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Some new refinements of the generalized Hölder inequality and applications

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Abstract

The main goal of this article is to present some new refinements of the generalized classical Hölder's inequality. As applications we present some refinements to several inequalities for the (q, s)-Polygamma functions, the s-Extension of Nielsen's β-function, the derivatives of the s-Extension of Nielsen's β-function, the extended Gamma function, the r-Gamma functions and the r-Riemann Zeta function.

Keywords: Generalized Hölder's inequality, (q, k)-Polygamma Functions, Nielsen's β-function, Gamma function, Zeta function
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1 Introduction

Let $(\Omega, \mathcal{F}, \mu)$ be a measure space (μ is a positive measure) and $f, g : \Omega \mapsto \mathbf{C}$ on Ω are two measurable functions. One of the most basic inequalities in Mathematics is the so called Hölder's inequality, which states

$$\int_{\Omega} |fg| d\mu(t) \leq \left(\int_{\Omega} |f|^p d\mu(t) \right)^{1/p} \left(\int_{\Omega} |g|^q d\mu(t) \right)^{1/q} \tag{1.1}$$

for all $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. The renowned inequality of Hölder (1.1) is well celebrated for its beauty and its wide range of important applications to real and complex analysis, functional analysis, as well as many disciplines in applied mathematics.

Let $n \geq 2$ be any integer. Hölder's inequality (1.1) can be generalized to the case involving n functions as follows.

Theorem 1.1 ([2]). [Generalized Hölder's inequality] Let $f_1, f_2, \dots, f_n$ be μ-measurable functions such that $f_k \in \mathcal{L}^{p_k}(\mu)$, for all $k = 1, \dots, n$, and let $p_k > 1$, such that $\sum_{k=1}^{n} \frac{1}{p_k} = 1$. Then we have $\prod_{k=1}^{n} f_k \in \mathcal{L}^1(\mu)$, and

$$\int_{\Omega} \prod_{k=1}^{n} |f_k(t)| d\mu(t) \leq \prod_{k=1}^{n} \left(\int_{\Omega} |f_k(t)|^{p_k} d\mu(t) \right)^{\frac{1}{p_k}}. \tag{1.2}$$

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Many generalizations and refinements have been obtained for the generalized classical Hölder inequality. See, for example [11] and the references therein. In [12], Y. I. Kim obtained the following refinements and generalizations of the generalized Hölder's inequality as follows:

Theorem 1.2. Let $g : [0,1] \to (0,+\infty)$ be defined as in (1.4). Then for $0 \le t_1 < t_2 < \cdots < t_k \le 1$, we have the following refinement and generalization of the Hölder's inequality for the integral form:

$$g(0) = \int_a^b \left(\prod_{k=1}^n f_k(x)\right) dx \le g(t_1) \le \cdots \le g(t_k) \le g(1) = \prod_{k=1}^n \left(\int_a^b f_k^{p_k}(x)dx\right)^{\frac{1}{p_k}}. \tag{1.3}$$

where

$$g(t) = \prod_{k=1}^n \left[\int_a^b \left(\prod_{j=1}^n f_j(x)\right)^{1-t} \left(f_k^{p_k}(x)\right)^t dx\right]^{\frac{1}{P_k}}. \tag{1.4}$$

Moreover, the function g satisfies $g(t) > 0$ for all $t \in (0,1)$ and is convex, that is, $g''(t) \ge 0$ for all $t \in (0,1)$.

The rest of the present paper is organized as follows. In the next section, we will present some new refinements of the generalized Hölder's inequality. Once we show our main inequalities, we will present some applications that include new refinements of some inequalities for certain special functions.

2 Some new refinements of the generalized Hölder's inequality

In this section, we present our main new result concerning the generalized Hölder's inequality. First, we list the following lemma and theorem that we will need in our analysis.

Lemma 2.1 ([6]). Let n and m be two integers and let $a_i \in \mathbb{R}^+$. Set $i_0 := m$, $i_n := 0$ and

$$A := \{(i_1, \dots, i_{n-1}) : 0 \le i_k \le i_{k-1},\ 1 \le k \le n-1\}.$$

Then, we have

$$\left(\sum_{k=1}^n \nu_k a_k\right)^m = \sum_{(i_1,\dots,i_{n-1})\in A} C_A \nu_1^{i_0-i_1}\nu_2^{i_1-i_2}\dots\nu_n^{i_{n-1}-i_n} a_1^{i_0-i_1} a_2^{i_1-i_2}\dots a_n^{i_{n-1}-i_n},$$

where, $C_A = \binom{i_1}{i_2}\dots\binom{i_{n-2}}{i_{n-1}}$, the $\binom{i_{k-1}}{i_k}$ is the binomial coefficient.

Theorem 2.2 ([6]). For $k = 1,2,\dots,n$, let $a_k \ge 0$ and let $\nu_k > 0$ satisfy $\sum_{k=1}^n \nu_k = 1$. Then for all integers $m \ge 1$, we have

$$\prod_{k=1}^n a_k^{\nu_k} + r_0^m \left(\sum_{k=1}^n a_k - n\sqrt[n]{\prod_{k=1}^n a_k}\right) \le \left(\sum_{k=1}^n \nu_k a_k^{\frac{1}{m}}\right)^m \le \sum_{k=1}^n \nu_k a_k,$$

where $r_0 = \min\{\nu_k : k = 1,\dots,n\}$.

Moreover, if we set $U_m := \left(\sum_{k=1}^n \nu_k a_k^{\frac{1}{m}}\right)^m$, then $\{U_m\}$ is a decreasing sequence and we have

$$\lim_{m\to\infty} U_m = \prod_{k=1}^n a_k^{\nu_k}.$$

Concerning the generalized Hölder's inequality, we establish the following result.

Theorem 2.3. Let n be a positive integer and let $f_1, f_2, \dots, f_n$ be μ-measurable functions such that $f_k \in \mathcal{L}^{p_k}(\mu)$, for all $k = 1, \dots, n$. Then for all integers $m \geq 2$, the inequalities

$$\begin{aligned}\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) \quad &+ \quad n r_0^m \prod_{k=1}^n ||f_k||_{p_k} \Big(1 - \prod_{k=1}^n ||f_k||_{p_k}^{\frac{-p_k}{n}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t)\Big) \\ &\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^n ||f_k||_{p_k}^{1-\frac{p_k(i_k-i_{k-1})}{m}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t) \\ &\leq \prod_{k=1}^n ||f_k||_{p_k},\end{aligned}$$

holds for $p_k > 1$, such that $\sum_{k=1}^n \frac{1}{p_k} = 1$, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \dots, n\}$.

Proof . Choose for $k = 1, \dots, n$, $a_k = \frac{|f_k(t)|^{p_k}}{||f_k||_{p_k}^{p_k}}$ and $\nu_k = \frac{1}{p_k}$. Then by Theorem 2.2, we have

$$\begin{aligned}\frac{\prod_{k=1}^n |f_k(t)|}{\prod_{k=1}^n ||f_k||_{p_k}} \quad &+ \quad r_0^m \left(\sum_{k=1}^n \frac{|f_k(t)|^{p_k}}{||f_k||_{p_k}^{p_k}} - n \sqrt[n]{\prod_{k=1}^n \frac{|f_k(t)|^{p_k}}{||f_k||_{p_k}^{p_k}}}\right) \\ &\leq \quad \Big(\sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{\frac{p_k}{m}}}{||f_k||_{p_k}^{\frac{p_k}{m}}}\Big)^m \leq \sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{p_k}}{||f_k||_{p_k}^{p_k}}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) \quad &+ \quad n r_0^m \prod_{k=1}^n ||f_k||_{p_k} \Big(1 - \prod_{k=1}^n ||f_k||_{p_k}^{\frac{-p_k}{n}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t)\Big) \\ &\leq \quad \prod_{k=1}^n ||f_k||_{p_k} \Big(\int_\Omega \big(\sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{\frac{p_k}{m}}}{||f_k||_{p_k}^{\frac{p_k}{m}}}\big)^m d\mu(t)\Big) \leq \prod_{k=1}^n ||f_k||_{p_k}.\end{aligned}$$

It follows that by Lemma 2.1,

$$\begin{aligned}&\prod_{k=1}^n ||f_k||_{p_k} \Big(\int_\Omega \big(\sum_{k=1}^n \frac{1}{p_k} \frac{|f_k(t)|^{\frac{p_k}{m}}}{||f_k||_{p_k}^{\frac{p_k}{m}}}\big)^m d\mu(t)\Big) \\ &= \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^n ||f_k||_{p_k}^{1-\frac{p_k(i_k-i_{k-1})}{m}} \int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t).\end{aligned}$$

This ends the proof. ☐ For the discrete case, we have the following theorem concerning the generalized Hölder's inequality for sums.

Theorem 2.4. Let n, N be two integers and $\{Q_{j,k}\} \subset \mathbb{R}$, where $k = 1, 2, \dots, n$ and $j = 1, 2, \dots, N$. Let $p_k > 1$, such that $\sum_{k=1}^n \frac{1}{p_k} = 1$, Then the inequalities

$$\begin{aligned}\sum_{j=1}^N \Big|\prod_{k=1}^n Q_{j,k}\Big| \quad &+ \quad n r_0^m \prod_{k=1}^n \Big(\sum_{j=1}^N |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}} \Big(1 - \prod_{k=1}^n \Big(\sum_{j=1}^N |Q_{j,k}|^{p_k}\Big)^{\frac{-1}{n}} \sum_{j=1}^N \prod_{k=1}^n |Q_{j,k}|^{\frac{p_k}{n}}\Big) \\ &\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Big(\sum_{j=1}^N |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k} - \frac{(i_k-i_{k-1})}{m}} \sum_{j=1}^N \prod_{k=1}^n |Q_{j,i}|^{\frac{p_k(i_k-i_{k-1})}{m}} \\ &\leq \prod_{k=1}^n \Big(\sum_{j=1}^N |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}},\end{aligned}$$

is valid, where $r_0 = \min\{\frac{1}{p_k} : k = 1, \dots, n\}$.

Proof . The inequality is obvious if $\sum_{j=1}^{N}|Q_{j,k}|^{p_k}=0$ for each k. So we assume $\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\neq 0$ and let for $k=1,\dots,n$,

$$a_k=\frac{|Q_{j,k}|^{p_k}}{\sum_{j=1}^{N}|Q_{j,k}|^{p_k}}$$

and $\nu_k=\frac{1}{p_k}$. Then by Theorem 2.2, we have

$$\begin{aligned}\frac{\prod_{k=1}^{n}|Q_{j,k}|}{\prod_{k=1}^{n}(\sum_{j=1}^{N}|Q_{j,k}|^{p_k})^{\frac{1}{p_k}}} &+ r_0^m\left(\sum_{k=1}^{n}\frac{|Q_{j,k}|^{p_k}}{\sum_{j=1}^{N}|Q_{j,k}|^{p_k}}-n\sqrt[n]{\prod_{k=1}^{n}\frac{|Q_{j,k}|^{p_k}}{\sum_{j=1}^{N}|Q_{j,k}|^{p_k}}}\right)\\ &\leq \left(\sum_{k=1}^{n}\frac{1}{p_k}\frac{|Q_{j,k}|^{\frac{p_k}{m}}}{(\sum_{j=1}^{N}|Q_{j,k}|^{p_k})^{\frac{1}{m}}}\right)^m\leq\sum_{k=1}^{n}\frac{1}{p_k}\frac{|Q_{j,k}|^{p_k}}{\sum_{j=1}^{N}|Q_{j,k}|^{p_k}}.\end{aligned}$$

By adding these N inequalities, we obtain

$$\begin{aligned}&\sum_{j=1}^{N}\Big|\prod_{k=1}^{n}Q_{j,k}\Big| + nr_0^m\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}}\Big(1-\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{-1}{n}}\sum_{j=1}^{N}\prod_{k=1}^{n}|Q_{j,k}|^{\frac{p_k}{n}}\Big)\\ &\quad\leq\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}}\left(\sum_{k=1}^{n}\frac{1}{p_k}\frac{|Q_{j,k}|^{\frac{p_k}{m}}}{\sum_{j=1}^{N}|Q_{j,k}|^{\frac{p_k}{m}}}\right)^m\\ &\quad\leq\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}},\end{aligned}$$

It follows that by Lemma 2.1,

$$\begin{aligned}&\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}}\left(\sum_{k=1}^{n}\frac{1}{p_k}\frac{|Q_{j,k}|^{\frac{p_k}{m}}}{\sum_{j=1}^{N}|Q_{j,k}|^{\frac{p_k}{m}}}\right)^m\\ &\quad=\sum_{(i_1,\dots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\Big(\sum_{j=1}^{N}|Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}-\frac{(i_k-i_{k-1})}{m}}\sum_{j=1}^{N}\prod_{k=1}^{n}|Q_{j,i}|^{\frac{p_k(i_k-i_{k-1})}{m}}.\end{aligned}$$

This ends the proof. □

3 Applications to refined certain inequalities for serval well-known special functions

In this section, we apply Theorems 2.3 and 2.4 to refined some inequalities for the (q,s)-Polygamma functions, the s-Extension of Nielsen's β-function, the derivatives of the s-Extension of Nielsen's β-function, the extended Gamma Function, the r-Gamma functions and the r-Riemann Zeta functions.

3.1 Refinements to the Turan-Type inequality for the (q,s)-Polygamma Functions

The (q,s)-analogue of the Gamma function, $\Gamma_{q,s}(x)$ is defined for $x>0$, $q\in(0,1)$ and $s>0$ by the following equality (see [5], [7]) and the references therein).

$$\Gamma_{q,s}(x)=\frac{1}{(1-q)^{\frac{x}{s}-1}}\prod_{k=0}^{+\infty}\frac{1-q^{(k+1)s}}{1-q^{ks+x}},$$

and the (q,s)-Polygamma functions, $\psi_{q,s}^{(N)}(x)$ are defined as follows (see [8]).

$$\psi_{q,s}^{(N)}(x)=(\ln(q))^{N+1}\sum_{k=1}^{\infty}\frac{(ks)^N q^{ksx}}{1-q^{ks}}:=\xi_N(x),$$

for $N \in \mathbb{N}$, where the function

$$\xi_\beta(x) := (\ln(q))^{\beta+1} \sum_{k=1}^{\infty} \frac{(ks)^\beta q^{ksx}}{1-q^{ks}}$$

is defined for all number $\beta \geq 1$ and for all positive number $x > 0$.

K. Nantomah in [9] proved the following Turan-type inequality involving the function $\psi_{q,k}^{(N)}(x)$.

Theorem 3.1 ([9]). For $k = 1, 2, \ldots, n$, let $p_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, $n_k \in \mathbb{N}$ and $\sum_{k=1}^n \frac{n_k}{p_k} \in \mathbb{N}$. Then the inequality

$$\psi_{q,s}^{(\sum_{k=1}^n \frac{n_k}{p_k})}\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) \leq \prod_{k=1}^n \Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}}, \tag{3.1}$$

holds for $x_k > 0$.

By using Theorem 2.4, we obtain the following refinements of the inequality (3.1).

Theorem 3.2. For $k = 1, 2, \ldots, n$, let $p_k > 1$, $\sum_{k=1}^n \frac{1}{p_k} = 1$, $m_k \in \mathbb{N}$ and $\sum_{k=1}^n \frac{n_k}{p_k} \in \mathbb{N}$. Then for all integers $m \geq 2$ and $x_k > 0$, we have

$$\begin{aligned}
&\psi_{q,s}^{(\sum_{k=1}^n \frac{n_k}{p_k})}\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) \\
&+ n r_0^m \prod_{k=1}^n \Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}} \Big[1 - \prod_{k=1}^n \Big(\psi_{q,k}^{(n_k)}(x_k)\Big)^{\frac{-p_k}{n}} \xi_{(\sum_{k=1}^n \frac{n_k}{n})}\Big(\sum_{k=1}^n \frac{x_k}{n}\Big)\Big] \\
&\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}} \xi_{(\sum_{k=1}^n \frac{(i_k-i_{k-1})n_k}{m})}\Big(\sum_{k=1}^n \frac{(i_k-i_{k-1})x_k}{m}\Big) \\
&\leq \prod_{k=1}^n \Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}},
\end{aligned}$$

where $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

Proof . To apply our Theorem 2.4, choose $Q_{j,k} = \frac{|\ln q|^{\frac{(n_k+1)}{p_k}} (ks)^{\frac{(n_k)}{p_k}} q^{\frac{jsx_k}{p_k}}}{(1-q^{js})^{\frac{1}{p_k}}}$, for $k = 1, 2, \ldots, n$. After some easy computations, we have the following equalities

$$\prod_{k=1}^n \Big(\sum_{j=1}^{\infty} |Q_{j,k}|^{p_k}\Big)^{\frac{1}{p_k}} = \prod_{k=1}^n \Big(\psi_{q,s}^{(n_k)}(x_k, z)\Big)^{\frac{1}{p_k}},$$

$$||Q_{j,k}||_{p_k}^{\frac{p_k}{n}} = \Big(\psi_{q,s}^{(n_k)}(x_k, z)\Big)^{\frac{1}{n}},$$

and

$$\sum_{j=1}^{+\infty} \Big|\prod_{k=1}^n Q_{j,k}\Big| = \psi_{q,s}^{(\sum_{k=1}^n \frac{n_k}{p_k})}\Big(\sum_{k=1}^n \frac{1}{p_k} x_k, z\Big),$$

$$\sum_{j=1}^{N} \prod_{k=1}^n |Q_{j,k}|^{\frac{p_k}{n}} = \xi_{(\sum_{k=1}^n \frac{n_k}{n})}\Big(\sum_{k=1}^n \frac{x_k}{n}\Big),$$

$$\sum_{j=1}^{+\infty} \prod_{k=1}^n |Q_{j,k}|^{\frac{p_k(i_k-i_{k-1})}{m}} = \xi_{(\sum_{k=1}^n \frac{(i_k-i_{k-1})n_k}{m})}\Big(\sum_{k=1}^n \frac{(i_k-i_{k-1})x_k}{m}\Big).$$

By virtue of Theorem 2.4, we have

$$\begin{aligned}
&\psi_{q,s}^{(\sum_{k=1}^{n}\frac{n_k}{p_k})}\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)\\
&+nr_0^m\prod_{k=1}^{n}\Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}}\Big[1-\prod_{k=1}^{n}\Big(\psi_{q,k}^{(n_k)}(x_k)\Big)^{\frac{-p_k}{n}}\xi_{(\sum_{k=1}^{n}\frac{n_k}{n})}\Big(\sum_{k=1}^{n}\frac{x_k}{n}\Big)\Big]\\
&\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}}\xi_{(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})n_k}{m})}\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\\
&\leq \prod_{k=1}^{n}\Big(\psi_{q,s}^{(n_k)}(x_k)\Big)^{\frac{1}{p_k}}.
\end{aligned}$$

This ends the proof. □

3.2 Refinements to the s-Extension of Nielsen's β-function inequality

We recall that [8] the Nielsen's β-function is defined by

$$\beta(x)=\int_0^1\frac{t^{x-1}}{1+t}dt,\quad x>0.$$

The s-Extension of Nielsen's β-function [8], is given by,

$$\beta_s(x)=\int_0^1\frac{t^{\frac{x}{s}-1}}{1+t}dt,\quad x>0.$$

K. Nantomah et al. [8] proved the following theorem involving the function $\beta_s(x)$.

Theorem 3.3 ([8]). Let $p,q>1$, be real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then for all real numbers $x,y\in(0,\infty)$,

$$\beta_s(\frac{x}{p}+\frac{y}{q})\leq\Big(\beta_s(x)\Big)^{\frac{1}{p}}\Big(\beta_s(y)\Big)^{\frac{1}{q}}. \tag{3.2}$$

By using Theorem 2.3, we obtain the following inequalities, of the s-Extension Nielsen's β-function.

Theorem 3.4. Let $p_k>1$ for $k=1,2,\ldots,n$ with $\sum_{k=1}^{n}\frac{1}{p_k}=1$ and $x_k\geq 0$. Then for all integers $m\geq 2$, we have

$$\begin{aligned}
&\beta_s\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)\quad+\quad nr_0^m\prod_{k=1}^{n}\beta_s^{\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^{n}\beta_s^{\frac{-1}{n}}(x_k)\beta_s\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big)\\
&\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\beta_s(x_k)^{1-\frac{i_k-i_{k-1}}{m}}\beta_s\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\\
&\leq \prod_{k=1}^{n}\beta_s^{\frac{1}{p_k}}(x_k),
\end{aligned}$$

where, $r_0=\min\{\frac{1}{p_k},\ k=1,\ldots,n\}$.

Proof . To apply our Theorem 2.3, we set $\Omega:=(0,1)$ and take the measure $d\mu(t):=\frac{1}{t(t+1)}dt$. Then we choose $f_k(t)=t^{\frac{x_k}{sp_k}}$, for $k=1,2,\ldots,n$. So we have the following equalities:

$$\int_\Omega\prod_{k=1}^{n}|f_k(t)|d\mu(t)=\beta_s\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big),$$

$$\int_{\Omega}\prod_{k=1}^{n}|f_k(t)|^{\frac{p_k}{n}}d\mu(t)=\beta_s\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big),$$

$$||f_k||_{p_k}=\Big[\beta_s(x_k)\Big]^{1/p_k},$$

and

$$\int_{\Omega}\prod_{k=1}^{n}|f_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}}d\mu(t)=\beta_s\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big).$$

By virtue of Theorem 2.3, we have

$$\begin{aligned}\beta_s\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big) \quad &+ \quad nr_0^m\prod_{k=1}^{n}\beta_s^{\frac{1}{p_k}}(x_k)\Big(1-\prod_{k=1}^{n}\beta_s^{\frac{-1}{n}}(x_k)\beta_s\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big)\\ &\leq \sum_{(i_1,\dots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\beta_s(x_k)^{1-\frac{i_k-i_{k-1}}{m}}\beta_s\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\\ &\leq \prod_{k=1}^{n}\beta_s^{\frac{1}{p_k}}(x_k).\end{aligned}$$

This ends the proof. □

3.3 Refinements of inequality involving derivatives of the s-Extension of Nielsen's β-function

The derivatives of the s-Extension of Nielsen's β-function [8], is given by,

$$\beta_s^{(N)}(x)=\frac{(-1)^N}{s^N}\int_0^{+\infty}\frac{t^Ne^{\frac{-xt}{s}}}{1+e^{-t}}dt,\quad x>0.$$

K. Nantomah et al. [8] proved the following theorem involving the function $\beta_s(x)$.

Theorem 3.5 ([8]). Let N be a positive integer and let $p,q>1$, be real numbers satisfying $\frac{1}{p}+\frac{1}{q}=1$. Then for all real numbers $x,y\in(0,\infty)$,

$$\Big|\beta_s^{(N)}(\frac{x}{p}+\frac{y}{q})\Big|\leq\Big|\beta_s^{(N)}(x)\Big|^{\frac{1}{p}}\Big|\beta_s^{(N)}(y)\Big|^{\frac{1}{q}}. \tag{3.3}$$

By using Theorem 2.3, we obtain the following inequalities, of the s-Extension Nielsen'sβ-function.

Theorem 3.6. Let $p_k>1$ for $k=1,2,\dots,n$ with $\sum_{k=1}^{n}\frac{1}{p_k}=1$ and $x_k\geq 0$. Then for all integers $m\geq 2$ and $N\geq 1$, we have

$$\begin{aligned}\Big|\beta_s^{(N)}\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)\Big| \quad &+ \quad nr_0^m\prod_{k=1}^{n}\Big|\beta_s^{(N)}(x_k)\Big|^{\frac{1}{p_k}}\Big(1-\prod_{k=1}^{n}\Big|\beta_s^{(N)}(x_k)\Big|^{\frac{-1}{n}}\Big|\beta_s^{(N)}\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big|\Big)\\ &\leq \sum_{(i_1,\dots,i_{n-1})\in A}C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}\Big|\beta_s^{(N)}(x_k)\Big|^{1-\frac{i_k-i_{k-1}}{m}}\Big|\beta_s^{(N)}\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\Big|\\ &\leq \prod_{k=1}^{n}\Big|\beta_s^{(N)}(x_k\Big|^{\frac{1}{p_k}},\end{aligned}$$

where, $r_0=\min\{\frac{1}{p_k},\ k=1,\dots,n\}$.

Proof . To apply our Theorem 2.3, we set $\Omega := (0, +\infty)$ and take the measure $d\mu(t) := \frac{t^N}{s^N(1+e^{-t})}dt$. Then we choose $f_k(t) = e^{\frac{-x_k t}{sp_k}}$, for $k = 1, 2, \dots, n$. Then we have the following identities:

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) = \Big|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{1}{p_k}x_k\Big)\Big|,$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \Big|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{1}{n}x_k\Big)\Big|,$$

$$||f_k||_{p_k} = \Big|\beta_s^{(N)}(x_k)\Big|^{1/p_k},$$

and

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_k - i_{k-1})}{m}} d\mu(t) = \Big|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big)\Big|.$$

By virtue of Theorem 2.3, we have

$$\begin{aligned}|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{1}{p_k}x_k\Big)| \quad &+ \quad nr_0^m \prod_{k=1}^n \Big|\beta_s^{(N)}(x_k)\Big|^{\frac{1}{p_k}}\Big(1 - \prod_{k=1}^n \Big|\beta_s^{(N)}(x_k)\Big|^{\frac{-1}{n}}\Big|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{1}{n}x_k\Big)\Big|\Big) \\ &\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^n \Big|\beta_s^{(N)}(x_k)\Big|^{1-\frac{i_k-i_{k-1}}{m}} \Big|\beta_s^{(N)}\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big)\Big| \\ &\leq \prod_{k=1}^n \Big|\beta_s^{(N)}(x_k\Big|^{\frac{1}{p_k}}.\end{aligned}$$

This ends the proof. □

3.4 Refinements to the extended Gamma Function inequality

In, 1994, Chaudhy and Zubair (see [3]) introduced the extended gamma function by setting

$$\Gamma_\omega(x) := \int_0^{+\infty} t^{x-1}e^{-t-wt^{-1}}dt, \quad \mathcal{R}(x) > 0, \quad w \in (0, +\infty).$$

M. Akkouchi and M. A. Ighachane [1], proved the following theorem:

Theorem 3.7 ([1]). Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real number $x, y \in [0, \infty)$ and all integers $m \geq 2$ we have

$$\begin{aligned}\Gamma_\omega\Big(\frac{x}{p} + \frac{y}{q}\Big) \quad &\leq \quad \Big(\frac{1}{p^m} + \frac{1}{q^m}\Big)\Big[\Gamma_\omega(x)\Big]^{\frac{1}{p}}\Big[\Gamma_\omega(y)\Big]^{\frac{1}{q}} \\ &+ \quad \sum_{k=1}^{m-1}\binom{k}{m}\frac{1}{p^k q^{m-k}}\Big[\Gamma_\omega(x)\Big]^{\frac{1}{p}-\frac{k}{m}}\Big[\Gamma_\omega(y)\Big]^{\frac{1}{q}-\frac{(m-k)}{m}} \times \Gamma_\omega\Big(\frac{k}{m}x + \frac{(m-k)}{m}y\Big) \\ &\leq \quad \Big[\Gamma_\omega(x)\Big]^{\frac{1}{p}}\Big[\Gamma_\omega(y)\Big]^{\frac{1}{q}}.\end{aligned}$$

By using Theorem 2.3, we obtain the following inequalities, for the extended gamma function.

Theorem 3.8. Let $p_k > 1$ for $k = 1, 2, \dots, n$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$, we have

$$\begin{aligned}
\Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) \quad &+ \quad n r_0^m \prod_{k=1}^{n} \Gamma_\omega^{\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_\omega^{\frac{-1}{n}}(x_k)\Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big) \\
&\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_\omega(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_\omega\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \\
&\leq \prod_{k=1}^{n} \Gamma_\omega^{\frac{1}{p_k}}(x_k),
\end{aligned}$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

Proof . To apply our Theorem 2.3, we set $\Omega := (0, +\infty)$ and take the measure $d\mu(t) := e^{-t-\omega t^{-1}} dt$. Then we choose $f_k(t) = t^{\frac{1}{p_k}(x_k-1)}$, for $k = 1, 2, \dots, n$. Then we have the following equalities:

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)| d\mu(t) = \Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big),$$

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big),$$

$$||f_k||_{p_k} = \Big[\Gamma_\omega(x_k)\Big]^{1/p_k},$$

and

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t) = \Gamma_\omega\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big).$$

By virtue of Theorem 2.3, we have

$$\begin{aligned}
\Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) \quad &+ \quad n r_0^m \prod_{k=1}^{n} \Gamma_\omega^{\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_\omega^{\frac{-1}{n}}(x_k)\Gamma_\omega\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big) \\
&\leq \sum_{(i_1,\dots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_\omega(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_\omega\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \\
&\leq \prod_{k=1}^{n} \Gamma_\omega^{\frac{1}{p_k}}(x_k).
\end{aligned}$$

This ends the proof. □

3.5 Refinements of inequality involving of the *r*-Gamma functions

In 2007, Diaz and Pariguan [3] also defined the r-analogue of the Gamma functions for $r > 0$ and $x \in \mathbb{C} \setminus r\mathbb{Z}^-$ as

$$\Gamma_r(x) = \int_0^{+\infty} t^{x-1} e^{-\frac{t^r}{r}} dt = \lim_{n \longrightarrow +\infty} \frac{n! r^n (nr)^{\frac{x}{r}-1}}{(x)_{n,r}}.$$

where $\lim_{r\longrightarrow 1} \Gamma_r(x) = \Gamma(x)$ and $(x)_{n,r} = x(x + r)(x + 2r)...(x + (n - 1)r)$ is the Pochhammer $r-$symbol.

In [10], W. T. Sulaiman, proved the following theorem:

Theorem 3.9 ([10]). Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real numbers $x, y \in [0, \infty)$ and for all $r > 0$, we have

$$\Gamma_r(\frac{x}{p} + \frac{y}{q}) \leq \Big(\Gamma_r(x)\Big)^{\frac{1}{p}} \Big(\Gamma_r(y)\Big)^{\frac{1}{q}}. \tag{3.4}$$

By using Theorem 2.3, we obtain the following generalization and refinements of the inequality (3.4):

Theorem 3.10. Let $p_k > 1$ for $k = 1, 2, \ldots, n$ with $\sum_{k=1}^{n} \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$, we have

$$\begin{aligned} \Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) \quad &+ \quad n r_0^m \prod_{k=1}^{n} \Gamma_r^{\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_r^{\frac{-1}{n}}(x_k)\Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big) \\ &\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_r(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_r\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \\ &\leq \prod_{k=1}^{n} \Gamma_r^{\frac{1}{p_k}}(x_k). \end{aligned}$$

where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \ldots, n\}$.

Proof . To apply our Theorem 2.3, we set $\Omega := (0, +\infty)$ and take the measure $d\mu(t) := dt$. Then we choose $f_k(t) = t^{\frac{1}{p_k}(x_{k-1})} e^{-\frac{t^r}{r p_k}}$, for $k = 1, 2, \ldots, n$. After some easy computations, we have the following equalities

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)| d\mu(t) = \Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big),$$

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big),$$

$$||f_k||_{p_k} = \Big[\Gamma_r(x_k)\Big]^{1/p_k},$$

and

$$\int_\Omega \prod_{k=1}^{n} |f_k(t)|^{\frac{p_k(i_k-i_{k-1})}{m}} d\mu(t) = \Gamma_r\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big).$$

By virtue of Theorem 2.3, we have

$$\begin{aligned} \Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{p_k} x_k\Big) \quad &+ \quad n r_0^m \prod_{k=1}^{n} \Gamma_r^{\frac{1}{p_k}}(x_k)\Big(1 - \prod_{k=1}^{n} \Gamma_r^{\frac{-1}{n}}(x_k)\Gamma_r\Big(\sum_{k=1}^{n} \frac{1}{n} x_k\Big)\Big) \\ &\leq \sum_{(i_1,\ldots,i_{n-1})\in A} C_A \frac{1}{p_1^{i_0-i_1} .. p_n^{i_{n-1}-i_n}} \prod_{k=1}^{n} \Gamma_r(x_k)^{1-\frac{i_k-i_{k-1}}{m}} \Gamma_r\Big(\sum_{k=1}^{n} \frac{(i_k - i_{k-1})x_k}{m}\Big) \\ &\leq \prod_{k=1}^{n} \Gamma_r^{\frac{1}{p_k}}(x_k). \end{aligned}$$

This ends the proof. □

3.6 Refinements of inequality involving of the r-Riemann zeta function

The $r-$Riemann zeta function is defined as

$$\zeta_r(x) = \frac{1}{\Gamma_r(x)} \int_0^{+\infty} \frac{t^{x-r}}{e^t - 1} dt.$$

In [10], W. T. Sulaiman, proved the following theorem:

Theorem 3.11 ([10]). Let $p, q > 1$, be real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for all real numbers $x, y \in [0, \infty)$ we have

$$\frac{\Gamma_r(\frac{x}{p} + \frac{y}{q})}{\Big(\Gamma_r(x)\Big)^{\frac{1}{p}} \Big(\Gamma_r(y)\Big)^{\frac{1}{q}}} \leq \frac{\Big(\zeta_r(x)\Big)^{\frac{1}{p}} \Big(\zeta_r(y)\Big)^{\frac{1}{q}}}{\zeta_r\Big(\frac{x}{p} + \frac{y}{q}\Big)}. \tag{3.5}$$

By using Theorem 2.3, we obtain the following generalization and refinements of the inequality (3.5):

Theorem 3.12. Let $p_k > 1$ for $k = 1, 2, \dots, n$ with $\sum_{k=1}^n \frac{1}{p_k} = 1$ and $x_k \geq 0$. Then for all integers $m \geq 2$, we have

$$\begin{aligned}
&\Gamma_r\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) \zeta_r\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) \\
&+ n r_0^m \prod_{k=1}^n [\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{1}{p_k}} \Big(1 - \prod_{k=1}^n [\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{-1}{n}} \Gamma_r\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big) \zeta_r\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big)\Big) \\
&\leq \sum_{(i_1, \dots, i_{n-1}) \in A} C_A \frac{1}{p_1^{i_0 - i_1} .. p_n^{i_{n-1} - i_n}} \prod_{k=1}^n [\Gamma_r(x_k)\zeta_r(x_k)]^{1 - \frac{i_k - i_{k-1}}{m}} \\
&\qquad \times \Gamma_r\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big) \zeta_r\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big) \\
&\qquad \leq \prod_{k=1}^n [\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{1}{p_k}}.
\end{aligned}$$

Where, $r_0 = \min\{\frac{1}{p_k},\ k = 1, \dots, n\}$.

Proof .

We apply Theorem 2.3, by setting $\Omega := (0, \infty)$ and considering the measure $d\mu(t) := dt$ in $(0, \infty)$, choose $f_k(t) = \frac{t^{\frac{x_k - r}{p_k}}}{\Big(e^t - 1\Big)^{\frac{1}{p_k}}}$. After some easy computations, we have the following equalities

$$\int_\Omega \prod_{k=1}^n |f_k(t)| d\mu(t) = \Gamma_r\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big) \zeta_r\Big(\sum_{k=1}^n \frac{1}{p_k} x_k\Big),$$

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k}{n}} d\mu(t) = \Gamma_r\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big) \zeta_r\Big(\sum_{k=1}^n \frac{1}{n} x_k\Big),$$

$$||f_k||_{p_k} = \Big[\Gamma_r(x_k)\zeta_r(x_k)\Big]^{1/p_k},$$

and

$$\int_\Omega \prod_{k=1}^n |f_k(t)|^{\frac{p_k(i_k - i_{k-1})}{m}} d\mu(t) = \Gamma_r\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big) \zeta_r\Big(\sum_{k=1}^n \frac{(i_k - i_{k-1})x_k}{m}\Big).$$

By virtue of Theorem 2.3, we have

$$\begin{aligned}
&\Gamma_r\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)\zeta_r\Big(\sum_{k=1}^{n}\frac{1}{p_k}x_k\Big)\\
&+nr_0^m\prod_{k=1}^{n}[\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{1}{p_k}}\Big(1-\prod_{k=1}^{n}[\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{-1}{n}}\Gamma_r\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\zeta_r\Big(\sum_{k=1}^{n}\frac{1}{n}x_k\Big)\Big)\\
&\le \sum_{(i_1,\ldots,i_{n-1})\in A} C_A\frac{1}{p_1^{i_0-i_1}..p_n^{i_{n-1}-i_n}}\prod_{k=1}^{n}[\Gamma_r(x_k)\zeta_r(x_k)]^{1-\frac{i_k-i_{k-1}}{m}}\\
&\qquad\qquad\times\Gamma_r\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\zeta_r\Big(\sum_{k=1}^{n}\frac{(i_k-i_{k-1})x_k}{m}\Big)\\
&\qquad\qquad\le\prod_{k=1}^{n}[\Gamma_r(x_k)\zeta_r(x_k)]^{\frac{1}{p_k}}.
\end{aligned}$$

This ends the proof. □

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References

[1] M. Akkouchi and M.A. Ighachane, *Some refinements to Hölder's inequality and applications,* Proyecciones J. Math. **39** (2020), no. 1, 153–166.

[2] E.F. Beckenbach and R. Bellman, *Inequalities,* Springer Verlag, Berlin, 1961.

[3] M.A. Chaudhry and S.M. Zubair, *On a class of incomplete gamma functions with applications,* Chapman and Hall/CRC, 2002.

[4] Z. Cvetkovski, *Inequalities: Theorems techniques and selected problems,* Springer: Berlin/Heidelberg, Germany, 2012.

[5] R. Díaz and C. Teruel, *q,k-generalized gamma and beta functions,* J. Nonlinear Math. Sci. **11** (2016), no. 1.

[6] M.A. Ighachane, M. Akkouchi and El. H. Benabdi, *A new generalized refinement of the weighted arithmetic-geometric mean inequality,* Math. Ineq. Appl. **23** (2020), no. 3, 1079–1085.

[7] F. Merovci, *Turan type inequalities for some (q,k)-special functions,* Acta Univer. Apulensis **34** (2013), 69–76.

[8] K. Nantomah, K. S. Nisar and K. S. Gehlot, *On a k-extension of the Nielsen's β-function* Int. J. Nonlinear Anal. Appl. **9** (2018) 191–201.

[9] K. Nantomah, *Generalized turan-type inequalities for the (q,k)-polygamma functions,* Commun. Mathe. Appl. **9** (2018), no. 2, 87–92.

[10] W. T. Sulaiman, *Turan inequalities for the Riemann Zeta functions,* AIP Conf. Proc. 1389 (2011), 1763–1797.

[11] Y. I. Kim and X.Yang *Turan inequalities for the Riemann Zeta functions,* Appl. Math. Lett. **25** (2012), no. 7, 1094–1097.

[12] X. Yang, *Hölder's inequality,* Appl. Math. Lett. **16** (2003), 897—903.