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Julia sets are Cantor circles and Sierpinski carpets for rational maps

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Abstract

In this work, we study the family of complex rational maps which is given by

$$Q_\beta\left(z\right)=2\beta^{1-d}z^d-\frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}},$$

where d greater than or equal to 2 and $\beta\in\mathbb{C}\backslash\{0\}$ such that $\beta^{1-d}\neq 1$ and $\beta^{2d-2}\neq 1$. We show that $J(Q_\beta)$ is a Cantor circle or a Sierpinski carpet or a degenerate Sierpinski carpet, whenever the image of one of the free critical points for Q_β is not converge to 0 or ∞.

Keywords: Julia sets, Cantor circle, Sierpinski carpet, degenerate Sierpinski carpet.
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1. Introduction

In 1980s, McMullen is the first discovery the Julia set is cantor of circles from the maps as $F_\lambda\left(z\right)=z^2+\frac{\lambda}{z^3}$, where the Lambda is small and not equal to zero, see [10]. The authors came the generalization of the McMullen maps $F_\lambda\left(z\right)=z^m+\frac{\lambda}{z^n}$, which was studied by some people through some dynamical phenomena, see [5, 8, 17, 7, 14]. Devaney and other authors give "The Escape Trichotomy Theorem" for the orbits of the free critical points, see [4]. Fei and Jianxun have studied the following maps $f_\lambda\left(z\right)=\frac{z^d(z^{2d}-\lambda^{d+1})}{z^{2d}-\lambda^{3d-1}}$. Fei and Jianxun got several things, including Julia sets is quasi circles or cantor circles or Sierpinski carpet for the iterate of the free critical points, see[6, 16]. The appearance of the Julia sets is the Sierpinski carpet or the cantor circle of the McMullen map

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or a generalization of the McMullen map as in (see [17]). In [12, Appendix F], the Sierpinski carpet Julia set of the rational map have given by Milnor and Tan. In [3], the quasi symmetric geometry on Julia sets for rational maps of post-critically-finite was studied by Lyubich, Bonk and Merenkov. For more, the Julia sets are Sierpinski carpets of the rational maps, see [4].

2. Background and The Main Results

For any rational map F with degree great than or equal to two on the Riemann sphere $\mathbb{C}_\infty$=$\mathbb{C} \cup \{\infty\}$, the Julia set is closure {all repelling periodic points of F}. Let F^m be the $m-th$ iteration of F, for $m \in \mathbb{N}$. The Fatou set of F $\mathbb{F}(F)$ is the set of points that the family $\{F^m : m \in \mathbb{N}\}$ is normal family from Montel theorem, also $(\mathbb{C}_\infty \setminus \mathbb{F}(F))^{\complement}$ is the Julia set. If Γ is a subset of $\mathbb{C}_\infty$ consists of uncountably many simple closed curves which are homeomorphic to (Cantor middle third $\times$ unit circle), (in short $\complement \times S^1$). Then Γ is called Cantor circles. The standard Sierpinski carpet fractal is homeomorphic to a planar set, this set is Sierpinski carpet. $A\zeta \subset \mathbb{C}_\infty$ is a Sierpinski carpet iff ζ has empty interior and $\zeta = \mathbb{C}_\infty \backslash \cup_{m \in \mathbb{N}} V_m$, where $V_m \subseteq \mathbb{C}$ are disjoint Jordan disks for $\partial V_m \cap \partial V_\ell = \emptyset$ for $\ell \neq m$ and diameter $V_m \to 0$ as $m \to \infty$. If a compact set $\zeta \subset \mathbb{C}$ be the Sierpinski carpet except for the condition $\partial V_m \cap \partial V_\ell \neq \emptyset$, then ζ is degenerate Sierpinski carpet. In previous paper [1], we discuss the concept "Quasicircle" if I_0 or I_∞ contains one of the free critical points, we get the Julia set is quasicircle . Now we study to find β such that the Julia set is cantor circles whenever the free critical points does not lie in I_∞ and I_0 .

The main result will offer as follows:

Theorem 2.1. *Assume that the orbit of $Q_\beta(e_\beta)$ is attracted by the infinity or the origin. Then*

(A) If $Q_\beta(e_\beta) \in I_\infty$ or $Q_\beta(e_\beta) \in I_0$ but $e_\beta \notin I_\infty$ or $e_\beta \notin I_0$, then $J(Q_\beta)$ is a Cantor set of circles.

(B) If ${Q_\beta}^m(e_\beta) \in I_\infty$ or ${Q_\beta}^m(e_\beta) \in I_0$ for $m \geq 2$ and ${Q_\beta}^\ell(e_\beta) \notin I_\infty$ or ${Q_\beta}^\ell(e_\beta) \notin I_0$ for $0 \leq \ell < m$ and further,

(B1) If $\partial\ I_0 \cap \partial\ I_\infty = \emptyset$, then $J(Q_\beta)$ is a Sierpinski carpet.

(B2) If $\partial\ I_0 \cap \partial\ I_\infty \neq \emptyset$, then $J(Q_\beta)$ is a degenerated Sierpiński carpet.

From Theorem A, from the comparison between our family and McMullen maps, if the free critical orbits are attracted by the cycle to the origin or to infinity, then $J(F_\lambda)$ is neither to be a degenerate Sierpinski carpet nor a quasicircle, see Figure 1.

Theorem 2.2. *Assume that $J(Q_\beta)$ is a Cantor circles. Therefore any McMullen map is not topologically conjugate to Q_β corresponding to Julia sets.*

The punched region is called the McMullen domain if the Julia set of Q_β is a Cantor set of circles and $\beta \neq 0$. From Figure 2.

Theorem 2.3. *The McMullen domain exists in the family Q_β if and only if $d \geq 4$.*

3. Preliminaries

In this section, we give some of the theories that are an introduction and we need for our work.

Lemma 3.1. *[1] Let $Q_\beta(z) = 2\beta^{1-d} z^d - \frac{z^d\left(z^{2d}-\beta^{d+1}\right)}{z^{2d}-\beta^{3d-1}}$ and ω be a complex number satisfying $\omega^{2d} = 1$. Then $Q_\beta^m(\omega z) = \omega^{d^m} Q_\beta^m(z)$ for $m \geq 1$.*

Lemma 3.2. *[1] For any $\zeta(z) = \frac{\beta^2}{z}$. Then Q_β satisfies the equation $\zeta \circ Q_\beta(z) = Q_\beta \circ \zeta(z)$ $\forall\ z \in \mathbb{C}_\infty$.*

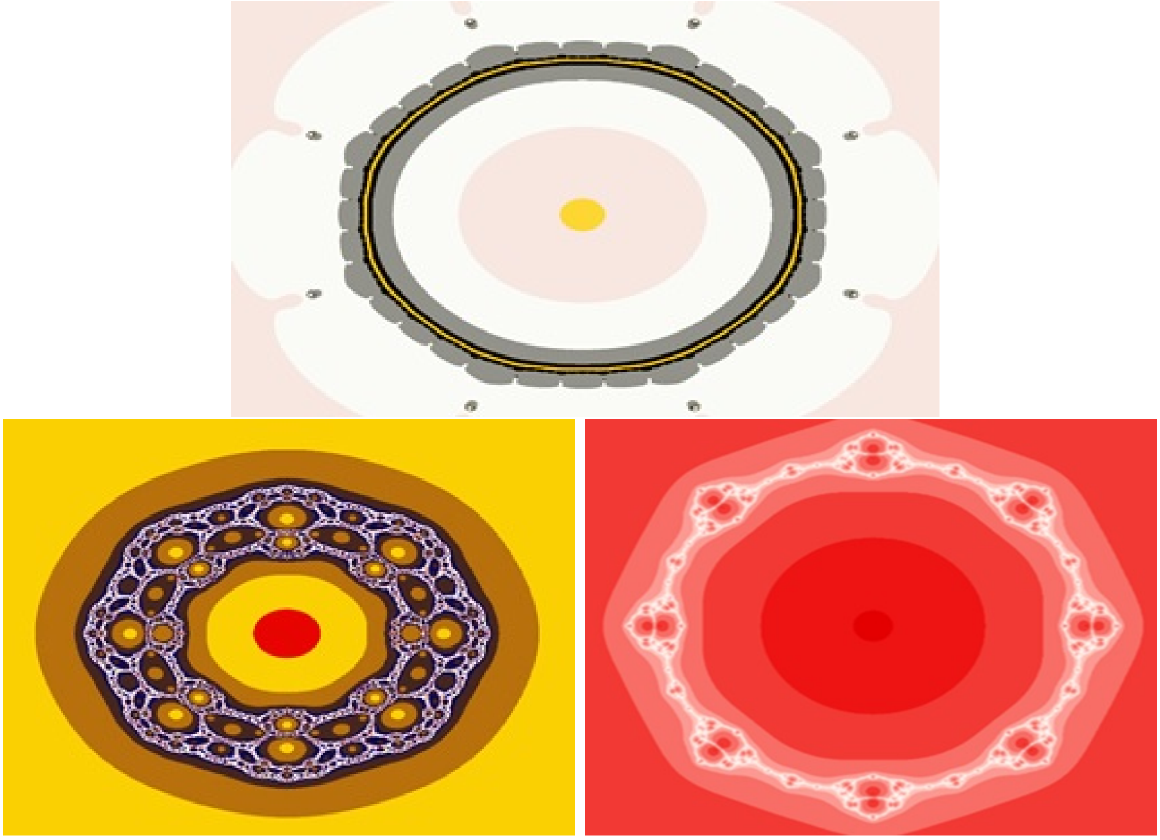
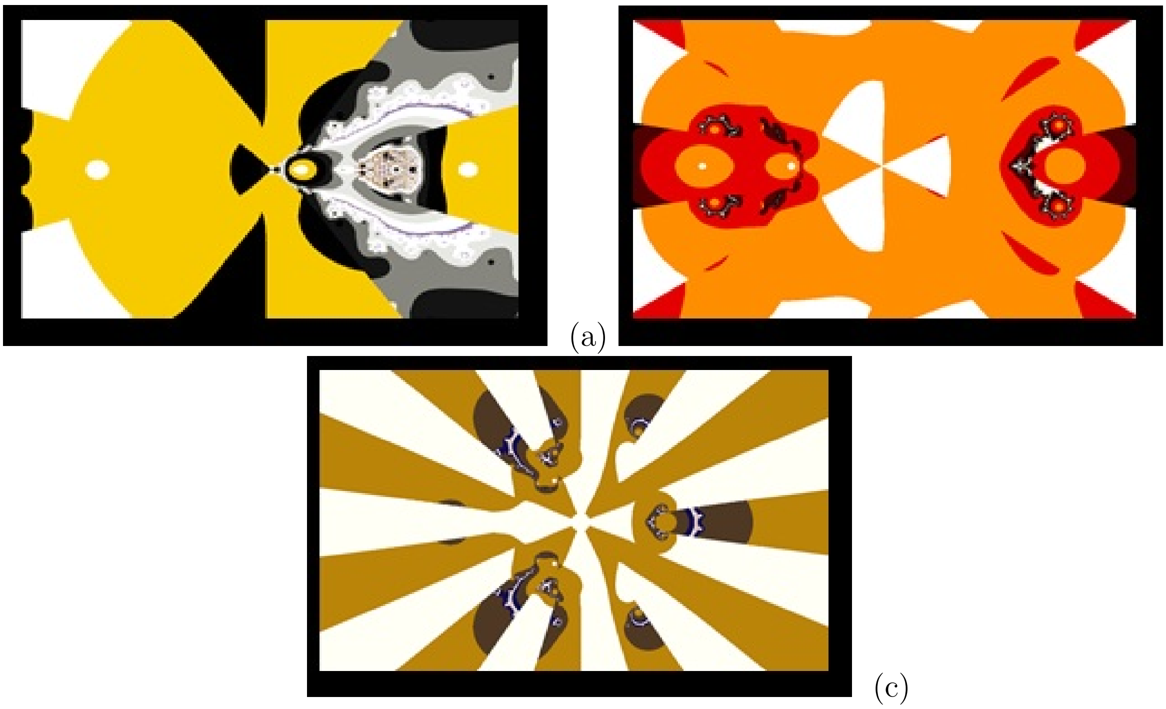


Figure 1: When $d = 4$. **Top:** $\beta = 0.799 + 0.8i$ **and** $J(Q_\beta)$ **is a Cantor circles; left Bottom:** $\beta \approx 1.151442$ **and** $J(Q_\beta)$ **is a Sierpinski carpet; right Bottom:** $\beta \approx 1.050$ **and** $J(Q_\beta)$ **is a degenerated carpet.**



(a) (b) (c)

Figure 2: The non-escaping loci of Q_β, **where** $d = 2, 3$ **and** 4. **(a) and (b) If** $d \leq 3$, **then** Q_β **has no McMullen domain .(c) If** $d \geq 4$, **then there is a McMullen domain centered at 0 this is a white disk.**

Corollary 3.3. *[1] Suppose that W is a Fatou component of Q_α, then $W = \eta(W)$. In special case , $\eta(I_0) = I_\infty$ and $\eta(I_\infty) = I_0$. Let $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$, we have CP $(Q_\beta) = \{0,\infty,e_\beta\}$, where e_β is the free critical points, there are two roots :*

$$e_\beta = \left(\frac{3\beta^{3d-1} - 4\beta^d - \beta^{d+1} - \sqrt{(3\beta^{3d-1} - 4\beta^d - \beta^{d+1})^2 - 4(1-2\beta^{1-d})(\beta^{4d} - 2\beta^{5d-1})}}{2-4\beta^{1-d}}\right)^{\frac{1}{8}}$$

From lemma 3.1 and 3.2, we can write the critical points as form

$$CP(Q_\beta) = \{\omega_0^m e_\beta, \omega_0^m \frac{\beta^2}{e_\beta} \ : 0 \le m \le 2d-1\},$$

where $\omega_0 = e^{\frac{pi}{d}}$, therefore Q_β contains only one free critical orbit.

Lemma 3.4. *[1] Both I_0 and I_∞ have $2d-$fold symmetry , i.e if $z \in I_0$ or I_∞ , then $\omega z \in I_0$ or I_∞ respectively, where ω satisfies $\omega^{2n} = 1$.*

Lemma 3.5. *[9] For any Fatou component V of Q_β. Assume that z_0 and $\omega^{m_0}z_0$ belong to V, where $\omega^{2d} = 1$ and $\omega^{m_0} \neq 1$. Then $\omega^m z_0 \in V$ for each m. In special case, V has $2d-$fold symmetry also surround 0.*

Proposition 3.6. *[1] Assume that $\beta \in \mathbb{R}$. Then $\mathcal{T}_{|\beta|} = \{z \in \mathbb{C}: |z| = |\beta|\}$ be the round circle and $Q_\beta : \mathcal{T}_{|\beta|} \to \mathcal{T}_{|\beta|}$. Moreover, $\mathcal{T}_{|\beta|} \subset J(Q_\beta)$ whenever the free critical orbits are attracted by ∞ and 0 .*

Theorem 3.7. *[1] Assume that $e_{\hat{a}} \in I_0$ or I_∞, then the $J(Q_{\hat{a}})$ is quasicircle.*

4. The Cantor of Circles

We will give the necessary and sufficient condition of Q_β such that $J(Q_\beta)$ is a cantor circles, by studying the location of the critical values and the critical points. We want to find β such that the Julia set of Q_β is the Cantor circles, the free critical points cannot remain in I_0 and I_∞ anymore.

In [1], if I_0 *or* I_∞ contains one of the free critical point, we get $J(Q_\beta)$ is quasicircle . Assume that $e_\beta \in \rho_0$ because ρ_0 is the complement of I_0 , where $\rho_0 = Q_\beta^{-1}(I_0)$ .Thus $\rho_0 \neq \emptyset$ and by Lemma 2.2 , it follows that $\frac{\beta^2}{e_\beta} \in \rho_\infty$. Therefore $\rho_\infty \neq \emptyset$. Q_β are both d *to* one on I_0 and I_∞.

Remark 4.1. *[1] Let $W \subset X$ be an open set of a topological space X and $V \Subset W$ an open, compactly contained set (i.e., $\overline{V}$ is compact and $\overline{V} \subset W$).*

Corollary 4.2. *[1] If I_0 is simple connected , then $\chi(I_0) = 1$. Where $\chi(.)$ is Euler characteristic.*

Proof . Since I_0 is simple connected. By [2, p.p 85] , hence $\chi(I_0) = 1$. □

Proposition 4.3. *I_0 and I_∞ are simply connected. However the two annuli ρ_0 and ρ_∞ around 0 are $2d-$fold symmetry .*

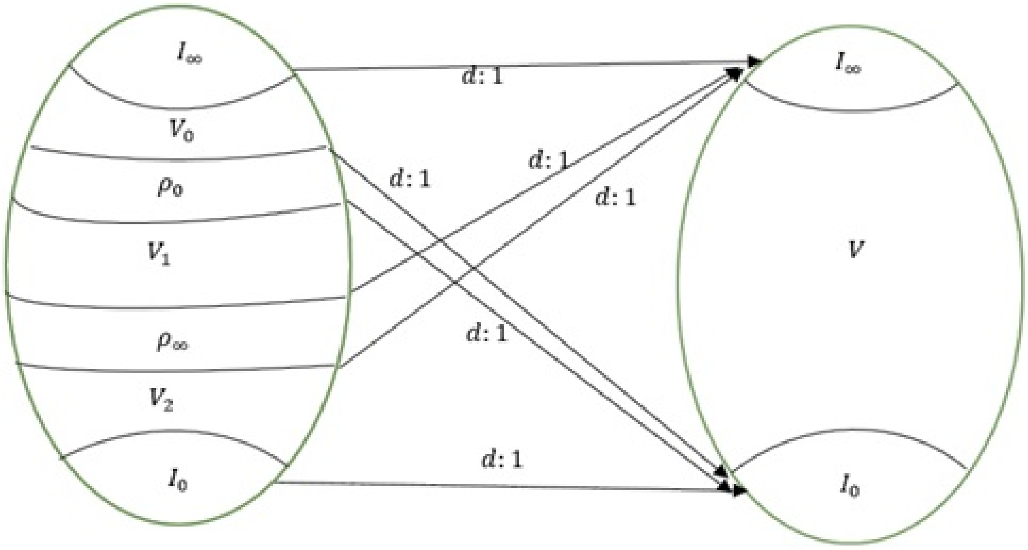


Figure 3: Sketch illustrating of the map relations of Q_β if $Q_\beta(e_\beta)\in I_0$,but $e_\beta\notin I_0$.

Proof . From [11, Theorem 8.9], the parabolic basin or the immediate attracting basin is either infinitely connected or simply connected. We assume that V_0 be small open disk around 0 by Remark 4.1 $Q_\beta(\overline{V_0}) \subset V_0 \subset I_0$ and the boundary of V_0 is Jordan curve containing no Q_β^m of critical points. Fix $V_m = Q_\beta^{-1}(V_0)$, where $Q_\beta^{-1}(V_0)$ are connected component which contains 0, hence $V_0 \subset V_1 \subset V_2 \ldots$,with $\bigcup_m V_m = I_0$. Now the map $Q_\beta : V_m\backslash\{0\} \to V_m\backslash\{0\}$ are covering by degree d^m, from the Riemann–Hurwitz's formula, $\chi(V_m) = \chi\left(\overset{\circ}{V_m}\right) + \chi(\partial V_m)$. Some finite Jordan curves are restricting for V_m and $(\mathbb{C}_\infty\backslash V_m)^{\complement}$ are disjoint union of Jordan disks, so $\chi(V_m) = 0$. So $V_m\backslash\{0\}$ is an annulus and from proof $V_m\backslash\{0\}$ is no containing Q_β^m of the critical points and $V_0\backslash\{0\}$ is an annulus, thus V_m is simply connected . Then $\bigcup_m V_m = I_0$ is simply connected. By Corollary 3.5, it follows that ρ_0 has either $2d$ component or one. Assume that ρ_0 has $2d$ component, then the origin has $5d$ preimages (that is $2d$ component + degree of Q_β is $3d$ =$5d$ preimages). Because the map from any component of ρ_0 to I_0 with degree two also there are $2d$ components . Which is contradict because degree of Q_β is $3d$. Thus ρ_0 has only one component and it is connected .By Riemann–Hurwitz's formula for $Q_\beta : \rho_0 \to I_0$ and by [11, Theorem 5.5.4], $\chi(\rho_0) + d_{Q_\beta}(\rho_0) = k\chi(I_0)$, so $\chi(\rho_0) + 2d = 2d\chi(I_0)$, since I_0 is simply connected .So by Corollary 4.2, $\chi(I_0) = 1$. Hence $\chi(\rho_0) = 0$ and ρ_0 is an annulus around 0 with $2d-$ fold symmetry . Similarly, by method we can show that the simply connected for I_∞ also ρ_∞ is an annulus around 0 with $2d-$ fold symmetry . □

For each M and N are two disjoint sets such that separate the origin and infinity . We define M precedes N ($M \prec N$) if the component $\mathbb{C}_\infty\backslash N$ contains M and the origin .

Proposition 4.4. *If ρ_0 , ρ_∞ are two annuli ,then $\rho_\infty \prec \rho_0$. However $\overline{\rho_\infty}, \overline{\rho_0}, \overline{I_0}$ and $\overline{I_\infty}$ are disjoint to each other.*

Proof . By definition of ρ_0 and ρ_∞ ,thus $\rho_\infty \cap \rho_0 = \emptyset$ and the intersection of I_∞ and I_0 is an empty set. Now we have two claims either $\rho_\infty \prec \rho_0$ or $\rho_0 \prec \rho_\infty$, since ρ_0 and ρ_∞ are separating the origin and the infinity. Assume that $\rho_0 \prec \rho_\infty$, for each V_0 is bounded component of $\mathbb{C}_\infty\backslash\rho_0$,$Q_\beta(\partial V_0) = Q_\beta(\partial\rho_0)$ because V_0 is compact set .Thus $Q_\beta(\partial V_0) = Q_\beta(\partial\rho_0) = \partial I_0 = Q_\beta(\partial I_0)$ and $Q_\beta(V_0) = Q_\beta(I_0) = I_0 \subseteq \overline{I_0}$ since $\rho_0 = Q_\beta^{-1}(I_0)\backslash I_0$ and $Q_\beta^{-1}(\infty) \subset I_\infty \cup \rho_\infty$, it follows $Q_\beta(V) \subset \overline{I_0}$. Therefore, the image of $\rho_0 \cup V_0$ is a subset of $\overline{I_0} \subset \rho_0 \cup V_0$, this mean $\rho_0 \cup V_0$ lies in $F(Q_\beta)$ and

$\rho_0 \cup V_0 = I_0$(in particular) .This is impossible because $\rho_0 \neq \emptyset$,hence $\rho_\infty \prec \rho_0$ .By Proposition 3.3 ,we have $\overline{I_\infty} \cap \overline{I_0} = \emptyset$.Now, we note that $Q_\beta(\overline{\rho_0}) = Q_\beta(\overline{I_0}) = \overline{I_0}$ and $Q_\beta(\overline{\rho_\infty}) = Q_\beta(\overline{I_\infty}) = \overline{I_\infty}$.Therefore $\overline{\rho_\infty} \cap \overline{\rho_0} = \emptyset$, $\overline{I_\infty} \cap \overline{\rho_0} = \emptyset$ and $\overline{\rho_\infty} \cap \overline{I_0} = \emptyset$. Thus $\overline{\rho_0} \cap \overline{I_0} = \emptyset$ and $\overline{I_\infty} \cap \overline{\rho_\infty} = \emptyset$ because $\rho_\infty \prec \rho_0$. □

Proposition 4.5. *[1] The boundaries of I_0 , I_∞ also each the preimages of I_0 , I_∞ are quasicircles around 0.*

Remark 4.6. *In previous work in source [1], we demonstrated that all Julia components of simple closed curves (quasicircles). Now we use the technique of symbol dynamics (in Σ_3). Let $V_\beta = \{v \in V_0 \cup V_1 \cup V_2 : Q_\beta^d(v) \in V_0 \cup V_1 \cup V_2$ for $d = 1, 2, 3, \dots\}$,all the points in the domain of Q_β either toward 0 or ∞ or stay in V_β . For any $v \in V_\beta$,then each iterate of v either V_0 or V_1 or V_2 ,so we can associate with v the forward sequence $v = (s_0 s_1 s_2, \dots)$, where*

$$v_d = \begin{cases} 0 & if \ \ Q_\beta^d \ \ is \ \ in \ \ V_0 \\ 1 & if \ \ Q_\beta^d \ \ is \ \ in \ \ V_1 \\ 2 & if \ \ Q_\beta^d \ \ is \ \ in \ V_2 \end{cases}$$

For each $\Sigma_3 = (v = (s_0 s_1 s_2, \dots); v_k \in \{0, 1, 2\}$ for every $m \geq 0$ be the space of one sided sequences of the symbols $\{0, 1, 2\}$. For $v = (s_0 s_1 s_2, \dots\ ) \in \Sigma_3$ and the shift map $s : \Sigma_3 \to \Sigma_3$ is denoted by $s(v) = (s_1 s_2, \dots)$. If there is an integer $i > 0$, such that $v_{m+i} = v_m$ for all $m \geq 0$. Suppose that $V_\beta \subset \Lambda_\beta = \{J_{j_0 j_1 \dots j_m} : 0 \leq j_m \leq 2\}$.

Proposition 4.7. *The set Λ_β is a Cantor set, also $s_\beta : \Lambda_\beta \to \Sigma_3$ the itinerary map is homeomorphism.*

Proof . First , to prove s_β is 1-1 map. If $z = (s_0 s_1 s_2, \dots)$ and $v = (v_0 v_1 v_2, \dots)$ such that $s_\beta(z) = s_\beta(v)$, it follows $s_0 = v_0$, $s_1 = v_1$, $s_2 = v_2$ ..... , so that z, v lie in the same V_β because the length of V_β is $1/3^d$ and go to 0 when $d \to \infty$. Hence s_β is one to one. Now if $(s_0 s_1 s_2, \dots)$ be the sequence of 0's, 1's and 2's, pick V_0 *or* V_1 *or* V_2 satisfying
z in $V_0 \to s_\beta(z) = s_0$
z in $V_1 \to s_\beta(z) = s_0 s_1$
z in $V_2 \to s_\beta(z) = s_0 s_1 s_2$. $V_0 \supseteq V_1 \supseteq V_2$ since each closed and bounded, by Heine-Borel Theorem $\exists\ z^* \in V_\beta$ and by definition of s_β .Therefore $s_\beta(z^*) = s_0 s_1 s_2$ and s_β is onto .To prove s_β is continuous. For any $e > 0$ and for any $z \in \Lambda_\beta$, let d be large so $1/2^d < e$. Fix $d > 0$ is small if $y \in \Lambda_\beta$ such that $|z - y| < d$,then z, y lie in the same V_β. For a y, the sequence $s_\beta(z)$ *and* $s_\beta(y)$ have the same initial d terms, since definition of s_β. Hence $\left|s_\beta(z) - s_{\beta(}y)\right| \leq 1/2^d < e$, therefore s_β is continuous. It follows ${s_\beta}^{-1}$ is continuous since s_β is 1-1 map . □

Theorem 4.8. *$J(Q_\beta)$ is a Cantor circles if $Q_\beta(e_\beta) \in I_0$ (or I_∞), where $Q_\beta(e_\beta)$ one of the free critical values but $e_\beta \notin I_0$ (or I_∞).*

Proof . For each closed set $V := \mathbb{C}_\infty \backslash I_\infty \cup I_0$ amidst I_0 and I_∞ divided into closed sets V_0, V_1 , V_2 between I_∞ and ρ_0, ρ_0 and ρ_∞, ρ_∞ and I_0 (see Figure 3). Each the map $Q_\beta : V_m \to V$ is covering by degree d, for $0 \leq m \leq 2$. So $J(Q_\beta)$ is equal to $\bigcup_{i \geq 0} Q_\beta^{-i}(V)$. For any $h : V \to V_m$ is the inverse branch of Q_β for $0 \leq m \leq 2$. Therefore

$$\forall j_{m_0, m_1, \dots, m_i, \dots} = \bigcap_{i=0}^{\infty} h_{m_i} \circ \cdots \circ h_{m_1} \circ h_{m_0}$$

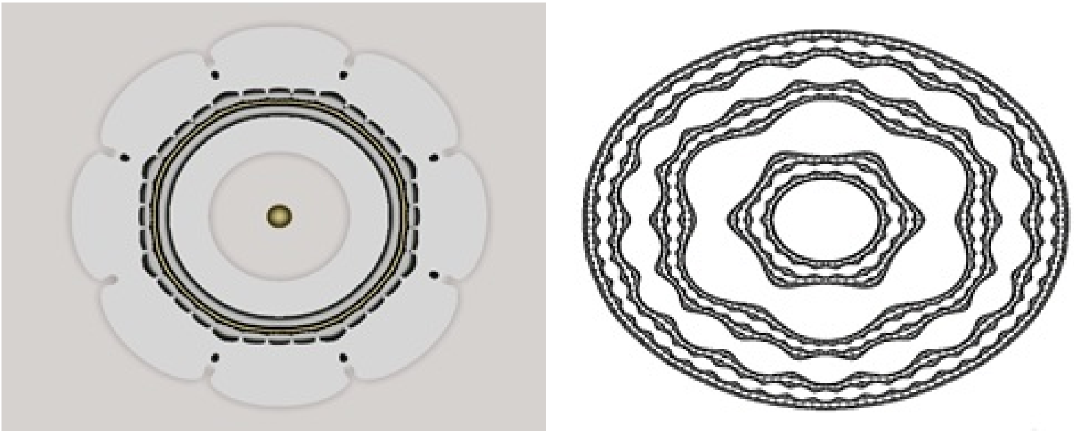


Figure 4: $J(Q_\beta)$ if d=4 ,β=0.7999+0.8i and $f\,(z)=z^3+\frac{0.01}{z^3}$. Are both of them Cantor circles. Hence f and Q_β are not topologically conjugate corresponding to Julia sets.

for $(m_0, m_1, \ldots, m_i, \ldots)$ be infinite sequence holding $0 \leq m \leq 2$. $\forall\ j_{m_0,m_1,\ldots,m_i,\ldots}$ is compact set separating the origin and the infinity. By [13, Corollary 2.3], thus $j_{m_0,m_1,\ldots,m_i,\ldots}$ is locally connected because Q_β is hyperbolic . Now, for any $E = \zeta \cup \eta$, $\rho = j_{2,2,\ldots,2,\ldots} = \partial I_0$ and $\rho = j_{0,0,\ldots,0,\ldots} = \partial I_\infty$. We note $V_m \subset V$, also $g : V_m \hookrightarrow V$ is identity map and not homotopic to a constant map . By [13, Lemma l2.4 and Proposition Case 2], we get $j_{m_0,m_1,\ldots,m_i,\ldots}$ is a simple closed curve . By Proposition 4.5, hence $j_{m_0,m_1,\ldots,m_i,\ldots}$ is a quasicircle since Q_β is hyperbolic. From Remark 4.6 and proposition 3.7 , it is clear that $s_\beta\,(Q_\beta\,(z)) = s(s_\beta\,(z))$ for $z \in \Lambda_\beta$. The one-sided shift on the space of 3 symbols $\Sigma_3 = \{s = (s_0 s_1 s_2, \ldots)\,; s_m \in \{0, 1, 2\}\}$ is isomorphic to the dynamics on the Julia components Λ_β. In special case, $J(Q_\beta)$ is homeomorphic to $\Sigma_3 \times S^1$, where this is a Cantor circles. □

Theorem 4.9. *Assume that one of the free critical values lies in I_0 or I_∞ but $e_\beta \notin I_0$ or I_∞ .Then any McMullen map is not topologically conjugate to Q_β corresponding to Julia sets*

Proof . From above Theorem4.8 , the one-sided shift on the space of 3 symbols $\Sigma_3 = \{s = (s_0 s_1 s_2, \ldots)\,; s_m \in \{0, 1, 2\}\}$ is isomorphic to the dynamics on the Julia components Λ_β. Notwithstanding, the dynamics of the one-sided shift on only two symbols $\Sigma_2 = \{s = (s_0 s_1, \ldots)\,; s_m \in \{0, 1\}\}$ is isomorphic to dynamics on the set of Julia components of any McMullen map. Hence, the McMullen map is not topologically conjugate to Q_β corresponding to Julia sets**,** see Figure4 . □ Now from Figure 2, we can define McMullen domain is the small region in the center corresponding to parameter values for which the Julia set is cantor set of simple closed curve.

Theorem 4.10. *The McMullen domain exists in the map Q_β iff $d \geq 4$.*

Proof . Assume that $J(Q_\beta)$ is a Cantor circles. Therefore I_0 and I_∞ are simply connected and *for any* Fatou components but except I_∞ and I_0 are annuli which separate ∞ from 0. By Proposition 4.3 , the Fatou components consists of two annular such that contain $2d$(critical points). From Riemann–Hurwitz's formula, the first preimage of I_∞ and I_0 contain all free critical points . However, each the free critical points does not lie in I_∞ and I_0 because $J(Q_\beta)$ is Cantor circles. By using Proposition 4.5 and from Figure 3, it follows that the conformal moduli of annuli holds $mod\,(V_0) = mod\,(V_1) = mod\,(V_2) = mod\,(V)\,/d$ because $Q_\beta : V_m \to V$ for $m = 0, 1, 2$ is a covering map d *to* 1. Moreover Vessentially contains on V_0, V_1, V_2 also $V \backslash (V_0 \cup V_1 \cup V_2) \neq \emptyset$. By the Grötzsch's modulus inequality, we get $mod\,(V_0) + mod\,(V_1) + mod\,(V_2) = \frac{3}{d} mod\,(V) < mod\,(V)$, that is $\frac{3}{d} < 1$. We need a $\frac{3}{d}$ of cycles to cover the circle which is equivalent iff $d \geq 4$. □ We have two values for the parameter

β as $A(\beta)$ and $\hat{a}(\beta)$. Then is said to be $A(\beta) \preccurlyeq \hat{a}(\beta)$ if there is $\varsigma \geq 0$ such that $A(\beta) \leq \rho.\hat{a}(\beta)$ for $0 \neq \beta$ is small.

Theorem 4.11. *Assume that $d \geq 4$. If β is a non-zero and small enough, then $J(Q_\beta)$ is a Cantor circles*

Proof . Let $Q_\beta(z) = 2\beta^{1-d}z^d - \frac{z^d(z^{2d}-\beta^{d+1})}{z^{2d}-\beta^{3d-1}}$ is a map, has one free point, say e_β such that

$$e_\beta = \left(\frac{3\beta^{3d-1} - 4\beta^d - \beta^{d+1} - \sqrt{(3\beta^{3d-1} - 4\beta^d - \beta^{d+1})^2 - 4(1-2\beta^{1-d})(\beta^{4d} - 2\beta^{5d-1})}}{2 - 4\beta^{1-d}}\right)^{\frac{1}{8}}.$$

If $|\beta|$ is small enough, it follows $e_\beta \asymp |\beta|^{\frac{d+1}{2d}}$. We define $b\rho|\beta|^{\frac{d+1}{2}}$ $\forall z \in \mathcal{T}_b$, where $\mathcal{T}_b$ is a round circle is defined as $\mathcal{T}_b = \{z : |z| = b\}$. We obtain

$$\begin{aligned}|Q_\beta(z)| &= 2|\beta|^{1-d}|z|^d - \frac{|z|^d \left|(z^{2d} - \beta^{d+1}\right|)}{|z^{2d} - \beta^{3d-1}|} \asymp\ 2|\beta|^{\frac{d+1}{2}}|\beta|^{-d} - \frac{|\beta|^{\frac{d+1}{2}} \left|(z^{2d} - \beta^{d+1}\right|)}{|\beta|^{d+1}} \\ &\preccurlyeq 2|\beta|^{\frac{1-d}{2}}|\beta|^{-d} - |\beta|^{\frac{d+1}{2}} \preccurlyeq 2|\beta|^{\frac{d+1}{2}} \preccurlyeq |\beta|^{\frac{d+1}{2}}\end{aligned}$$

t For $d \geq 4$, therefore $5d-3 < d(d+1)\ \forall\ a > 0$ satisfying

$$\frac{5d-3}{2d} < \frac{5d-2}{2d} < \frac{5d-1}{2d} < \frac{5d}{2d} = \frac{d+1}{2}.$$

Hence $d+1 < 3d-1 < 5d-3 < 2da$, define $U = \{z : |z| < |\beta|^a, \quad \forall z \in U$ and $|\beta|$ is small, it follows $\left|z^{2d} - \beta^{d+1}\right| \asymp |\beta|^{d+1}$ and $\left|z^{2d} - \beta^{3d-1}\right| \asymp |\beta|^{3d-1}$, we have

$$\begin{aligned}|Q_\beta(z)| &\asymp 2|\beta|^{1-d}|z|^d - \frac{|z|^d}{|\beta|^{2d-2}} \asymp 2|\beta|^{1-d}|z|^d - |z|^d|\beta|^{-2d+2} \\ &< 2|\beta|^{1-d}|\beta|^{ad} - |\beta|^{ad}|\beta|^{-2d+2} = 2|\beta|^{ad-d} - |\beta|^{da-2d+2} < |\beta|^{da-2d+2} \\ &< |\beta|^{\frac{5d-3}{2}-2d+2} = |\beta|^{d+1}.\end{aligned}$$

Thus $Q_\beta(U) \subset U$ if β is small enough. Therefore U is lies in I_0 by definition of U. By using that $|Q_\beta(z)| \preccurlyeq |\beta|^{\frac{d+1}{2}}$ and $\frac{5d-3}{2d} < a < \frac{d+1}{2}$, we have $Q_\beta(\mathcal{T}_b) \subset U \subset I_0$ and $\mathcal{F}(Q_\beta)$ is contains $\mathcal{T}_b$ if β is small . Thus $Q_\beta(e_\beta) \in I_0$, hence $e_\beta \notin I_0$ whenever β is small and $Q_\beta(\mathcal{T}_b) \subset I_0$ and $|e_\beta| > b$. Now, assume that Q_β has critical point $\acute{e_\beta}$ such that if $|\acute{e_\beta}| \asymp |\beta|^{\frac{3d-1}{2d}}$ and by lemma 3.1, where $|\beta|$ is small. Then $Q_\beta(\acute{e_\beta}) \in I_\infty$ and $e_\beta \acute{\notin} I_\infty$ because $|\acute{e_\beta}| < b$ and $\mathcal{T}_b \subset {Q_\beta}^{-1}(I_0)$. Hence there is critical point is not contains in I_∞ or I_0 but the image of this critical point by Q_β contains in I_∞ or I_0. Therefore from theorem 4.8, $J(Q_\beta)$ is a Cantor circles . □

5. Sierpinski Carpet and degenerated carpet

We will study the technique of escaping to the free critical points also to prove $J(Q_\beta)$ is a Sierpinski carpet. Also we give the degenerated Sierpinski carpet if the intersection of the boundaries of complementary domains are non-empty.

Proposition 5.1. *Assume that e_β be a free critical point lies in q_0^m for $m \geq 2$. Therefore each Fatou components of Q_β are simply connected and $J(Q_\beta)$ is compact, connected, nowhere dense and locally connected .*

Proof . I_∞ and I_0 are simply connected from Proposition 4.3 Suppose that $q_0^1 = Q_\beta^{-1}(I_0)\backslash I_0$ of I_0 consists of Fatou components with $2d$- symmetry. Since Q_β maps each one of them onto I_0 is conformal and $e_\beta \in q_0^m$ for $m \geq 2$, it follows all component of q_0^i is simply connected $1 \leq i \leq m-1$ $\forall i$, the number of components in q_0^m is at least $2d$ and by Proposition 4.5 these component $2d$-symmetry surround 0. For any V is simply connected component in the $(m-1$) preimages of I_0. Suppose that the critical orbits does not lie in V, thus all components of $Q_\beta^{-1}(V)$ are simply connected. Now the critical value lies in V also there is U component of $Q_\beta^{-1}(V)$ such that cannot simply connected, therefore U has two critical points at least. Hence there is $2d-1$ different Fatou component from the symmetric Fatou components $\rho_0^i V$ where $\rho_0 = e^{\frac{ip}{d}}$, $1 \leq i \leq m-1$.Thus $\omega_0^i V$ has two critical points at least. Therefore Q_β has $4d$ free critical points, this is impossible. Thus each components of $Q_\beta^{-1}(V)$ are simply connected and V has critical value. Hence each components in q_0^m are simply connected. Therefore all components of $Q_\beta^{-1}(I_0)$ are simply connected because q_0^m has no critical values. By Corollary3.3, thus each Fatou components of Q_β are simply connected. Notice that $J(Q_\beta) = \left(\bigcup_{i\geq 0} Q_\beta^{-i}(I_0 \cup I_\infty)\right)^{\complement}$, since $I_0 \cup I_\infty$ are simply connected, then $J(Q_\beta)$ is connected and by definition of the Julia set is bounded and closed sets, thus $J(Q_\beta)$ is compact set. Since $J(Q_\beta) \neq \mathbb{C}_\infty$ and by [11, Corollary 4.11], thus $\overline{J(Q_\beta)}^{\circ} = \emptyset$ and $J(Q_\beta)$ is nowhere dense. By [11, Theorem 3.19], it follows $J(Q_\beta)$ is locally connected since Q_β is hyperbolic map. □

Theorem 5.2. *Suppose that $e_\beta \in q_0^m$(or q_∞^m) for $m \geq 2$. Therefore each Fatou components of Q_β are Jordan disks. However, if $\partial I_0 \cap \partial I_\infty = \emptyset$, then $J(Q_\beta)$ is a Sierpinski carpet. Otherwise $J(Q_\beta)$ is a degenerate Sierpinski carpet.*

Proof . From Corollary 3.3 and also Proposition 5.1 , we must to prove the boundary of I_∞ is a simple closed curve. Because the boundary of I_∞ is locally connected and connected, then $\left(\mathbb{C}_\infty \backslash \overline{I_\infty}\right)^{\complement}$ has at most countable Jordan disks. For any Ω_0 component of $\mathbb{C}_\infty \backslash \overline{I_\infty}$ contains 0. Therefore $\partial\Omega_0$ is a simple closed curve. We claim that ${Q_\beta}^{-1}(\Omega_0) \subset \Omega_0$. Suppose that $0 \in I_0 \subset \Omega_0$, to show that ${Q_\beta}^{-1}(0) \subset \Omega_0$. From Lemmas 3.1 and3.5 , we have $2d$-roots for ${Q_\beta}^{-1}(0)\backslash\{0\}$ have either in γ_0 (Fatou component) around 0 or contain in $2d$ different components of Q_β. For the previous case if ${Q_\beta}^{-1}(0)$ is not contain in Ω_0, it follows γ_0 separate I_∞ from $\overline{\Omega_0}$, which is contradict because $\partial\Omega_0 \subset \partial I_\infty$. Now there is case that $2d$ Fatou component ought contain in $2d$ different component $U_0, \ldots, U_{2d-1}$ of $\mathbb{C}_\infty \backslash (\overline{I_\infty} \cup \overline{\Omega_0})$. However ${Q_\beta}^{-1}(\infty)\backslash\{\infty\} \subset \bigcup_{i=0}^{2d-1} \zeta(U_i) \subset \Omega_0$, $(\zeta(z) = \frac{\beta^2}{z})$. Hence $Q_\beta(U_i) = \Omega_0$ $\forall i = 0, \ldots, 2d-1$.Therefore $Q_\beta\left(\bigcup_{i=0}^{2d-1} \partial U_i\right) = \partial\Omega_0 \subset \partial I_0 \Rightarrow \partial\Omega_0 \subset \partial I_0$ has $2d-$preimages on the boundary of I_0, because $Q_\beta : \partial I_\infty \to \partial I_\infty$ has degree d. This is impossible. Hence ${Q_\beta}^{-1}(\Omega_0) \subset \Omega_0$ and $\Omega_0 = \mathbb{C}_\infty \backslash \overline{I_\infty}$. Suppose that $z \in \partial\Omega_0$, since ${Q_\beta}^{-1}(\overline{\Omega_0}) \subset \overline{\Omega_0}$ and we have $\partial\Omega_0 \subset \partial I_\infty$. It follows $\partial I_\infty \subset J(Q_\beta) = \overline{\bigcup_{m\geq 0} Q_\beta^{-m}(z)} \subset \overline{\Omega_0}$, *so* $\partial I_\infty \subset \overline{\partial\Omega_0}$ and $\partial I_\infty \subset \partial\Omega_0$. Thus $\partial I_\infty = \partial\Omega_0$ is simple closed curve and Q_β is hyperbolic. By theorem 4.4, then ∂I_∞ is quasicircle. Now we have three cases and we discuss of these cases.

Case one : For any M and N are distinct components of q_0^i(*or* q_∞^i) for $i \geq 1$, such that $\overline{M}$ intersect with $\overline{N}$. Let $z \in \overline{M} \cap \overline{N}$, it follows that ${Q_\beta}^{i-1}(z)$ is a critical point of Q_β because ${Q_\beta}^{i}(M) = {Q_\beta}^{i}(N) = I_0$, also ${Q_\beta}^{i-1}(\overline{M}) \cap {Q_\beta}^{i-1}(\overline{N}) \neq \emptyset$. Which is contradict because that all critical points escape to either the infinity or the origin.

Case two : For any M and N are components of q_0^i and q_0^k (*or* q_∞^i and q_∞^k) for $0 \leq k < i$ such that $\overline{M}$ intersect $\overline{N}$ is a non-empty. Therefore ${Q_\beta}^{i-1}(\overline{M} \cap \overline{N})$ are critical point of Q_β Which is contradict.

Case three : For any M and N are components of q_0^i and q_∞^k for $0 \neq k, 0 \neq i$. Because $\partial I_0 \cap \partial I_\infty = \emptyset$, it follows $\partial M \cap \partial N = \emptyset$. Therefore $\overline{M} \cap \overline{N} = \emptyset$. By Proposition 4.1 $J(Q_\beta)$ is a Sierpinski carpet. Otherwise, if $\partial I_0 \cap \partial I_\infty \neq \emptyset$, then $J(Q_\beta)$ is a degenerate Sierpinski carpet. □

Theorem 5.3. *For each $d = 4$ and $\beta \approx 1.15144239$ such that*

$$Q_\beta^2(e_\beta) = 0, \tag{5.1}$$

where $e_{\hat{a}} \approx 1.1592 + 0.4802i$ is a free critical point of $Q_{\hat{a}}$. Therefore $J(Q_{\hat{a}})$ is a Sierpinski carpet.

Proof . From (5.1), it follows that the free critical orbits are escaping to 0 also $Q_{\hat{a}}$ is critically-finite. From Proposition 3.6, $\mathcal{T}_\beta = \{z : |z| = \beta\}$ is contained in $J(Q_{\hat{a}})$. We have from a direct calculation, $|e_\beta| \approx 1.254707 > \beta$ and $|Q_\beta(e_\beta)| \approx 3.90962576 > \beta$. Therefore, $Q_\beta(e_\beta) \in q_0^1$ and $e_\beta \in q_0^2$ because $\mathcal{T}_\beta$ is contained in $J(Q_{\hat{a}})$. Now, we prove that $\partial \mathrm{I}_0 \cap \partial \mathrm{I}_\infty = \emptyset$. Because $\mathcal{T}_\beta$ has no critical values, thus ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ include of finitely many disjoint simple closed curves. From the Argument Principle and since in the $\mathbb{D}_\beta = \{z : |z| < \beta\}$ $\exists$ $d - roots$ and $2d$ poles, thus $Q_\beta : \mathcal{T}_\beta \to \mathcal{T}_\beta$ has degree d. Therefore ${Q_\beta}^{-1}(\mathcal{T}_\beta) \backslash \mathcal{T}_\beta \neq \emptyset$.Now, we claim each components of ${Q_\beta}^{-1}(\mathcal{T}_\beta) \backslash \mathcal{T}_\beta$ around 0 . But if the converse of the claim is satisfy and from by Lemma 3.1and3.2 , we have inside of $\mathcal{T}_\beta$ are $2d$ components of ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ and outside of $\mathcal{T}_\beta$ are $2d$ components. Which is contradict with degree of Q_β. Hence the each components of ${Q_\beta}^{-1}(\mathcal{T}_\beta)$ are disjoint and around 0. Therefore $J(Q_{\hat{a}})$ contain at least there are 3 disjoint simple closed curves and $\partial I_0 \cap \partial I_\infty = \emptyset$. $J(Q_{\hat{a}})$ is a Sierpinski carpet from Theorem 4.2. See Figure 1. □

Theorem 5.4. *For any $d = 4$ and $\beta \approx 1.050$ such that*

$$Q_\beta^2(e_\beta) = \infty, \tag{5.2}$$

where $e_{\hat{a}} \approx -1.8774 - 2.0208i$ is a free critical point of $Q_{\hat{a}}$. Then $J(Q_{\hat{a}})$ is a degenerated Sierpinski carpet.

Proof . From (5.2), thus the critical orbits are escaping also $Q_{\hat{a}}$ is critically-finite. By Theorem 4.5, to prove $e_\beta \in q_\infty^2$ and the boundary of I_0 intersect with the boundary of I_∞ are a non-empty. Because $Q_\beta^2(e_\beta) = \infty$, it follows that $e_\beta \in q_\infty^2$ if $J(Q_{\hat{a}})$ is not cantor circles and not quasicircles from Theorems 4.8 and3.7.To show that $-\beta \in \partial I_0 \cap \partial I_\infty$, for $- - \beta$ is a repelling fixed point of $Q_{\hat{a}}$. Since if d is odd, we have $Q_\beta(-\beta) = -\beta$. ${Q_\beta}'(-\beta) = \frac{d\beta^{2d-2} - 4d\beta^{d-1} - 3d}{1 + 2\beta^{d-1} + \beta^{2d-2}} \approx -5.696895521$. Then $|{Q_\beta}'(-\beta)| > 1$ and $-\beta$ is a repelling fixed point. Our procedure can be analyzed into three steps:
Step one. To find V_0 is a neighborhood of 0 such that $Q_\beta(V_0) \subset V_0$. Therefore $V_0 \subset I_0$.
Step two. To find U_1 and U_2 are two open neighborhoods of $-\beta$ such that

(1) $U_1 \Subset Q_\beta(U_1) \Subset U_2$.

(2)critical values and poles of Q_β not lie in U_2.

(3) the map restriction on U_1 of Q_β is conformal.

Step three. To find $v \in V_0$ and $u_1 \in U_1$ such that I_0 contains the segment $[v, u_1]$.
Now we prove these steps. If $e_{\hat{a}} \approx -1.8774 - 2.0208i$, then $|e_{\hat{a}}| \approx 2.75830$. For any $V_0 = \{z \in \mathbb{C}_\infty : |z| < 0.4\}$ be the disk center zero and radius is 0.4. Suppose that $z \in V_0$, thus

$$|z|^{2d} - |\beta|^{3d-1} < -1.709684 < 0$$

and therefore

$$|Q_\beta(z)| = \left| 2\beta^{1-d} z^d - \frac{z^d(z^{2d} - \beta^{d+1})}{z^{2d} - \beta^{3d-1}} \right| < 2|\beta|^{1-d}|z|^d + \frac{|z|^d \left(|z|^{2d} - |\beta|^{d+1} \right)}{|z|^{2d} - |\beta|^{3d-1}} < 0.0633 < 0.4.$$

Therefore $Q_\beta(\overline{V_0}) \subset V_0$ and $\overline{V_0} \subset I_0$. By using CP $(Q_\beta) = \{\eta_0^m e_\beta, \eta_0^m \frac{\beta^2}{e_\beta} : 0 \le m \le 2d-1\}$, where $\eta_0 = e^{\frac{pi}{d}}$. The set of critical values : CV (Q_β) $=\{(\pm(23.6577 + 6.16768i), \pm(-0.02868 - 0.000138i), 0, \infty\}$. The distance from $-\beta$ to CV (Q_β) is 1.038267. By according step two, fix $v_0 = 0.4$, $u_1 = 0.5$, $a = 1.76$ and $A = 2.5$. Define

$$U_1 = \mathbb{D}_a(-\beta) = \{z \in \mathbb{C}: |z+\beta| < a\}$$

and

$$U_2 = \mathbb{D}_A(-\beta) = \{z \in \mathbb{C}: |z+\beta| < A\}.$$

Thus

$$\max_{z\in[v_0,u_1]} |Q_\beta(z)| < \max_{y\in[0.4,0.5]} 2|\beta|^{-3}y^4 + \frac{y^d\left|y^{2d} - \beta^{d+1}\right|}{\left|y^{2d} - \beta^{3d-1}\right|} \approx 0.154578 < 0.4.$$

It follows that $Q_\beta([v_0, u_1]) \subset V_0$ and thus $[v_0, u_1] \subset I_0$. Since $|u_1 + \beta| \approx 1.55 < a$, therefore $u_1 \in U_1$. Now to show that $U_1 \Subset Q_\beta(U_1) \Subset U_2$. It means that if $u_1 \in U_1$ and $z \in \overline{\mathbb{D}}_a(-\beta)$, then $|Q_\beta(u_1) + \beta| < A$. Also if $z \in \overline{\partial\mathbb{D}}_a(-\beta)$, $so\, |Q_\beta(u_1) + \beta| > a$. We take a value of $u_1 = 0.6$, thus

$$\max_{z\in\overline{\mathbb{D}}_a(-\beta)} |Q_\beta(z) + \beta| \approx |0.315938 + 1.05| \approx 1.36 < A.$$

Also if $z \in \overline{\partial\mathbb{D}}_a(-\beta)$, we take $u = 0.75 \notin U_1$, therefore $min_{z\in\partial\overline{\mathbb{D}}_a(-\beta)} |Q_\beta(z) + \beta| \approx |0.731 + 1.05| \approx 1.78 > a$. Hence $U_1 \Subset Q_\beta(U_1) \Subset U_2$, also U_2 has no critical values and poles of Q_β. Because $u_1 \in U_1$ and the segment $[v, u_1]$ lies in I_0, thus $u_0 = Q_\beta(u_1) \in Q_\beta(U_1) \cap I_0$. $Q_\beta^{-1} : Q_\beta(U_1) \to U_1$ is the inverse of the conformal map Q_β: $U_1 \to Q_\beta(U_1)$ is a strict contraction map for the unique fixed point $-\beta$. For each η_0 lies in I_0 is a smooth curve linking u_1 and u_0. Let $m \ge 1$, such that u_m is the $m-th$ preimage of u_0 for Q_β, also η_m is the $m-th$ preimage of η_0 for Q_β linking u_{m+1} and u_m. Therefore $\eta_m \subset I_0\ \forall m \ge 0$. Thus $\bigcup_{m\ge0} \eta_m \subset I_0$. Because $lim_{m\to\infty} u_m = -\beta$, then $-\beta$ lies in ∂I_0. By lemma 3.2, therefore $-\beta$ lies in ∂I_∞. Hence $-\beta \in \partial I_\infty \cap \partial I_0 \ne \emptyset$. Because $\partial I_\infty \cap \partial I_0 \ne \emptyset$, then $J(Q_\beta)$ is not cantor circles. Therefore $e_\beta \notin q_\infty^1$, to show that $J(Q_\beta)$ is not quasicircle. Assume that $J(Q_\beta)$ is quasicircle. From (#), thus $e_\beta \in I_0$ and $Q_\beta(e_\beta) \in I_0$ but I_0 is Fatou component of superattracting fixed point $\infty \ne e_\beta$. From [11, Bottcher's Theorem], we have ${Q_\beta}^d(e_\beta)$ in I_0 is infinite. This is impossible with $Q_{\hat{a}}$ is critically-finite. It follows $J(Q_\beta)$ is not quasicircle. Since $e_\beta \in q_\infty^2$ and use Theorem 5.2, therefore $J(Q_\beta)$ is degenerate Sierpinski carpet . □

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