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Some new results on differential subordinations and superordinations for analytic univalent functions defined by Rafid-Jassim operator

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Abstract

In the present paper, we obtain sandwich theorems for univalent functions by using some results of differential subordination and superordination for univalent functions involving the Rafid-Jassim operator.

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1. Introduction

Let $H = H(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For $n \in N$ and $a \in \mathbb{C}$. Let $H[a, n]$ be the subclass of H of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C})$$

Let A denote the subclass of H of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \ (z \in U), \tag{1.1}$$

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which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let f and g are analytic functions in H, f is said to be subordinate to g, or g is said to be superordinate to f in U and write $f \prec g$, if there exists a Schwarz function K in U, which with $K(0) = 0$, and $|K(z)| < 1$, $(z \in U)$ where $f(z) = g(K(z))$. In such a case we write $f \prec g$ or $f(z) \prec g(z)$ $(z \in U)$. If g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$ [14, 15].

Definition 1.1. *[14] Let $\emptyset : \mathbb{C}^3 \times U \to \mathbb{C}$ and $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the second-order differential subordination:*

$$\emptyset(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.2}$$

then $p(z)$ is called a solution of the differential subordination (1.2), and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.2), or more simply dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.2). A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominant $q(z)$ of (1.2) is said to be the best dominant is unique up to a relation of U.

Definition 1.2. *[14] Let $p, h \in A$ and $\emptyset(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\emptyset(p(z), zp'(z), z^2p''(z); z)$ are univalent function in U and if p satisfies*

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called a solution of the differential superordination (1.3). An analytic function $q(z)$ which is called a subordinat of the solutions of the differential superordination (1.3), or more simply a subordinant if $p \prec q$ for all the functions p satisfying (1.3). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinats q of (1.3) is said to be the best subordinat.

Several authors [1, 2, 9, 14, 16] obtained sufficient conditions on the functions h, p and $\emptyset$ for which the following implication holds

$$h(z) \prec \emptyset(p(z), zp'(z), z^2p''(z); z),$$

then

$$q(z) \prec p(z) \tag{1.4}$$

Using the results (see [3, 4, 5, 6, 10, 11, 15]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$$

where q_1 and q_2 are given univalent functions in U and $q_1(0) = q_2(0) = 1$. Also, several authors (see[1, 3, 5, 6, 7, 8]) derived some differential subordination and superordination results with some sandwich theorems.

For $f \in A$, Buti and Jassim [13] defined the following generalized integral operator:

$$p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z) = \frac{\theta k(\lambda - \beta + 2)^{\mu-\alpha+1}}{\ell^{\mu-\alpha+1}\Gamma(\mu - \alpha + 1)} \int_0^1 (\log \frac{1}{\tau^\ell})^{\mu-\alpha} f(\frac{z\tau}{\theta k}) d_\tau, \tag{1.5}$$

where

$$\lambda - \alpha < 1, \ell > 0, \tau > 0, \theta > 0, k > 0$$

For $f(z) \in A$ given by (1.1), we have

$$p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z) = z + \sum_{n=2}^{\infty} \left[\frac{\lambda - \beta + 2}{\lambda - \beta + n + 1} \right]^{\mu-\alpha+1} a_n z^n \tag{1.6}$$

From (1.6), we note that

$$z \left(p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z) \right)' = (\mu - \beta + 2) p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,k} f(z) - (\lambda - \beta + 1) p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z). \tag{1.7}$$

The main object of the present investigation is to find sufficient conditions for certan normalized analytic function f to satisfy:

$$q_1(z) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z)}{z} \right]^{\Upsilon} \prec q_2(z).$$

and

$$q_1(z) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,k} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z)} \right]^{\Upsilon} \prec q_2(z).$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.
In this paper, we derive some sandwich theorems, involving the operator $p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,k} f(z)$.

2. Preliminaries

We need the following definitions and lemmas to prove our results.

Definition 2.1. *[14] Denote by Q the set of all functions q that are analytic and injective on $\bar{U}|E(q)$, where $\bar{U} = U \cup \{z \in \partial U\}$, and*

$$E(q) = \{\varepsilon \in \partial U : \lim_{z \to \varepsilon} q(z) = \infty\}$$

and are such that $q'(\varepsilon) \neq 0$ for $\varepsilon \in \partial U | E(q)$. Further, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, and $Q(0) = Q_0, Q(1) = Q_1 = \{q \in Q : q(0) = 1\}$.

Lemma 2.2. *[15] Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}|\{0\}$ with*

$$Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\{0, -Re(\frac{\alpha}{\beta})\}.$$

If p is analytic in U and

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z), \tag{2.1}$$

then $p \prec q$ and q is the best dominant of (2.1).

Lemma 2.3. *[4] Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$, when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

- *$Q(z)$ is starlike univalent in U,*

- $Re\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for* $z \in U$.

If p *is analytic in* U, *with* $p(0) = q(0)$, $p(U) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p \prec q$ *and* q *is the best dominant of* (2.2).

Lemma 2.4. *[15] Let* q *be a convex univalent in* U *and let* $\beta \in \mathbb{C}$, *that* $Re(\beta) > 0$. *If* $p \in H[q(0), 1] \cap Q$ *and* $p(z) + \beta zp'(z)$ *is univalent in* U, *then*

$$q(z) + \beta zq'(z) \prec p(z) + \beta zp'(z), \tag{2.3}$$

which implies that $q \prec p$ *and* q *is the best subordinant of* (2.3).

Lemma 2.5. *[12] Let* q *be univalent in the unit disk* U *and let* θ *and* ϕ *be analytic in a domain* D *containing* $q(U)$. *Suppose that*

- $Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$,
- $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* U.

If $p \in H[q(0), 1] \cap Q$, *with* $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ *is univalent in* U *and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.4}$$

then $q \prec p$ *and* q *is the best subordinant of* (2.4).

3. Differential Subordination Results

Here, we introduce some differential subordination results by using the Rafid-Jassim operator.

Theorem 3.1. *Let* q *be convex univalent function in* U *with* $q(0) = 1$, $0 \neq \varepsilon \in \mathbb{C}, \gamma > 0$ *and suppose that* q *satisfies:*

$$Re\left\{1 - \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -Re(\frac{\gamma}{\varepsilon})\}. \tag{3.1}$$

If $f \in A$ *satisfies the subordination*

$$(\lambda - \beta + 2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)} - 1\right) + \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}{z}\right]^{\gamma} \prec q(z) + \frac{\varepsilon}{\gamma} zq'(z) \tag{3.2}$$

then

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}{z}\right]^{\gamma} \prec q(z), \tag{3.3}$$

and q *is the best dominant of* (3.2).

Proof . Define the function p by

$$p(z)=\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}, \tag{3.4}$$

then the function $p(z)$ is analytic in U and $p(0)=1$, therefore, differentiating (3.4) with respect to z and using the identity (1.7) in the resulting equation, we obtain

$$\frac{zp'(z)}{p(z)}=\gamma\left[\frac{z\left(p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)\right)'}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right]. \tag{3.5}$$

Hence

$$\frac{zp'(z)}{p(z)}=\gamma\left[(\lambda-\beta+2)\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)\right].$$

Therefore,

$$\frac{zp'(z)}{\gamma}=\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left[(\lambda-\beta+2)\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)\right].$$

The subordination (3.2) from the hypothesis becomes

$$p(z)+\frac{\varepsilon}{\gamma}zp'(z)\prec q(z)+\frac{\varepsilon}{\gamma}zq'(z).$$

An application of lemma 2.2 with $\beta=\frac{\varepsilon}{\gamma}$ and $\alpha=1$, we obtain (3.3). □

Putting $q(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.2. *Let* $0\neq\varepsilon\in\mathbb{C},\ \gamma>0$ *and*

$$Re\left\{1+\frac{2z}{1-z}\right\}>\max\{0,-Re(\frac{\gamma}{\varepsilon})\}.$$

If $f\in A$ *satisfies the subordination*

$$(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\prec\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right),$$

then

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\prec\left(\frac{1+z}{1-z}\right)$$

and $q(z)=\left(\frac{1+z}{1-z}\right)$ *is the best dominant.*

Theorem 3.3. *Let q be convex univalent function in U with $q(0) = 1,\ q'(z) \neq 0\ (z \in U)$ and assume that q satisfies*

$$Re\left\{1 + \frac{m}{\varepsilon}(q(z))^m + \frac{m-1}{\varepsilon}(q(z))^{m-1} - z\frac{q'(z)}{q(z)} + z\frac{q''(z)}{q'(z)}\right\} > 0, \tag{3.6}$$

where $m \in \mathbb{C}, \varepsilon \in \mathbb{C}|\{0\}$ and $z \in U$. Suppose that $z\frac{q'(z)}{q(z)}$ is starlike univalent in U. If $f \in A$ satisfies

$$\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m;z) \prec (1+q(z))q(z)^{m-1} + \varepsilon z\frac{q'(z)}{q(z)}, \tag{3.7}$$

where,

$$\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z) = \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma m} + \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma(m-1)} + \varepsilon\gamma(\lambda-\beta+2)\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-2,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)} - \frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right), \tag{3.8}$$

then

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma} \prec q(z) \tag{3.9}$$

and q is the best dominant of (3.7).

Proof . Define the function p by

$$p(z) = \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma}. \tag{3.10}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$ differenitating (3.10) with respect to z and using the identity (1.7), we get,

$$\frac{zp'(z)}{p(z)} = \gamma\left[(\mu-\beta+2)\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-2,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)} - \frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right)\right].$$

By setting
$\theta(w) = (1+w)w^{m-1}$ and $\phi(w) = \frac{\varepsilon}{w},\ w \neq 0,$
we see that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}|\{0\}$ and that $\phi(w) \neq 0,\ w \in \mathbb{C}|\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = (1+q(z))q(z)^{m-1} + \varepsilon z\frac{q'(z)}{q(z)}$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1+\frac{m}{\varepsilon}(q(z))^{m}+\frac{m-1}{\varepsilon}(q(z))^{m-1}-z\frac{q'(z)}{q(z)}+z\frac{q''(z)}{q'(z)}\right\}>0.$$

By a straightforward computation, we obtain

$$\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z) = (1+p(z))(p(z))^{m-1}+\varepsilon z\frac{p'(z)}{p(z)}, \tag{3.11}$$

where $\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8).
From (3.7) and (3.11), we have

$$(1+p(z))(p(z))^{m-1}+\varepsilon z\frac{p'(z)}{p(z)} \prec (1+q(z))(q(z))^{m-1}+\varepsilon z\frac{q'(z)}{q(z)}. \tag{3.12}$$

Therefore, by Lemma 2.3, we get $p(z)\prec q(z)$. By using (3.10), we obtain the result. □
Putting $q(z)=\left(\frac{1+Az}{1+Bz}\right)$, $(-1\le B<A\le 1)$ in Theorem 3.3, we obtain the following corollary:

Corollary 3.4. *Let $-1\le B<A\le 1$ and*

$$Re\left\{\frac{m}{\varepsilon}\left(\frac{1+Az}{1+Bz}\right)^{m}+\frac{m-1}{\varepsilon}\left(\frac{1+Az}{1+Bz}\right)^{m-1}+\frac{1+Bz(4+3Az)}{(1+Bz)(1+Az)}\right\}>0$$

where $\varepsilon\in\mathbb{C}|\{0\}$ and $z\in U$, if $f\in A$ satisfies

$$\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)\prec\left[1+\left(\frac{1+Az}{1+Bz}\right)\right]\left(\frac{1+Az}{1+Bz}\right)^{m-1}+\varepsilon z\frac{A-B}{(1+Az)(1+Bz)},$$

where is given $\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)$ by (3.8),
then

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma}\prec\left(\frac{1+Az}{1+Bz}\right)$$

and $q(z)=\left(\frac{1+Az}{1+Bz}\right)$ is the best dominant.

4. Differential Superordination Results

Theorem 4.1. *Let q be convex univalent function in U with $q(0)=1$, $\gamma>0$ and $Re\{\varepsilon\}>0$. Let $f\in A$ satisfies*

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\in H[q(0),1]\cap Q$$

and

$$(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}$$

be univalent in U. If

$$q(z)+\frac{\varepsilon}{\gamma}zq'(z)\prec(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}, \tag{4.1}$$

then

$$q(z)\prec\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma} \tag{4.2}$$

and q is the best subordinant of (4.1).

Proof . Define the function p by

$$p(z)=\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}. \tag{4.3}$$

Differentiating (4.3) with respect to z, we get

$$\frac{zp'(z)}{p(z)}=\gamma\left[\frac{z\left(p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)\right)'}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right]. \tag{4.4}$$

After some computations and using (1.7), from (4.4), we obtain

$$(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}=p(z)+\frac{\varepsilon}{\gamma}zp'(z),$$

and now, by using Lemma 2.4, we get the desired result. □
Putting $q(z)=\left(\frac{1+z}{1-z}\right)$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.2. *Let $\gamma>0$ and $Re\{\varepsilon\}>0$. If $f\in A$ satisfies*

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\in H[q(0),1]\cap Q$$

and

$$(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}$$

be univalent in U. If

$$\left(\frac{1-z^2+2\frac{\varepsilon}{\gamma}z}{(1-z)^2}\right)\prec(\lambda-\beta+2)\left[\frac{P^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma}\left(\frac{P^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{P^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{P^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^{\gamma},$$

then

$$\left(\frac{1+z}{1-z}\right) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}{z}\right]^{\gamma}$$

and $q(z) = \left(\frac{1+z}{1-z}\right)$ *is the best subordinant.*

Theorem 4.3. *Let* q *be convex univalent function in* U *with* $q(0) = 1,\ q'(z) \neq 0$ *and assume that* q *satisfies*

$$Re\left\{\frac{m}{\varepsilon}(q(z))^m q'(z) + \frac{m-1}{\varepsilon}(q(z))^{m-1} q'(z)\right\} > 0, \tag{4.5}$$

where $m \in \mathbb{C},\ \varepsilon \in \mathbb{C}|\{0\}$ *and* $z \in U$.
Suppose that $z\frac{q'(z)}{q(z)}$ *is starlike univalent in* U. *Let* $f \in A$ *satisfies*

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}\right]^{\gamma} \in H[q(0), 1] \cap Q,$$

and $\Psi(\gamma, \mu, \beta, \ell, \lambda, \theta, k, m, \varepsilon; z)$ *is univalent in* U, *where is given* $\Psi(\gamma, \mu, \beta, \ell, \lambda, \theta, k, m, \varepsilon; z)$ *by* (3.8).
If

$$(1 + q(z))(q(z))^{m-1} + \varepsilon z\frac{q'(z)}{q(z)} \prec \Psi(\gamma, \mu, \beta, \ell, \lambda, \theta, k, m, \varepsilon; z), \tag{4.6}$$

then

$$q(z) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}\right]^{\gamma} \tag{4.7}$$

and q *is the best subordinant of* (4.6).

Proof . Define the function p by

$$p(z) = \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}\right]^{\gamma}. \tag{4.8}$$

Differentiating (4.8) with respect to z, we get

$$\frac{zp'(z)}{p(z)} = \gamma\left[(\mu - \beta + 2)\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-2,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)} - \frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K} f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K} f(z)}\right)\right].$$

By setting
$\theta(w) = (1 + w)w^{m-1}$ and $\phi(w) = \frac{\varepsilon}{w},\ w \neq 0,$
we see that $\theta(w)$ and $\phi(w)$ are analytic in $\mathbb{C}|\{0\}$ and that $\phi(w) \neq 0,\ w \in \mathbb{C}|\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = \varepsilon z\frac{q'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = Re\left\{\frac{m}{\varepsilon}(q(z))^m q'(z) + \frac{m-1}{\varepsilon}(q(z))^{m-1}q'(z)\right\} > 0$$

By a straightforward computation, we obtain

$$\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z) = (1+p(z))(p(z))^{m-1} + \varepsilon z\frac{p'(z)}{p(z)}, \tag{4.9}$$

where $\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)$ is given by (3.8).
From (4.6) and (4.9), we have

$$(1+q(z))(q(z))^{m-1} + \varepsilon z\frac{q'(z)}{q(z)} \prec (1+p(z))(p(z))^{m-1} + \varepsilon z\frac{p'(z)}{p(z)}. \tag{4.10}$$

Therefore, by Lemma 2.5, we get $q(z) \prec p(z)$. □

5. Sandwich Results

Theorem 5.1. *Let q_1 be convex univalent function in U with $q_1(0) = 1$, $\gamma > 0$ and $Re\{\varepsilon\} > 0$ and q_2 be univalent U, $q_2(0) = 1$ and satisfies* (3.1)*. Let $f \in A$ satisfies*

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma \in H[1,1]\cap Q$$

and

$$(\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma$$

be univalent in U. If

$$q_1(z)+\frac{\varepsilon}{\gamma}zq_1'(z) \prec (\lambda-\beta+2)\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma\left(\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}-1\right)+\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma \prec q_2(z)+\frac{\varepsilon}{\gamma}zq_2'(z),$$

then

$$q_1(z) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}{z}\right]^\gamma \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

Theorem 5.2. *Let q_1 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies* (4.5) *and q_2, satisfies* (3.6)*. Let $f \in A$ satisfies*

$$\left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^\gamma \in H[1,1]\cap Q$$

and $\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)$ is univalent in U, where is given $\Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z)$ by (3.8)*.*

if

$$(1+q_1(z))(q_1(z))^{m-1}+\varepsilon z\frac{q_1'(z)}{q_1(z)} \prec \Psi(\gamma,\mu,\beta,\ell,\lambda,\theta,k,m,\varepsilon;z) \prec (1+q_2(z))(q_2(z))^{m-1}+\varepsilon z\frac{q_2'(z)}{q_2(z)}$$

then

$$q_1(z) \prec \left[\frac{p^{\mu,\beta,\ell}_{\lambda,\alpha-1,\theta,K}f(z)}{p^{\mu,\beta,\ell}_{\lambda,\alpha,\theta,K}f(z)}\right]^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

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