a Smarandache n-fold p-filter, Smarandache q-filter, a Smarandache-n-fold q-filter and of a Smarandache BH-Algebra of a Smarandache BH-Algebra are introduced. Also, some properties of these notions are studied.

Definition 3.1. *Let X be a S. BH-algebra and F be a Smarandache filter of X. Then F is called a Smarandache p-filter of X and denoted by* ***S.p-f*** *of X if it satisfies:*

$$\textit{If } x, y\in F \textit{ imply } (x\,\square\, z)\,\square\,(y\,\square\, z)\in F \textit{ for all } z\in Q.$$

Further F is a Smarandache closed p-filter if $0\,\square\, x\in F$, for all $x\in F$.

Example 3.2. *Let $X=\{0,1,\ 2,\ 3\}$. Define $\square$ as follows:*

$\square$	0	1	2	3
0	0	0	2	3
1	1	0	1	2
2	2	2	0	1
3	3	3	2	0

where $Q=\{0,1\}$, the subset $F=\{0,1,2\}$ is a S.P.f of X. But is not p.f of X, since $z=3, x=3, y=0$, $(3\,\square\,3)\,\square\,(0\,\square\,3)=3\notin F$.

Proposition 3.3. *Let X be a S. BH-algebra and F be a p-f of X. Then F is a S.p-f of X.*

Proof . Directly since $Q\subseteq X$. □

Theorem 3.4. *Let X be a S. BH-algebra, and F be a S.p-f of X such that $x\,\square\, y\neq 0$, $y\notin F$ if $(x\,\square\, z)\,\square\,(y\,\square\, z)\notin F$ and $x\in F, z\in X$. Then F is a p.f of X.*

Proof . Let F be a S.p-f of X it follows that By Definition3.1 is a S.f of $\mathcal{X}$. Since $x \Box y \neq 0, y \notin F, x \in F$, By Theorem 2.14, F is a filter of $\mathcal{X}$.

Now, let $x, y \in F, z \in X$, then we have two cases:

Case (I): If $z \in Q$, imply $(x \Box z) \Box (y \Box z) \in F$ because by definition 3.1 F is S.p-f of $\mathcal{X}$,

Cases(II): If $z \notin Q$, then either $(x \Box z) \Box (y \Box z) \in \notin F$ or $(x \Box z) \Box (y \Box z) \in F$.

Suppose $(x \Box z) \Box (y \Box z) \notin F$, then $y \notin F$, this is a contradiction. Thus $(x \Box z) \Box (y \Box z) \in F$. Therefore, is a p.f of $\mathcal{X}$. $\square$

Proposition 3.5. *Let $\mathcal{X}$ be a Smarandache BH-algebra, and let $\{F_\beta,\ \beta \in \Omega\}$ be a family of S.p-fs of $\mathcal{X}$. Then $\bigcap_{\beta \in \Omega} F_\beta$ is a S.p-f of $\mathcal{X}$.*

Proof . Let $\{F_\beta,\ \beta \in \Omega\}$ be a family of S.p-fs of $\mathcal{X}$, imply $\{F_\beta, \beta \in \Omega\}$ be a family of Smarandache filters of X. Hence, By Proposition 2.12, $\bigcap_{\beta \in \Omega} F_\beta$ is a S.f of $\mathcal{X}$. Now, let $x, y \in \bigcap_{\beta \in \Omega} F_\beta$ and $z \in Q$. Then $x, y \in F_\beta$ and $z \in Q, \forall \beta \in \Omega$ implies that $(x \Box z) \Box (y \Box z) \in F_\beta$, $\forall \beta \in \Omega$, because F_β is a S.p-f of $\mathcal{X}$, for all $\beta \in \Omega$, this mean that $(x \Box z) \Box (y \Box z) \in \bigcap_{\beta \in \Omega} F_\beta$. Therefore $\bigcap_{\beta \in \Omega} F_\beta$ is a S.p-f of $\mathcal{X}$. $\square$

Example 3.6. *Let $\mathcal{X} = \{0, 1, 2, 3, 4, 5\}$. Define $\Box$ as follows:-*

$\Box$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	1
2	2	2	0	0	1	0
3	3	2	2	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

where $Q = \{0, 2\}$. The subset $F_1 = \{0, 2, 3\}$ and $F_2 = \{0, 2, 5\}$ are two S.p-f of X, but $F_1 \cup F_2 = \{0, 2, 3, 5\}$ is not a S.p-f of $\mathcal{X}$, since $x = 3, y = 5, z = 0 \in Q$ but $(3 \Box 0) \Box (5 \Box 0) = 1 \notin F_1 \cup F_2$,

Proposition 3.7. *Let $\mathcal{X}$ be a S. BH-algebra, and let $\{F_\beta,\ \beta \in \Omega\}$ be a chain of S.P.f of X. Then $\bigcup_{\beta \in \Omega} F_\beta$ is a S.P.f of $\mathcal{X}$.*

Proof . Let $\{F_\beta,\ \beta \in \Omega\}$ be a chain of S.P.f of X. it follows that $\{F_\beta,\ \beta \in \Omega\}$ be a chain of Smarandache filters of X [By definition 3.1]. This together with Proposition (2.13) implies that $\bigcup_{\beta \in \Omega} F_\beta$ is a Smarandache filter of X.

Now, let $x, y \in \bigcup_{\beta \in \Omega} F_\beta, z \in Q$, then there exists $F_n, F_m \in \{F_\beta,\ \beta \in \Omega\}$, such that $x \in F_j$ and $y \in F_k$. Then either $F_n \subseteq F_m$ or $F_m \subseteq F_n$. If $F_n \subseteq F_m$, it follows that $x, y \in F_m$ and $z \in Q$. So, there exists $m \in \Omega$ such that $(x \Box z) \Box (y \Box z) \in F_m$, because F_i is a S.P.f of X,($\forall \beta \in \Omega$). Then $(x \Box z) \Box (y \Box z) \in \bigcup_{\beta \in \Omega} F_\beta$. Similarly, $F_m \subseteq F_n$ implies that $\bigcup_{\beta \in \Omega} F_\beta$ is a S.P.f of $\mathcal{X}$. $\square$

Definition 3.8. *Let F be a Smarandache filter of a S. BH-algebra $\not{X}$. If $x, y \in F$ and there exists a fixed $n \in N$ such that $z^n \in Q$ imply $(x \square z^n) \square (y \square z^n) \in F$, for all $z \in Q$. Then F is said to be a Smarandache n-fold p-filter of X, denoted by a* ***S. n-fold. p-f*** *of $\not{X}$.*

Example 3.9. *Let $\not{X} = \{0, 1, 2, 3, 4\}$ be as in example 3.6. The filter $F = \{0, 2, 3\}$ is a S. 2-fold. p-f of $\not{X}$.*

Theorem 3.10. *Let $\not{X}$ be a S. BH-algebra, and F be a S. n-fold. p-f of X such that $x \square y \neq 0, y \notin F$ if $(x \square z^n) \square (y \square z^n) \notin F$ and $x \in F, z^n \in \not{X}$, for a fixed $n \in N$. Then F is a n-fold p-filter of $\not{X}$.*

Proof . Let F be a S. n-fold. P.f of $\not{X}$, then By Definition 3.8, F is a S.f of $\not{X}$. Since $x \square y \neq 0, y \notin F, x \in F$, By Theorem 2.14, F is a filter of X. Now, let $x, y \in F, z^n \in \not{X}$, then we have the following two cases:

Case (I): If $z^n \in Q$, then $(x \square z^n) \square (y \square z^n) \in F$, because by Definition 3.8, F is S. n-fold. P.f of $\not{X}$,

Cases(II): If $z^n \notin Q$, then either $(x \square z^n) \square (y \square z^n) \notin F$ or $(x \square z^n) \square (y \square z^n) \in F$.

Suppose that $(x \square z^n) \square (y \square z^n) \notin F$, then $y \notin F$, this a contradiction. Thus $(x \square z^n) \square (y \square z^n) \in F$, consequently F is a n-fold p-filter of $\not{X}$. □

Proposition 3.11. *Let X be a S. BH-algebra, and let $\{F_\beta,\ \beta \in \Omega\}$ be a family of S. n-fold. p-f of $\not{X}$. Then $\bigcap_{\beta \in \Omega} F_\beta$ is a S. n-fold. p-f of $\not{X}$.*

Proof . Straightforward. □

Proposition 3.12. *Let $\not{X}$ be a Smarandache BH-algebra, and let $\{F_\beta,\ \beta \in \Omega\}$ be a chain of S. n-fold. p-f of X. Then $\bigcup_{\beta \in \Omega} F_\beta$ is a* ***S. n-fold. p-f*** *of $\not{X}$.*

Proof . Straightforward. □

Definition 3.13. *Let X be a S. BH-algebra and F be a Smarandache filter of X. Then F is called* ***a Smarandache q-filter*** *and denoted by a* ***S.q-f*** *of X if it satisfies:- If $x \square z \in F, y \in F$ imply $x \square (y \square z) \in F$ for all $x, z \in Q$.*

Example 3.14. *Let $\not{X} = \{0, 1, 2, 3, 4\}$. Define $\square$ as follows:*

$\square$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	2
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Where $Q = \{0, 2\}$. The subset $F = \{0, 1, 2\}$ is a S.q-f of $\not{X}$ but it is not a q-filter of X. Since $x = 3, y = 0, z = 3$ and $3 \square (0 \square 3) = 3 \notin F$

Proposition 3.15. *Let X be a S. BH-algebra and F is a q-filter of X. Then F is a S.q.f of X .*

Proof . Since $Q \subseteq X$, the proof is clear. □

Remark 3.16. *Consider the Q_1- S. BH-algebra and Q_2-Smarandache BH-algebra X such that $Q_1 \subseteq Q_2$. The Q_1-Smarandache q-filter of X may be not a Q_2-Smarandache q-filter of X as in the following example. Consider $X = \{0,1,2,3\}$ in example 3.14, where $Q_1 = \{0,1\}$, $Q_2 = \{0,2,3\}$ are BCK-algebras and $Q_1 \subseteq Q2$: $F = \{0,1,2\}$ is a Q_1-Smarandache q-filter of X, but it is not Q_2-Smarandache q-filter of X. Since $x = 3, y = 2, z = 3$ implies that $3 \square (2 \square 3) = 3$, but $3 \notin F$.*

Proposition 3.17. *Let X be a S. BH-algebra and F be a S.q-f of X, such that $F \subseteq Q$. Then F is a subalgebra of X.*

Proof . Let $x, y \in F$. Since $z \in Q$, choose $z = 0$, we have $x = x \square 0 \in F, y \in F, x, 0 \in Q$, because $F \subseteq Q$. This Implies that $x \square (y \square 0) \in F$, because by Definition 3.13, F is a S.q.f of X. Then $x \square y \in F$. Hence, F is a subalgebra. □

Theorem 3.18. *Let X be a S. BH-algebra, and be a S.q-f of X such that $x \square y \neq 0, x \square z \notin F$, and $y \notin F if x \square (y \square z) \notin F$ and $x \in F, z \in X$. Then F is a q-filter of X.*

Proof . Let F be a S.q.f of X, then By Definition 3.13, it is a S.f of X. Since $x \square y \neq 0$, $y \notin F$, $x \in F$, By Theorem 2.14, F is a filter of X.

Now, let $x \square z \in F, y \in F, x, z \in X$, then we have the following two cases:

Case (I): If $x, z \in Q$, then by Definition 3.13, $x \square (y \square z) \in F$,

Cases(II): If $x, z \notin Q$, then either $x \square (y \square z) \notin F$ or $x \square (y \square z) \in F$.

If $x \square (y \square z) \notin F$, then $y = \in F$, or x$\square z \notin F$, contradiction. Since $x \square z \in F, y \in F$, we have $x \square (y \square z) \in F$. Hence, it is a q-filter of X. □

Proposition 3.19. *Let X be a S. BH-algebra, and let $\{F_\beta,\ \beta \in \Omega\}$ be a family of S.q-f of X. Then $\bigcap_{\beta \in \Omega} F_\beta$ is a S.q-f of X.*

Proof . Let $\{F_\beta,\ \beta \in \Omega\}$ be a family of S.q-fs of X, then By Definition 3.13, $\{F_\beta,\ \beta \in \Omega\}$ be a family of S.f of X. Thus, By Proposition 2.12, $\bigcap_{\beta \in \Omega} F_\beta$ is a S.f of X.

Now, let $x \square z \in \bigcap_{\beta \in \Omega} F_\beta,\ y \in \bigcap_{\beta \in \Omega} F_\beta$ such that $x, z \in Q$, it follows that $x \square z \in F_\beta, y \in F_\beta$, such that $x, z \in Q$, imply $x \square (y \square z) \in F_\beta,\ (\forall \beta \in \Omega)$, bevause F_i is a S.q-f of X. Hence, $x \square (y \square z) \in \bigcap_{\beta \in \Omega} F_\beta$. Therefore, $\bigcap_{\beta \in \Omega} F_\beta$ is a S.q-f of X. □

Remark 3.20. *Let X be a S. BH-algebra and let f_1, f_2 be a S.q.f of X. Then $f_1 \cup f_2$ is not necessary a S.q.f of X.*

Example 3.21. *Consider $X = \{0,1,2,3,4,5\}$ be as in example 3.6, where $Q = \{0,1\}$. The subset $F_1 = \{0,1,3\}$ and $F_2 = \{0,1,4\}$ are two S.q-fs of X, but $F_1 \cup F_2 = \{0,1,3,4\}$ is not a S.q-f of X, because $3, 4 \in F_1 \cup F_2$, but $3 \square (3 \square 4) = 2 \notin F_1 \cup F_2$. Then $F_1 \cup F_2$ it is not a S.q-f.*

Proposition 3.22. *Let $\mathcal{X}$ be a S. BH-algebra and let $\{F_\beta,\ \beta \in \Omega\}$ be a chain of S.q.f of X. Then $\bigcup_{\beta\in\Omega} F_\beta$ is a S.q.f of $\mathcal{X}$.*

Proof . Let $\{F_\beta,\ \beta \in \Omega\}$ be a chain of S.q.f of $\mathcal{X}$. Then by Definition 3.13 $\{F_\beta,\ \beta \in \Omega\}$ is a chain of S.f of X. Thus, by Proposition 2.13, $\bigcup_{\beta\in\Omega} F_\beta$ is a S.f of X,

Now, let $x \square z \in \bigcup_{\beta\in\Omega} F_\beta,\ y \in \bigcup_{\beta\in\Omega} F_\beta$, such that $x, z \in Q$, then there exist $F_n, F_m \in \{F_\beta :\ \beta \in \Omega\}$, such that $x \square z \in F_n$ and $y \in F_m$. Thus either $F_n \subseteq F_m$ or $F_m \subseteq F_n$.

If $F_n \subseteq F_m$, then $x \square z \in F_m, y \in F_m$, such that $x, z \in Q$, thus there exists $m \in \Omega$ such that $x \square (y \square z) \in F_m$, because F_β is a S.q.f of X, for all $\beta \in \Omega$. Consequently, $x \square (y \square z) \in \bigcup_{\beta\in\Omega} F_\beta$. Similarly, $F_m \subseteq F_n$. Hence, $\bigcup_{\beta\in\Omega} F_\beta$ is a S.q-f of X. □

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