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Vibration analysis of an n^{th} order shear deformation nanoplate using the modified couple stress theory

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Abstract

This paper attempts to study the vibration characteristics of a simply supported n^{th} order rectangular nanoplate using the modified couple stress theory. The modified couple stress theory, which has only one length scale parameter, has been used to consider the small-scale effects. The basic and auxiliary equations of the nanoplate are obtained after determining the strain energy, kinetic energy, and external work and substituting them into Hamilton's equation. Then, the vibrations of the simply supported n^{th} order rectangular nanoplate with a thickness of h are investigated by substituting the boundary and force conditions into the governing equations. Navier's method is used for the solution. The results indicate that the frequencies of the different modes of the n^{th} order nanoplate decrease with an increase in the length-to-thickness ratio of the nanoplate. Furthermore, the frequency is the smallest when the effect of the size parameter is not considered (classical theory), and it increases with an increase in the size effect. In addition, the frequency is smallest for the first mode and increases for the subsequent modes. Also, the vibrational frequency increases with an increase in the order of the nanoplate.

Keywords: modified couple stress theory, n^{th} order nanoplate, vibrations, Navier solution method
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1 Introduction

The most reliable method to study materials on a small scale is the experimental method on atomic and molecular scales. In this method, the nanostructure is studied in its real dimensions. In this technique, an atomic force microscope (AFM) is used to apply different mechanical loads on the nanoplates, and the responses are measured to determine the mechanical properties of the nanostructures. The fundamental problems of this method are the difficulty in controlling the experimental conditions on this scale, the high costs, and the time-consuming nature of the method. Hence, this method is used only for validating other simple and inexpensive methods.

Atomic simulation is another approach in the study of small-scale structures. In this method, the behavior of the atoms and molecules are investigated by considering the intermolecular and interatomic effects on their movement, which ultimately leads to an overall deformation in the object. Using this method involves a high computational cost

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and is not economical when the problem includes large deformations or when the scale is larger than one or more atoms. Hence, this method is used only for problems with small deformations.

Given the limitations mentioned in the above methods for studying nanostructures, researchers have sought simpler approaches for investigating nanostructures. Small-scale modeling of structures using continuum mechanics is another approach to the study of these materials. There are various size-dependent continuum theories that have considered size effects, including the following: Micromorphic theory, microstructure theory, micropolar theory, Cosserat theory, nonlinear theory, modified couple stress theory, and strain gradient elasticity theory. These are the extended versions of the classical field theories, where the size effect has been considered.

2 Modified couple stress theory:

In 2002, Yang et al. [9] modified the stress couple theory presented by Toupin [7], Mindlin and Tiersten [4], Koiter [2], and Mindlin [3] to propose a modified couple stress theory with only one material length scale parameter for representing the size effect as opposed to the classical couple stress theory, which has two material length scale parameters.

In the modified couple stress theory, the strain energy density in three-dimensional vertical coordinates for an object limited to a volume V and a surface Ω is expressed as follows [8]:

$$U = \frac{1}{2}\int_V (\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij})\mathrm{d}V \qquad i,j = 1,2,3 \tag{2.1}$$

where

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.2}$$

$$\chi_{ij} = \frac{1}{2}(\theta_{j,i} + \theta_{j,i}) \tag{2.3}$$

Also χ_{ij} and ε_{ij} are the symmetric part of the curvature tensor and the strain tensor, respectively. θ_i and u_i are defined as the displacement vector and the rotation vector, respectively.

$$\theta = \frac{1}{2}\,\mathrm{Curl}\,u \tag{2.4}$$

Moreover, σ_{ij} and m_{ij} are the stress tensor and the deviatoric part of the couple stress tensor, respectively, which are defined as follows:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \tag{2.5}$$

$$m_{i,j} = 2\mu l^2\chi_{ij} \tag{2.6}$$

Where λ and μ are the Lamé parameters, δ_{ij} is the Kronecker delta, and l is the material length scale parameter. It can be inferred from Eq. (2.3) and (2.6) that χ_{ij} and m_{ij} are symmetric.

3 n^{th} order plate model:

The displacement equations of an nth order plate are defined as follows:

$$\begin{aligned}
u_1(x,y,z,t) =& z\varphi_x(x,y,t) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n\left(\frac{\partial w(x,y,t)}{\partial x} + \varphi_x(x,y,t)\right) \quad n = 3,5,7,9,\ldots \\
u_2(x,y,z,t) =& z\varphi_y(x,y,t) - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n\left(\frac{\partial w(x,y,t)}{\partial y} + \varphi_y(x,y,t)\right) \quad n = 3,5,7,9,\ldots \\
u_3(x,y,z,t) =& w(x,y,t)
\end{aligned} \tag{3.1}$$

where φ_x and φ_y are the rotations of the normal vector about the x and y axes, and w is the displacement of the midpoint of the plate along the z-axis. The symmetric part of the curvature tensor, the strain tensor, the stress tensor, and the rotation vector for the nth order plate model are as follows:

$$\varepsilon_{xx} = z\frac{\partial \varphi_x}{\partial x} - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_x}{\partial x}\right) \tag{3.2}$$

$$\varepsilon_{yy} = z\frac{\partial \varphi_y}{\partial y} - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial^2 w}{\partial y^2} + \frac{\partial \varphi_y}{\partial y}\right) \tag{3.3}$$

$$\varepsilon_{zz} = 0 \tag{3.4}$$

$$\varepsilon_{xy} = \varepsilon_{yx} = \frac{1}{2}z\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x}\right) - \frac{1}{2n}\left(\frac{2}{h}\right)^{n-1} z^n \left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial \varphi_y}{\partial x} + 2\frac{\partial^2 w}{\partial x \partial y}\right) \tag{3.5}$$

$$\varepsilon_{xz} = \varepsilon_{zx} = \frac{1}{2}\left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial w}{\partial x} + \varphi_x\right) \tag{3.6}$$

$$\varepsilon_{yz} = \varepsilon_{zy} = \frac{1}{2}\left(1 - 2\left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial w}{\partial y} + \varphi_y\right) \tag{3.7}$$

$$\theta_x = \frac{\partial w}{\partial y} - \frac{1}{2}\left(1 - 2\left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial w}{\partial y} + \varphi_y\right) \tag{3.8}$$

$$\theta_y = -\frac{\partial w}{\partial x} - \frac{1}{2}\left(1 - 2\left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial w}{\partial x} + \varphi_x\right) \tag{3.9}$$

$$\theta_z = \frac{1}{2}\left(z - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n\right)\left(\frac{\partial \varphi_y}{\partial x} - \frac{\partial \varphi_x}{\partial y}\right) \tag{3.10}$$

$$x_{xx} = \frac{\partial^2 w}{\partial x \partial y} - \frac{1}{2}\left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial \varphi_y}{\partial x}\right) \tag{3.11}$$

$$x_{yy} = -\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{2}\left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial \varphi_x}{\partial y} + \frac{\partial^2 w}{\partial x \partial y}\right) \tag{3.12}$$

$$x_{zz} = \frac{1}{2}\left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial \varphi_y}{\partial x} - \frac{\partial \varphi_x}{\partial y}\right) \tag{3.13}$$

$$x_{xy} = = \frac{1}{2}\left(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}\right) + \frac{1}{4}\left(1 - \left(\frac{2z}{h}\right)^{n-1}\right)\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \varphi_x}{\partial x} - \frac{\partial^2 w}{\partial y^2} - \frac{\partial \varphi_y}{\partial y}\right) \tag{3.14}$$

$$x_{xz} = \frac{1}{4}\left(z - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n\right)\left(\frac{\partial^2 \varphi_y}{\partial x^2} - \frac{\partial^2 \varphi_x}{\partial y \partial x}\right) - \frac{1}{4}\left(\left(\frac{2}{h} - \frac{2n}{h}\right)\left(\frac{2n}{h}\right)^{n-2}\right)\left(\frac{\partial w}{\partial y} + \varphi_y)\right) \tag{3.15}$$

$$x_{yz} = \frac{1}{4}\left(\left(\frac{2}{h} - \frac{2n}{h}\right)\left(\frac{2z}{h}\right)^{n-2}\right)\left(\frac{\partial w}{\partial x} + \varphi_x\right) + \frac{1}{4}\left(z - \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} z^n\right)\left(\frac{\partial^2 \varphi_y}{\partial x \partial y} - \frac{\partial^2 \varphi_x}{\partial y^2}\right) \tag{3.16}$$

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx} + \lambda\varepsilon_{yy} \tag{3.17}$$

$$\sigma_{yy} = \lambda\varepsilon_{xx} + (\lambda + 2\mu)\varepsilon_{yy} \tag{3.18}$$

$$\sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) \tag{3.19}$$

$$\sigma_{yx} = \sigma_{xy} = 2\mu\varepsilon_{xy} \tag{3.20}$$

$$\sigma_{xz} = \sigma_{zx} = 2\mu\varepsilon_{xz} \tag{3.21}$$

$$\sigma_{yz} = \sigma_{zy} = 2\mu\varepsilon_{yz} \tag{3.22}$$

The changes in the strain energy are expressed as follows:

$$\begin{aligned}\delta U = \int_V (&\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{yy}\delta\varepsilon_{yy} + 2\sigma_{xy}\delta\varepsilon_{xy} + 2\sigma_{xz}\delta\varepsilon_{xz} + 2\sigma_{yz}\delta\varepsilon_{yz} + m_{xx}\delta x_{xx} \\ &+ m_{yy}\delta x_{yy} + m_{zz}\delta x_{zz} + 2m_{xy}\delta x_{xy} + 2m_{xz}\delta x_{xz} + 2m_{yz}\delta x_{yz})\mathrm{d}V\end{aligned} \tag{3.23}$$

For simplification, the coefficients of the variables can be named from F_1 to F_{15} according to Eq. (3.24) and obtained individually.

$$\begin{aligned}\delta U = \int_V (&F_1\delta w,_{xx} + F_2\delta w,_{yy} + F_3\delta w,_{xy} + F_4\delta w,_x + E_5\delta w,_y + E_6\delta\varphi_{x,yy} \\ &+F_7\delta\varphi_{y,xx} + F_8\delta\varphi_{y,xy} + F_9\delta\varphi_{x,yx} + F_{10}\delta\varphi_{x,x} + F_{11}\delta\varphi_{y,y} + F_{12}\delta\varphi_{x,y} \\ &+F_{13}\delta\varphi_{y,x} + F_{14}\delta\varphi_x + F_{15}\delta\varphi_y)\mathrm{d}V\end{aligned} \tag{3.24}$$

where:

$$\begin{aligned}F_1 =& \frac{\partial^2 w}{\partial x^2}\left[(\lambda+2\mu)(A_3 - A_1A_2) + \tfrac{1}{2}\mu l^2(1+A_4) - \tfrac{1}{4}\mu l^2(1+A_4)(1-A_4)\right] \\ &+ \frac{\partial^2 w}{\partial y^2}\left[\lambda(A_3 - A_1A_2) - \tfrac{1}{2}\mu l^2(1+A_4) + \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_x}{\partial x}\left[-(\lambda+2\mu)(A_2A_1) - \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_y}{\partial y}\left[-\lambda(A_2A_1) - \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right]\end{aligned} \tag{3.25}$$

$$\begin{aligned}F_2 =& \frac{\partial^2 w}{\partial y^2}\left[(\lambda+2\mu)(A_3 - A_1A_2) + \tfrac{1}{2}\mu l^2(1+A_4) - \tfrac{1}{4}\mu l^2(1+A_4)(1-A_4)\right] \\ &+ \frac{\partial^2 w}{\partial x^2}\left[\lambda(A_3 - A_1A_2) - \tfrac{1}{2}\mu l^2(1+A_4) + \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_y}{\partial y}\left[-(\lambda+2\mu)(A_2A_1) - \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_x}{\partial x}\left[-\lambda(A_2A_1) - \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right]\end{aligned} \tag{3.26}$$

$$\begin{aligned}F_3 =& \frac{\partial^2 w}{\partial x\partial y}\left[4\mu A_2^2 + \mu l^2(1+A_4)^2\right] + \frac{\partial\varphi_x}{\partial y}\left[-2\mu A_2A_1 - \tfrac{1}{2}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_y}{\partial x}\left[-2\mu A_2A_1 - \tfrac{1}{2}\mu l^2(1-A_4)(1+A_4)\right]\end{aligned} \tag{3.27}$$

$$F_4 = \left(\frac{\partial w}{\partial x} + \varphi_x\right)\left[\mu(1-A_4)^2 + \tfrac{1}{4}\mu l^2A_5^2\right] + \left(\frac{\partial^2\varphi_y}{\partial x\partial y} - \frac{\partial^2\varphi_x}{\partial y^2}\right)\left[\tfrac{1}{4}\mu l^2A_5A_1\right] \tag{3.28}$$

$$F_5 = \left(\frac{\partial w}{\partial y} + \varphi_y\right)\left[\mu(1-A_4)^2 + \tfrac{1}{4}\mu l^2A_5^2\right] + \left(\frac{\partial^2\varphi_y}{\partial x\partial y} - \frac{\partial^2\varphi_y}{\partial x^2}\right)\left[\tfrac{1}{4}\mu l^2A_5A_1\right] \tag{3.29}$$

$$F_6 = F_8\left(\frac{\partial w}{\partial x} + \varphi_x\right)\left[\tfrac{1}{4}\mu l^2A_5^2A_1\right] + \left(\frac{\partial^2\varphi_y}{\partial x\partial y} - \frac{\partial^2\varphi_x}{\partial y^2}\right)\left[\tfrac{1}{4}\mu l^2A_1^2\right] \tag{3.30}$$

$$F_7 = F_9 = \left(\frac{\partial w}{\partial y} + \varphi_y\right)\left[-\tfrac{1}{4}\mu l^2A_5^2A_1\right] + \left(\frac{\partial^2\varphi_y}{\partial^2 x} - \frac{\partial^2\varphi_x}{\partial x\partial y}\right)\left[\tfrac{1}{4}\mu l^2A_1^2\right] \tag{3.31}$$

$$\begin{aligned}F_{10} =& \frac{\partial^2 w}{\partial x^2}\left[(\lambda+2\mu)(A_1^2 - zA_1) - \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial^2 w}{\partial y^2}\left[\lambda A_1(-z + A_1) + \tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right] \\ &+ \frac{\partial\varphi_x}{\partial x}\left[(\lambda+2\mu)A_1^2 + \tfrac{1}{4}\mu l^2(1-A_4)^2\right] + \frac{\partial\varphi_y}{\partial y}\left[\lambda A_1^2 - \tfrac{1}{4}\mu l^2(1-A_4)^2\right]\end{aligned} \tag{3.32}$$

$$\begin{aligned}F_{11} =&\frac{\partial^2 w}{\partial y^2}\left[(\lambda+2\mu)(A_1^2-zA_1)-\tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right]\\&+\frac{\partial^2 w}{\partial x^2}\left[\lambda A_1(-z+A_1)+\tfrac{1}{4}\mu l^2(1-A_4)(1+A_4)\right]\\&+\frac{\partial \varphi_y}{\partial y}\left[(\lambda+2\mu)A_1^2+\tfrac{1}{4}\mu l^2(1-A_4)^2\right]+\frac{\partial \varphi_x}{\partial x}\left[\lambda A_1^2-\tfrac{1}{4}\mu l^2(1-A_4)^2\right]\end{aligned} \tag{3.33}$$

$$\begin{aligned}F_{12} =&\frac{\partial^2 w}{\partial x\partial y}\left[-2\mu A_2A_1-\tfrac{1}{2}\mu l^2(1-A_4)(1+A_4)\right]\\&+\frac{\partial \varphi_x}{\partial y}\left[\mu A_1^2+\mu l^2(1-A_4)^2\right]+\frac{\partial \varphi_y}{\partial x}\left[\mu A_1^2-\tfrac{1}{2}\mu l^2(1-A_4)^2\right]\end{aligned} \tag{3.34}$$

$$\begin{aligned}F_{13} =&\frac{\partial^2 w}{\partial x\partial y}\left[-2\mu A_2A_1-\tfrac{1}{2}\mu l^2(1-A_4)(1+A_4)\right]\\&+\frac{\partial \varphi_x}{\partial y}\left[\mu A_1^2-\tfrac{1}{2}\mu l^2(1-A_4)^2\right]+\frac{\partial \varphi_y}{\partial x}\left[\mu A_1^2+\mu l^2(1-A_4)^2\right]\end{aligned} \tag{3.35}$$

$$F_{14}=\left(\frac{\partial w}{\partial x}+\varphi_x\right)\left[\mu(1-A_4)^2+\tfrac{1}{4}\mu l^2A_5^2\right]+\left(\frac{\partial^2\varphi_y}{\partial x\partial y}-\frac{\partial^2\varphi_x}{\partial y^2}\right)\left[\tfrac{1}{4}\mu l^2A_5A_1\right] \tag{3.36}$$

$$F_{15}=\left(\frac{\partial w}{\partial y}+\varphi_y\right)\left[\mu(1-A_4)^2+\tfrac{1}{4}\mu l^2A_5^2\right]+\left(\frac{\partial^2\varphi_x}{\partial x\partial y}-\frac{\partial^2\varphi_y}{\partial x^2}\right)\left[\tfrac{1}{4}\mu l^2A_5A_1\right] \tag{3.37}$$

The coefficients A are as follows:

$$A_1=z-\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n \tag{3.38}$$

$$A_2=\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^n \tag{3.39}$$

$$A_3=\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}z^{n+1} \tag{3.40}$$

$$A_4=4\left(\frac{2z}{h}\right)^{n-1} \tag{3.41}$$

$$A_5=\left(\frac{2}{h}-\frac{2n}{h}\right)\left(\frac{2z}{h}\right)^{n-2} \tag{3.42}$$

$$A_6=\frac{1}{n}\left(\frac{2}{h}\right)^{n-1} \tag{3.43}$$

$$A_7=\mu\left(\frac{2}{h}\right)^{n-1}I_{n-1} \tag{3.44}$$

$$A_8 = \mu \left(\frac{2}{h}\right)^{2n-2} I_{2n-2} \tag{3.45}$$

$$A_9 = (\lambda + 2\mu) \left(\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}\right)^2 I_{2n} \tag{3.46}$$

$$A_{10} = (\lambda + 2\mu)\frac{1}{n}\left(\frac{2}{h}\right)^{n-1} I_{n+1} \tag{3.47}$$

$$A_{11} = \frac{1}{4}\mu l^2 \left(\frac{1}{n}\left(\frac{2}{h}\right)^{n-1}\right)^2 (n^2 - n)^2 I_{2n-4} \tag{3.48}$$

$$A_{12} = \frac{1}{4}\mu l^2 h \tag{3.49}$$

$$I_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} Z^i \mathrm{d}z \qquad (i = 0, 1, 2, n-1, n, n+1, 2n-4, 2n-2, 2n) \tag{3.50}$$

4 Virtual work equation:

The virtual work done by the external force consists of three parts:

1. The virtual work done by the body forces on $V = \Omega * (-h/2, h/2)$
2. The virtual work done by the surface tractions at the top and bottom surfaces (Ω)
3. The virtual work done by the surface traction on the side surfaces $S = \Gamma * (-h/2, h/2)$ where Ω is the mid-plane of the plate, and Γ is the mid-perimeter of the plate.

If (f_x, f_y, f_z) is the body force, (c_x, c_y, c_z) is the body couple, (q_x, q_y, q_z) are forces exerted on the surface Ω, (t_x, t_y, t_z) is the Cauchy tractions, and (S_x, S_y, S_z) is the surface couple, the variations of virtual work will be as follows:

$$\begin{aligned} \delta w = -\Big[&\int_\Omega (f_x\delta u + f_y\delta V + f_z\delta w + q_x\delta u + q_y\delta V + q_z\delta w + c_x\delta\theta_x + c_y\delta\theta_y + c_z\delta\theta_z)\mathrm{d}x\mathrm{d}y \\ &+ \int_\Gamma (t_x\delta u + t_y\delta V + t_z\delta w + s_x\theta_x + s_y\delta\theta_y + s_z\delta\theta_z)\mathrm{d}\Gamma\Big] \end{aligned} \tag{4.1}$$

Since the only external force applied in this research is q_z, the virtual work is as follows:

$$\delta w = \int_0^a \int_0^b q(x, y)\delta w(x, y)\mathrm{d}x\mathrm{d}y \tag{4.2}$$

The changes in the kinetic energy are expressed as follows:

$$\delta T = \int_A \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\dot{u}_1\delta\dot{u}_1 + \dot{u}_2\delta\dot{u}_2 + \dot{u}_3\delta\dot{u}_3)\mathrm{d}A\mathrm{d}z \tag{4.3}$$

where ρ is the density. Furthermore, according to Hamilton's principle [1], the following holds:

$$\int_0^T (\delta T - (\delta U - \delta w))\mathrm{d}t = 0 \tag{4.4}$$

where T is the kinetic energy, U is the strain energy, and W is the work done by the external forces.

5 Plate equations

Using Hamilton's principle (Eq. (4.4)), the basic equations are obtained as follows

$$\begin{aligned}&\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial^2 F_1}{\partial x^2}-\frac{\partial F_4}{\partial x}+\frac{\partial^2 F_2}{\partial y^2}+\frac{\partial^2 F_3}{\partial x\partial y}-\frac{\partial F_5}{\partial y}\right)\mathrm{d}z\right]=q(x,y)+\rho I_0 w_{,tt}\\&-A_6^2\rho I_{2n}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)_{,tt}+A_6^2\rho J_{n+1}\left(\frac{\partial\varphi_x}{\partial x}+\frac{\partial\varphi_y}{\partial y}\right)_{,tt}\end{aligned}\tag{5.1}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial^2 F_6}{\partial y^2}+\frac{\partial^2 F_9}{\partial x\partial y}-\frac{\partial F_{12}}{\partial y}+\frac{\partial F_{10}}{\partial x}+F_{14}\right)\mathrm{d}z=\rho k_2\varphi_{x,tt}-A_6\rho J_{n+1}\left(\frac{\partial w}{\partial x}\right)_{,tt}\tag{5.2}$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\frac{\partial^2 F_7}{\partial x^2}-\frac{\partial F_{13}}{\partial x}+\frac{\partial^2 F_8}{\partial x\partial y}-\frac{\partial F_{11}}{\partial y}+F_{15}\right)\mathrm{d}z=\rho k_2\varphi_{y,tt}-A_6\rho J_{n+1}\left(\frac{\partial w}{\partial y}\right)_{,tt}\tag{5.3}$$

in which we have the following:

$$J_1=I_1-A_6 I_n\tag{5.4}$$

$$J_{n+1}=I_{n+1}-A_6 I_{2n}\tag{5.5}$$

$$K_2=I_2-2A_6 I_{n+1}-A_6^2 I_{2n}\tag{5.6}$$

6 Determination of the general equations of an n^{th} order plate

Considering the following parameters:

$$B_1=2A_{12}+l^2A_7+\frac{1}{2}l^2A_8+2A_9\tag{6.1}$$

$$B_2=\frac{1}{2}B_1=A_{12}+A_9+\frac{1}{2}l^2A_7+\frac{1}{4}l^2A_8\tag{6.2}$$

$$B_3=-\mu h+2A_7-A_8-A_{11}\tag{6.3}$$

$$B_4=A_9-A_{10}+\frac{1}{4}l^2A_8-A_{12}\tag{6.4}$$

$$B_5=3A_{12}-\frac{3}{2}l^2A_7+\frac{3}{4}l^2A_8-(\lambda+\mu)I_2+2(\lambda+\mu)A_6I_{n+1}-(\lambda+\mu)A_6^2I_{2n}\tag{6.5}$$

$$B_6=-\mu I_2+2\mu A_6 I_4-\mu A_6^2 I_6-4A_{12}+2l^2A_7-l^2A_8\tag{6.6}$$

$$B_7=\frac{1}{4}\mu l^2 I_2-\frac{1}{2}\mu l^2 A_6 I_{n+1}+\frac{1}{4}\mu l^2 A_6^2 I_{2n}\tag{6.7}$$

$$B_8=-(\lambda+2\mu)I_2+2A_{10}-A_9-A_{12}+\frac{1}{2}l^2A_7-\frac{1}{4}l^2A_8\tag{6.8}$$

$$\begin{aligned} B_9 =& \frac{5}{4}l^2A_8 - \frac{1}{2}\mu l^2A_6^2I_{2n-2} - l^2A_7 - \frac{1}{2}l^2nA_7 + 3A_{12} - (\lambda+\mu)I_2 - (\lambda+\mu)A_6^2I_{2n} \\ &+ 2(\lambda+\mu)A_6I_{n+1} \end{aligned} \tag{6.9}$$

$$B_{10} = \frac{3}{2}l^2A_7 + \frac{1}{2}l^2nA_7 - \frac{3}{2}l^2A_8 + \frac{1}{2}\mu l^2A_6^2I_{2n-2} - \mu I_2 - \mu A_6^2I_{2n} + 2\mu A_6I_{2n+1} - 4A_{12} \tag{6.10}$$

$$B_{11} = \rho A_6^2I_{2n} \tag{6.11}$$

$$B_{12} = \rho A_6I_{n+1} - \rho A_6^2I_{2n} \tag{6.12}$$

$$B_{13} = \rho I_2 - 2\rho A_6I_{n+1} - \rho A_6^2I_{2n} \tag{6.13}$$

The general equations of an n^{th} order plate will be as follows:

$$\begin{aligned} &B_1\frac{\partial^4w}{\partial x^2\partial y^2} + B_2\frac{\partial^4w}{\partial x^4} + B_2\frac{\partial^4w}{\partial y^4} + B_3\frac{\partial^2w}{\partial x^2} + B_3\frac{\partial^2w}{\partial y^2} + B_4\frac{\partial^3\varphi_x}{\partial x^3} + B_4\frac{\partial^3\varphi_x}{\partial x\partial y^2} \\ &+B_4\frac{\partial^3\varphi_y}{\partial y\partial x^2} + B_3\frac{\partial\varphi_x}{\partial x} + B_3\frac{\partial\varphi_y}{\partial y} + B_4\frac{\partial^3\varphi_y}{\partial y^3} = q(x,y) + \rho h\frac{\partial^2w}{\partial t^2} \\ &-B_{11}\left(\frac{\partial^4w}{\partial x^2\partial t^2} + \frac{\partial^4w}{\partial y^2\partial t^2}\right) + B_{12}\left(\frac{\partial^3\varphi_x}{\partial x\partial t^2} + \frac{\partial^3\varphi_y}{\partial y\partial t^2}\right) \end{aligned} \tag{6.14}$$

$$\begin{aligned} &-B_4\frac{\partial^3w}{\partial x\partial y^2} + B_5\frac{\partial^2\varphi_y}{\partial y\partial x} + B_6\frac{\partial^2\varphi_x}{\partial y^2} + B_7\frac{\partial^4\varphi_y}{\partial x\partial y^3} - B_7\frac{\partial^4\varphi_x}{\partial y^4} + B_7\frac{\partial^4\varphi_y}{\partial y\partial x^3} - B_7\frac{\partial^4\varphi_x}{\partial y^2\partial x^2} \\ &-B_3\frac{\partial w}{\partial x} - B_3\varphi_x - B_4\frac{\partial^3w}{\partial x^3} + B_8\frac{\partial^2\varphi_x}{\partial x^2} = -B_{12}\frac{\partial^3w}{\partial x\partial t^2} + B_{13}\frac{\partial^2\varphi_x}{\partial t^2} \end{aligned} \tag{6.15}$$

$$\begin{aligned} &-B_4\frac{\partial^3w}{\partial y\partial x^2} + B_9\frac{\partial^2\varphi_x}{\partial y\partial x} + B_{10}\frac{\partial^2\varphi_y}{\partial x^2} + B_7\frac{\partial^4\varphi_y}{\partial x^4} + B_7\frac{\partial^4\varphi_x}{\partial x^2\partial y^2} - B_7\frac{\partial^4\varphi_x}{\partial y\partial x^3} - B_7\frac{\partial^4\varphi_x}{\partial x\partial y^3} \\ &-B_4\frac{\partial^3w}{\partial y^3} - B_3\frac{\partial w}{\partial y} - B_3\varphi_y + b_8\frac{\partial^2\varphi_y}{\partial y^2} = -B_{12}\frac{\partial^3w}{\partial y\partial t^2} + B_{13}\frac{\partial^2\varphi_y}{\partial t^2} \end{aligned} \tag{6.16}$$

7 Navier solution method

The Navier solution method can be used for rectangular plates with simply supported boundary conditions on all edges. Since the boundary conditions are automatically satisfied in this method, the unknown functions at the mid-plane are expressed as dual trigonometric series, as follows [6, 10]:

$$W(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin\alpha x\sin\beta y e^{i\omega t} \tag{7.1}$$

$$\varphi_x(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn}\cos\alpha x\sin\beta y e^{i\omega t} \tag{7.2}$$

$$\varphi_y(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn}\sin\alpha x\cos\beta y e^{i\omega t} \tag{7.3}$$

The force is calculated as follows:

$$q = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin\alpha x \sin\beta y e^{i\omega t} \tag{7.4}$$

$$Q_{mn} = \frac{4}{ab}\int_0^a\int_0^b q(x,y)\sin\alpha x\sin\beta y \mathrm{d}x\mathrm{d}y \tag{7.5}$$

$$Q_{m,n} = \begin{cases} q_0 & \text{for a sinusoidal force} \\ \dfrac{16q_0}{mn\pi^2} & \text{for a uniform force} \\ \dfrac{4Q_0}{ab} & \text{for a point force at the center of the plate} \end{cases} \tag{7.6}$$

where

$$\alpha = \frac{\pi m}{a}, \quad \beta = \frac{\pi n}{b}, \quad i = \sqrt{-1} \tag{7.7}$$

The simply supported boundary conditions are satisfied by the Navier method according to the following equations:

$$\begin{cases} x = 0 \quad w(0,y) = w(a,y) = \sum\sum w_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y = 0 \\ , \\ x = a \quad \varphi_y(0,y) = \varphi_y(a,y) = \sum\sum y_{mn} \sin\frac{m\pi}{a}x\cos\frac{n\pi}{b}y = 0 \end{cases} \tag{7.8}$$

$$\begin{cases} y = 0 \quad w(x,0) = w(x,b) = \sum\sum w_{mn} \sin\frac{m\pi}{a}x \sin\frac{n\pi}{b}y = 0 \\ , \\ y = b \quad \varphi_x(x,0) = \varphi_x(x,b) = \sum\sum X_{mn} \cos\frac{m\pi}{a}x\sin\frac{n\pi}{b}y = 0 \end{cases} \tag{7.9}$$

8 Equation matrix of the n^{th} order plate:

After solution using the Navier method and naming the coefficients of the equation variables as follows:

$$M_1 = B_1\alpha^2\beta^2 + B_2\alpha^4 + B_2\beta^4 - B_3\alpha^2 - B_3\beta^2 \tag{8.1}$$

$$M_2 = M_4 = B_4\alpha^3 + B_4\alpha\beta^2 - B_3\alpha \tag{8.2}$$

$$M_3 = M_7 = B_4\beta^3 + B_4\alpha^2\beta - B_3\beta \tag{8.3}$$

$$M_5 = -B_7\beta^4 - B_7\alpha^2\beta^2 - B_6\beta^2 - B_8\alpha^2 - B_3 \tag{8.4}$$

$$M_6 = B_7\alpha\beta^3 + B_7\alpha^3\beta - B_5\alpha\beta \tag{8.5}$$

$$M_8 = -B_7\alpha^3\beta - B_7\alpha\beta^3 - B_9\alpha\beta \tag{8.6}$$

$$M_9 = B_7\alpha^4 + B_7\alpha^2\beta^2 - B_{10}\alpha^2 - B_8\beta^2 - B_3 \tag{8.7}$$

$$N_1 = -B_{11}\alpha^2 - B_{11}\beta^2 - \rho h \tag{8.8}$$

$$N_2 = N_4 = B_{12}\alpha \tag{8.9}$$

Table 1: Comparison between the frequencies of the first mode for different length-to-thickness ratios and length-scale-parameter-to-thickness ratios for an nth order nanoplate (MHZ)($a/b = 1$).

l/h	a/h					
	5	10	20	30	40	50
0	444.5855	121.6342	31.2161	13.9441	7.8576	5.0330
0.5	653.6982	173.8911	44.2699	19.7447	11.1202	7.1210
1	1055.3211	276.5826	70.1049	31.2407	17.5894	11.2621
2	1963.4588	511.3107	129.3539	57.6223	32.4387	20.7686

$$N_3 = N_7 = B_{12}\beta \tag{8.10}$$

$$N_5 = N_9 = -B_{13} \tag{8.11}$$

$$N_5 = N_9 = -B_{13} \tag{8.12}$$

The general matrix of the equations of an n^{th} order plate and the auxiliary equations will be obtained as follows:

$$\left(\begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{bmatrix} - \omega^2 \begin{bmatrix} N_1 & N_2 & N_3 \\ N_4 & N_5 & N_6 \\ N_7 & N_8 & N_9 \end{bmatrix}\right) \begin{bmatrix} w_{mn} \\ X_{mn} \\ y_{mn} \end{bmatrix} = \begin{bmatrix} Q_{mn} \\ 0 \\ 0 \end{bmatrix} \tag{8.13}$$

The plate is considered to be made of various materials, including epoxy, graphene, copper, etc. In this paper, the plate material is considered to be graphene. A thin single-layer graphene plate has the following properties [1]:

$$E = 1.06\,TPa, \quad \nu = 0.25, \quad h = 0.34nm, \quad \rho = 2250kg/m^3.$$

Moreover, the relationship between E and μ and ν can be written as follows:

$$\lambda = \frac{vE}{(1+\nu)(1-2\nu)}\,, \qquad \mu = \frac{E}{2(1+\nu)} \tag{8.14}$$

Where E is Young's modulus, and μ and λ are Lamé parameters [5]. Also, the force has been considered to be $q = 1N/m^2$.

9 Results and discussion:

The computational program was has been written in MATLAB software, and the results have been obtained from this software. All the boundary conditions have been considered to be simply supported.

Tables 1 to 4 indicate that the frequencies of the different modes ($\omega_{11} - \omega_{12} - \omega_{21} - \omega_{22}$) of the nth order nanoplate decrease with an increase in the length-to-diameter ratio. Furthermore, the frequency is the smallest when the effect of the size parameter is not considered (classical theory), and it increases with an increase in the size effect. In addition, the frequency is smallest for the first mode and increases for the subsequent modes.

Table 1 indicates that the frequencies of the different modes of the nth order nanoplate increase with an increase in the ratio of the length scale parameter to the thickness of the nanoplate.

Table 5 compares the different mode frequencies for the nth order nanoplate with different orders. As seen in the table, the vibrational frequency increases with an increase in the order of the nanoplate.

Tables 6 to 9 display the different mode frequencies ($\omega_{11} - \omega_{12} - \omega_{21} - \omega_{22}$) for different nanoplates. According to the table, the frequency is the highest for the Kirchhoff nanoplate when the length scale parameter is not considered.

Table 2: Comparison of the frequency ω_{12} for different length-to-thickness ratios and length-scale-parameter-to-thickness-ratios for an nth order nanoplate (MHZ)$(a/b = 1)$.

a/h	l/h			
	0	0.5	1	2
5	969.9190	1491.1907	2460.6235	4620.1898
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613
40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 3: Comparison of the frequency ω_{21} for different length-to-thickness ratios and length-scale-parameter-to-thickness-ratios for an nth order nanoplate (MHZ)$(a/b = 1)$.

a/h	l/h			
	0	0.5	1	2
5	969.9190	1491.1907	2460.6235	4620.1898
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613
40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 4: Comparison of the frequency ω_{22} for different length-to-thickness ratios and length-scale-parameter-to-thickness-ratios for an nth order nanoplate (MHZ)$(a/b = 1)$.

a/h	l/h			
	0	0.5	1	2
5	1400.0932	2231.9203	3742.7704	7073.5822
10	444.5855	653.6982	1055.3211	1963.4588
20	121.6342	173.8911	276.5826	511.3107
30	55.1098	78.3225	124.1752	229.2384
40	31.2161	44.2699	70.1049	129.3539
50	20.0437	28.3971	44.9443	82.9090

Table 5: Comparison of the different mode frequencies for the n^{th} order nanoplate with different orders (MHZ) $(a/b = 1, a/h = 5)$

Mode	n						
	3	5	7	9	11	13	15
ω_{11}	1055.3211	1058.2515	1063.3141	1067.3951	1070.5480	1073.0140	1074.9821
ω_{12}	2460.6235	2471.3233	2491.3787	2507.9604	2520.9555	2531.2138	2539.4538
ω_{21}	2460.6235	2471.3233	2491.3787	2507.9604	2520.9555	2531.2138	2539.4538
ω_{22}	3742.7704	3760.0712	3795.5355	3825.2651	3848.6823	3867.2063	3882.0979
ω_{33}	7581.5671	7601.9951	7668.8776	7724.6763	7767.7158	7801.0760	7827.4309

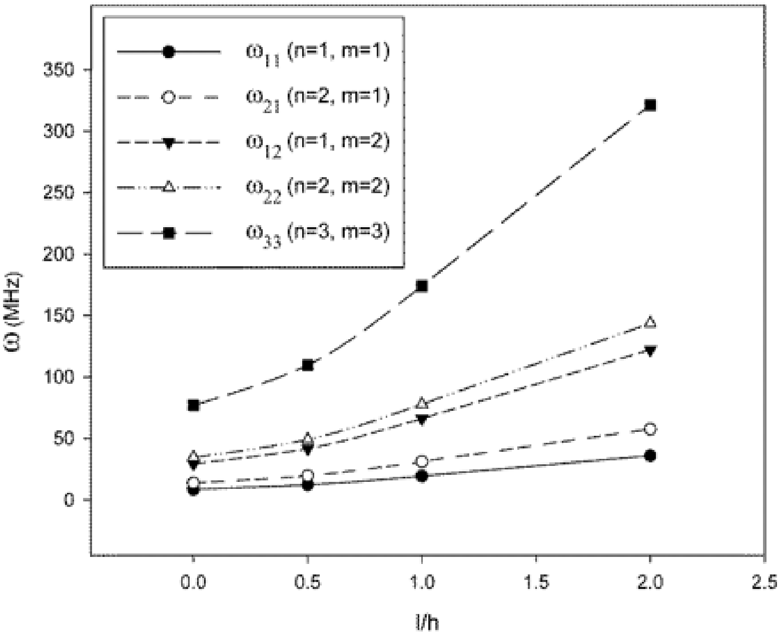

Figure 1: Comparison of the different mode frequencies for different length-parameter-to-thickness rations of the nth order nanoplate (MHZ) $(a/b = 2, a/h = 30)$

Table 6: Comparison of the frequency ω_{11} for different nanoplates (MHZ)$(a/b = 1)$

l/h	a/h				
	10	20	30	40	50
	Kirchhoff plate				
0	124.98838	31.43847	13.98857	7.87172	5.03883
0.5	176.76026	44.46071	19.78283	11.13229	7.12598
1	279.48251	70.29855	31.27940	17.60169	11.26717
2	515.34028	129.62412	57.67637	32.45591	20.77563
	Mindlin plate				
0	121.5505	31.2102	13.9429	7.8572	5.0329
0.5	249.5297	64.2570	28.7266	16.1924	10.3732
1	436.5378	115.4757	51.9052	29.3145	18.7965
2	722.2379	215.5686	99.1252	56.4382	36.3253
	Third-order shear deformation plate				
0	121.6342	31.2161	13.9441	7.8576	5.0330
0.5	173.8911	44.26997	19.7447	11.1202	7.1210
1	276.5826	70.1049	31.2407	17.5894	11.2621
2	511.3107	129.3539	57.6223	32.4387	20.7686

Table 7: Comparison of the frequency ω_{12} for different nanoplates (MHZ)($a/b = 1$)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	308.7461	436.6329	690.3772	1272.9926
20	78.3559	110.8119	175.2090	323.0695
30	34.9237	49.3895	78.0917	143.9940
40	19.6641	27.8093	43.9704	81.0774
50	12.5909	17.8062	28.1540	51.9135
	Mindlin plate			
10	289.5156	592.8023	988.5087	1223.2879
20	76.9722	158.2169	280.4153	492.9660
30	34.6425	71.3140	128.0217	237.9174
40	19.5743	40.3193	72.7219	137.8488
50	12.5539	25.8663	46.7575	89.4593
	Third-order shear deformation plate			
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613
40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 8: Comparison of the frequency ω_{21} for different nanoplates (MHZ)($a/b = 1$)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	308.7461	436.6329	690.3772	1272.9926
20	78.3559	110.8119	175.2090	323.0695
30	34.9237	49.3895	78.0917	143.9940
40	19.6641	27.8093	43.9704	81.0774
50	12.5909	17.8062	28.1540	51.9135
	Mindlin plate			
10	289.5156	592.8023	988.5087	1223.2879
20	76.9722	158.2169	280.4153	492.9660
30	34.6425	71.3140	128.0217	237.9174
40	19.5743	40.3193	72.7219	137.8488
50	12.5539	25.8663	46.7575	89.4593
	Third-order shear deformation plate			
10	289.9375	420.6563	674.3836	1250.9755
20	77.0069	109.6563	174.0385	321.4395
30	34.6497	49.1546	77.8533	143.6613
40	19.5766	27.7342	43.8941	80.9709
50	12.5548	17.7753	28.1226	51.8696

Table 9: Comparison of the frequency ω_{22} for different nanoplates (MHZ)($a/b = 1$)

a/h	l/h			
	0	0.5	1	2
	Kirchhoff plate			
10	488.2420	690.4785	1091.7424	2013.0735
20	124.9884	176.7603	279.4825	515.3403
30	55.8018	78.9157	124.7766	230.0767
40	31.4385	44.4607	70.2985	129.6241
50	20.1355	28.4759	45.0243	83.0207
	Mindlin plate			
10	443.6884	908.3644	1444.5250	1088.1654
20	121.5505	249.5297	436.5378	722.2379
30	55.0918	113.3246	202.1703	365.8010
40	31.2102	64.2570	115.4757	215.5686
50	20.0412	41.2804	74.4444	141.0270
	Third-order shear deformation plate			
10	444.5855	653.6982	1055.3211	1963.4588
20	121.6342	173.8911	276.5826	511.3107
30	55.1098	78.3225	124.1752	229.2384
40	31.2161	44.2699	70.1049	129.3539
50	20.0437	28.3971	44.9443	82.9090

10 Conclusion

This paper addressed the vibrations of an n^{th} order nanoplate using the modified couple stress theory. As observed in the tables and figures, the frequencies of the different modes of the n^{th} order nanoplate decrease with an increase in the length-to-thickness ratio of the nanoplate. Moreover, the frequency is the smallest when the effect of the size parameter is not considered (classical theory), and it increases with an increase in the size effect. Also, the frequency is smallest for the first mode and increases for the subsequent modes. Finally, the vibrational frequency increases with an increase in the order of the nanoplate.

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