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System of bipolar max-drastic product fuzzy relation equations with a drastic negation

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Abstract

This paper investigates the consistency of bipolar max-T_D Fuzzy Relation Equations (FREs) where T_D is the drastic product with a specific drastic negation operator. We firstly study the bipolar max-T_D fuzzy relation equation. Then the special characterizations of its feasible domain and its maximal and minimal solutions are presented. Furthermore, some necessary conditions and sufficient conditions are proposed for the solvability of a system of bipolar max-T_D FREs. Some examples are also given to illustrate them.

Keywords: Bipolar Fuzzy Relation Equation, Max-drastic product operator, Drastic negations, Feasible domain
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1 Introduction

Since fuzzy relation equations were presented in 1976 by Sanchez [21], they have extensively been used to formulate the applied problems. The most important applications of FREs can be considered in the engineering and medical sciences. FREs have an important role in engineering sciences such as fuzzy control [5], image processing [16, 18, 5], decision sciences and systems analysis [20], wireless communication [26], and operations research [12, 26]. Moreover, the relationship between cause and effect in medical sciences has been studied by FREs [21, 22]. Due to their repeatedly appearance in the applications, researchers focused on their theoretical aspects with the emphasis on the consistency criteria, the structure of solution set, and designing efficient algorithms for finding the solution set of the system of FREs [11, 17, 19, 23, 25, 28]. The variables involved in the systems of FREs have a unipolar character. In some applications, researchers encountered with systems of FREs containing the variables with bipolar characters. These systems contain both the variable vector and its logical negation, simultaneously. Such systems were applied to formulate the public awareness from products in the revenue management by Freson et al [10]. They investigated the Bipolar Fuzzy Relation Equations (BFREs) with the max-min composition accompanied by the residuated negation of the Lukasiewicz operator as $n_L(x) = 1 - x$, where $x \in [0, 1]$. Its solution set was determined by a finite set of maximal and minimal solution pairs when its feasible domain is non-empty. As Li and Jin show in Ref. [13], the check of consistency of BFRE system is NP-complete. Therefore, the resolution of BFRE system is computationally NP-hard. Then BFRE system using Lukasiewicz t-norm together with its negation was investigated to study a linear optimization problem. They firstly converted the system to a 0-1 integer system and then solved the corresponding 0-1 integer programming problem [14, 15]. Moreover, the system was considered by Yang [27] and its feasible domain

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was characterized by conservative bipolar paths and an algorithm was designed based on it. Some optimization problems subject to the BFRE were investigated using the max-parametric hamacher [2] and max-product [1, 3] with the negation of Lukasiewicz n_L. Recently, the system of bipolar max-product FRE has been studied with the product negation in [8] to extend the flexibility of the usual FRE. Its consistency and feasible domain have been discussed based on the number of variables and equations. The system has also been investigated for other t-norms in [6, 7, 24]. In this paper, the bipolar max-drastic product FREs are studied with one of the two drastic negation operators. This paper discusses about the consistency of the system of bipolar max-drastic product FREs with a specific drastic negation operator. The main contributions of this paper are the determination of the following items: (1) The necessary and sufficient conditions for consistency of a bipolar max-drastic product FRE with a specific drastic negation operator, (2) the solution set of a bipolar max-drastic product FRE with a specific drastic negation operator, (3) the maximal and minimal solutions of the solution set of a bipolar max-drastic product FRE with a specific drastic negation operator, and (4) some necessary conditions and sufficient conditions for solvability of a system of the bipolar max-drastic product FREs with a specific drastic negation operator.
The structure of the paper is as follows. Section 2 presents some required preliminary definitions and points. A bipolar max-drastic product FRE along with its negation is introduced with several variables and the necessary and sufficient conditions are given for its consistency in Section 3. The structure of solution set of one equation with several variables is determined in Section 4. The minimal solutions of the equation are discussed in Section 5. The structure of maximal solutions of the equation are characterized in Section 6. The system of bipolar max-drastic FREs with its negation is investigated and the necessary and sufficient conditions for its consistency are proposed in Section 7. Finally, the conclusions are given in Section 8.

2 Preliminaries

In this section, the required basic definitions are expressed.

Definition 2.1. [8] A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a triangular norm (t-norm) if the following properties are satisfied, for all $x, y, z \in [0,1]$:
1) $T(x,y) = T(y,x)$ (commutativity);
2) If $x \le y$ then $T(x,z) \le T(y,z)$ (monotonicity);
3) $T(x,1) = x$ (neutral element);
4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity).

Definition 2.2. [29] Mapping $T_D : [0,1] \times [0,1] \to [0,1]$ (drastic product) is a kind of t-norm which is defined as follows:

$$T_D(x,y) = \begin{cases} \min\{x,y\} & \text{if} \quad max\{x,y\} = 1, \\ 0 & otherwise. \end{cases}$$

Also, we can rewrite its simple form below:

$$T_D(x,y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & otherwise. \end{cases}$$

Definition 2.3. [4] A decreasing function $N : [0,1] \to [0,1]$ is called a fuzzy negation, if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is called
(i) strict, if it is strictly decreasing and continuous;
(ii) strong, if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0,1]$;
(iii) non-vanishing, if $N(x) = 0 \Leftrightarrow x = 1$.

Definition 2.4. [9, 4] (Drastic negations) We call n_D and n'_D drastic negations when

$$n_D(x) = \begin{cases} 1 & if \quad x \neq 1, \\ 0 & if \quad x = 1, \end{cases} \quad or \quad n'_D(x) = \begin{cases} 1 & if \quad x = 0, \\ 0 & if \quad x \neq 0. \end{cases} \tag{2.1}$$

which are the greatest and least fuzzy negations. They are non-strong negations. They are non-continuous negations.

We are now ready to present the necessary and sufficient conditions for the solvability of one bipolar fuzzy relation equation using the max-drastic product composition along with the n_D negation operator.

3 Bipolar max-T_D FRE with one equation

In this section, we introduce the bipolar max-T_D FRE system with n_D negation in (2.1) and a finite number of different unknown variables. Furthermore, we will find its feasible domain and maximal and minimal solutions for different cases.

Definition 3.1. Let $a_j^+, a_j^-, b \in [0,1]$ and T_D be the t-norm introduced in Definition 2.2. Also, the notations $\vee$ and n_D be the maximum operator and the negation in (2.1), respectively. Now, we will find the vectors $x = (x_1, \ldots, x_m) \in [0,1]^m$ such that the following equation holds:

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = b. \tag{3.1}$$

Equation (3.1) is called a simple bipolar max-T_D FRE with the n_D negation.

Theorem 3.2. Suppose that $b = 0$ in Equation (3.1). Equation (3.1) is solvable if and only if for each $j \in J$, either $a_j^+ = 0$ or $a_j^- = 0$ holds.

Proof. Firstly, suppose that for each $j \in J$, either $a_j^+ = 0$ or $a_j^- = 0$. We show that Equation (3.1) is solvable.
Under the these assumptions, we will show that vector $x = (x_1, x_2, \ldots, x_m)$ where $x_j = 1$ if $a_j^+ = 0$ and $x_j = 0$ if $a_j^- = 0$, is a solution for Equation (3.1). If $a_j^+ = 0$, then $x_j = 1$ and with regard to Definitions 2.2 and 2.4, we have:

$$T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = T_D(0,1) \vee T_D(a_j^-, n_D(1)) = 0 \vee 0 = 0,$$

If $a_j^- = 0$, then $x_j = 0$ and with regard to Definitions 2.2 and 2.4, we have:

$$T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = T_D(a_j^+, 0) \vee T_D(0, n_D(0)) = 0 \vee 0 = 0,$$

Therefore, the obtained vector $x = (x_1, x_2, \ldots, x_m)$ is feasible for the following equation:

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = 0.$$

Now, we suppose that Equation (3.1) is solvable and vector $x = (x_1, x_2, \ldots, x_m)$ is a solution for this equation. By contradiction, assume that there exists $k \in J$ such that $a_k^+ > 0$ and $a_k^- > 0$. With regard to Definition 2.4, we discuss two cases $x_k = 1$ and $x_k \neq 1$. If $x_k = 1$, then the following inequalities hold:

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) \geq T_D(a_k^+, 1) \vee T_D(a_k^-, n_D(1)) = a_k^+ \vee 0 = a_k^+ > 0,$$

which contradicts feasibility of vector $x = (x_1, x_2, \ldots, x_m)$ for Equation (3.1). Now, if $x_k \neq 1$, then the following inequalities hold:

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) \geq T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, x_k) \vee a_k^- \geq a_k^- > 0,$$

which also contradicts the hypothesis of feasibility of vector x.
In following Theorem, we will consider the solvability conditions of the bipolar max-T_D FRE with n_D negation with one constraint for $b \in (0,1]$.

Theorem 3.3. Consider Equation (3.1) with $b \in (0,1]$. Equation (3.1) is solvable if and only if for each $j \in J$, $a_j^+ \leq b$ or $a_j^- \leq b$ is held and there exists $k \in J$ such that at least one of the following conditions holds:
1) $b = a_k^+$, 2) $b = a_k^-$, and 3)($a_k^+ = 1$ and $a_k^- \leq b$).

Proof. At first, we will suppose that for each $j \in J$, $a_j^+ \leq b$ or $a_j^- \leq b$ and at least one of the conditions 1, 2 or 3 is satisfied, then we show that Equation (3.1) is solvable. For this purpose, we prove that $x = (x_1, x_2, \ldots, x_m)$ with

$$x_k = \begin{cases} 1 & (a_k^+ = b) \text{ or } (a_k^+ < b \text{ and } a_k^- > b), \\ 0 & (a_k^- = b) \text{ or } (a_k^+ < b \text{ and } a_k^- < b) \text{ or } (a_k^+ > b \text{ and } a_k^- < b), \\ b & a_k^+ = 1 \text{ and } a_k^- \leq b, \end{cases}$$

for each $k \in \{1, 2, \ldots, m\}$, is a solution of Equation (3.1).
If $a_k^+ = b$ and $x_k = 1$, then with regard to Definitions 2.2 and 2.4, the following equations hold:

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(b, 1) \vee T_D(a_k^-, n_D(1)) = b \vee 0 = b. \tag{3.2}$$

If $a_k^- = b$ and $x_k = 0$, then with regard to Definitions 2.2 and 2.4, we have

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, 0) \vee T_D(b, n_D(0)) = 0 \vee b = b. \tag{3.3}$$

If $(a_k^+ = 1,\ a_k^- \leq b)$ and $x_k = b$, then with regard to Definitions 2.2 and 2.4, the following equalities are true.

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(1, b) \vee T_D(a_k^-, n_D(b)) = b \vee a_k^- = b. \tag{3.4}$$

If $\big((a_k^+ < b,\ a_k^- < b) \text{ or } (a_k^+ > b, a_k^- < b)\big)$ and $x_k = 0$, then with regard to Definitions 2.2 and 2.4, we have the following results:

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, 0) \vee T_D(a_k^-, n_D(0)) = 0 \vee a_k^- = a_k^- < b. \tag{3.5}$$

If $(a_k^+ < b,\ a_k^- > b)$ and $x_k = 1$, with regard to Definitions 2.2 and 2.4, we have

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, 1) \vee T_D(a_k^-, n_D(1)) = a_k^+ \vee 0 = a_k^+ < b. \tag{3.6}$$

According to the assumption of satisfying at least one of the conditions 1, 2, or 3 and relations (3.2), (3.3), and (3.4) we have:

$$\exists\, j \in J \quad s.t \quad T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = b,$$

and also with regard to the relations (3.2)-(3.6), for each $j' \in J$, we have

$$T_D(a_{j'}^+, x_{j'}) \vee T_D(a_{j'}^-, n_D(x_{j'})) \leq b,$$

and

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) = b.$$

In order to prove its converse, suppose that equation (3.1) is solvable then we show for each $j \in J$, $(a_j^+ \leq b \text{ or } a_j^- \leq b)$ and at least one of the conditions 1, 2, or 3 is satisfied.
By contradiction, assume that there exists $k \in J$ such that $a_k^+ > b$ and $a_k^- > b$. With regard to Definition 2.4, two cases $x_k = 1$ and $x_k \neq 1$ are considered. If $x_k = 1$, then the following relations hold:

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) \geq T_D(a_k^+, 1) \vee T_D(a_k^-, n_D(1)) = a_k^+ \vee 0 = a_k^+ > b,$$

which contradicts the solvability of equation (3.1). Also, if $x_k \neq 1$, then the following inequalities hold

$$\bigvee_{j=1}^{m} T_D(a_j^+, x_j) \vee T_D(a_j^-, n_D(x_j)) \geq T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, x_k) \vee a_k^- \geq a_k^- > b,$$

which contradicts the assumption. Thus we can ensure that $a_k^+ \leq b$ or $a_k^- \leq b$. Now, we show that at least one of the conditions 1, 2, or 3 is satisfied. Since equation (3.1) is solvable, there exists $k \in J$ such that

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = b. \tag{3.7}$$

With regard to Definition 2.4, two cases $x_k = 1$ and $x_k \neq 1$ are considered. In equation (3.7), if $x_k = 1$, we have:

$$T_D(a_k^+, 1) \vee T_D(a_k^-, n_D(1)) = a_k^+ \vee 0 = a_k^+,$$

So b should be equal to a_k^+. Hence, condition 1 is satisfied. On the other hand, in equation (3.7), if $x_k \neq 1$, we have:

$$T_D(a_k^+, x_k) \vee T_D(a_k^-, n_D(x_k)) = T_D(a_k^+, x_k) \vee a_k^-,$$

In this relation, if $a_k^+ \neq 1$, then $T_D(a_k^+, x_k) = 0$. Therefore, b should be equal to a_k^- and condition 2 is satisfied and if $a_k^+ = 1$, then $T_D(a_k^+, x_k) = x_k$ and we have:

$$x_k \vee a_k^- = b.$$

This relation for $x_k = b$ and $a_k^- \leq b$ is satisfied. Thus, condition 3 is satisfied. This completes the proof.
Two numerical examples are given to illustrate Theorem 3.3.

Example 3.4. Two Examples of the bipolar max-T_D FRE are provided and we will apply Theorem 3.3 to check whether the equations are solvable or not.

(1) $T_D(0.4, x_1) \vee T_D(0.6, n_D(x_1)) \vee T_D(0.5, x_2) \vee T_D(0.9, n_D(x_2)) \vee T_D(0.7, x_3) \vee T_D(0.3, n_D(x_3)) = 0.5$

Since the assumptions in Theorem 3.3 hold (i.e., $a_2^+ = b$ and $a_j^+ \leq b$ or $a_j^- \leq b$ for $j = 1, 2, 3$), this equation is solvable and $x = (1, 1, 0)$ is a solution for this equation.

(2) $T_D(1, x_1) \vee T_D(0.6, n_D(x_1)) \vee T_D(0.4, x_2) \vee T_D(0.5, n_D(x_2)) \vee T_D(0.8, x_3) \vee T_D(0.9, n_D(x_3)) = 0.7$

In this equation $a_1^+ = 1$ and $a_1^- \leq 0.7$ (i.e., condition 3 in Theorem 3.3 is satisfied) but $a_3^+ > 0.7$ and $a_3^- > 0.7$. Therefore, this equation is not solvable.

4 The Solution Set of a Bipolar Max-T_D FRE with the n_D Negation

Let equation (3.1) be solvable, we intend to obtain solution set for two cases $b = 0$ and $b \neq 0$.

Case 1: The first, let $b = 0$. In this case, the feasible domain is given by $D = \prod_{j=1}^{m} D_j$ which for each $j \in J$, D_j defined as follows:

$$D_j = \begin{cases} \{1\} & a_j^+ = 0 \text{ and } a_j^- \neq 0, \\ [0, 1) & a_j^+ \neq 0, a_j^+ \neq 1 \text{ and } a_j^- = 0, \\ \{0\} & a_j^+ = 1 \text{ and } a_j^- = 0, \\ [0, 1] & a_j^+ = 0 \text{ and } a_j^- = 0. \end{cases}$$

Case 2: Now, let $b \neq 0$. In this case, the feasible domain is given by $D = D^{(1)} \cup D^{(2)} \cup D^{(3)}$ where $D^{(1)}$, $D^{(2)}$, and $D^{(3)}$ are feasible domains corresponding to conditions 1, 2, and 3 in Theorem 3.3, respectively. Each of the sets $D^{(1)}$, $D^{(2)}$, and $D^{(3)}$ will be obtained based on the conditions.
1- Condition 1 holds. Some index sets are defined as follow:

$$S^1 = \{k \in J | a_k^+ = b\}, \qquad K^1 = \{k \in S^1 | a_k^- \geq b\}, \qquad S_1 = \{k \in J | a_k^+ \leq b, a_k^- \leq b\},$$

$$S_2 = \{k \in J | a_k^+ \leq b, a_k^- > b\}, \quad S_3 = \{k \in J | 1 \neq a_k^+ > b, a_k^- \leq b\}, \quad \text{and} \quad S_4 = \{k \in J | a_k^+ = 1, a_k^- \leq b\}.$$

we determine the feasible solution set based on the index sets in the following cases:
(1-1) If $K^1 \neq \emptyset$, then for each $j \in J$, we define

$$D_j = \begin{cases} [0, 1] & j \in S_1, \\ \{1\} & j \in S_2, \\ [0, 1) & j \in S_3, \\ [0, b] & j \in S_4. \end{cases}$$

In this case, the feasible domain is given by $D^{(1)} = \prod_{j=1}^{m} D_j$.

(1-2) If $K^1 = \emptyset$, then the feasible domain is given by $D^{(1)} = \bigcup_{k \in S^1} D^k$, where $D^k = \prod_{j=1}^{m} D_j^k$ and set D_j^k is defined as follows:

$$D_j^k = \begin{cases} D_j & k \neq j, \\ \{1\} & k = j, \end{cases} \quad \text{where} \quad D_j = \begin{cases} [0,1] & j \in S_1, \\ \{1\} & j \in S_2, \\ [0,1) & j \in S_3, \\ [0,b] & j \in S_4. \end{cases}$$

2- Condition 2 holds. Consider index sets S_1, S_2, S_3, and S_4, and index sets that defined as follows:

$$S^2 = \{k \in J | a_k^- = b\}, \quad \text{and} \quad K^2 = \{k \in S^2 | a_k^+ \geq b\}.$$

We determine the feasible solution set based on the index sets in the following cases:
(2-1) If $K^2 \neq \emptyset$, then for each $j \in J$, we define

$$D_j = \begin{cases} [0,1] & j \in S_1, \\ \{1\} & j \in S_2, \\ [0,1) & j \in S_3, \\ [0,b] & j \in S_4. \end{cases}$$

In this case, the feasible solution set is given by $D^{(2)} = \prod_{j=1}^{m} D_j$.

(2-2) If $K^2 = \emptyset$, then feasible domain is given by $D^{(2)} = \bigcup_{k \in S^2} D^k$ where $D^k = \prod_{j=1}^{m} D_j^k$ and D_j^k is defined as follows:

$$D_j^k = \begin{cases} D_j & k \neq j \\ [0,1) & k = j \end{cases} \quad \text{where} \quad D_j = \begin{cases} [0,1] & j \in S_1, \\ \{1\} & j \in S_2, \\ [0,1) & j \in S_3, \\ [0,b] & j \in S_4. \end{cases}$$

3- Condition 3 holds. Consider index sets S_1, S_2, S_3, and S_4. In this case, the fesible domain is given by $D^{(3)} = \bigcup_{k \in S_4} D^k$ where $D^k = \prod_{j=1}^{m} D_j^k$ and D_j^k is defined as follows:

$$D_j^k = \begin{cases} D_j & k \neq j, \\ \{b\} & k = j, \end{cases} \quad \text{where} \quad D_j = \begin{cases} [0,1] & j \in S_1, \\ \{1\} & j \in S_2, \\ [0,1) & j \in S_3, \\ [0,b] & j \in S_4. \end{cases}$$

Example 4.1. Consider the following equation.

$$T_D(1, x_1) \vee T_D(0.2, n_D(x_1)) \vee T_D(0.6, x_2) \vee T_D(0.8, n_D(x_2)) \vee T_D(1, x_3) \vee T_D(0.4, n_D(x_3)),$$

$$\vee T_D(0.9, x_4) \vee T_D(0.5, n_D(x_4)) = 0.7,$$

with regard to conditions of Theorem 3.3, this equation is solvable. Therefore, we can obtain its feasible domain. Since $S^1 = S^2 = \emptyset$, we have $D^{(1)} = D^{(2)} = \emptyset$ and $D = D^{(3)}$. With regard to the method of calculation $D^{(3)}$ and the sets of $S_1 = \emptyset, S_2 = \{2\}, S_3 = \{4\},\ and\ S_4 = \{1,3\}$, we have $D^{(3)} = \bigcup_{k \in S_4} D^k = D^1 \cup D^3$, where

$$D^1 = \prod_{j=1}^{4} D_j^1 = \{0.7\} \times \{1\} \times [0, 0.7] \times [0, 1),$$

$$D^3 = \prod_{j=1}^{4} D_j^1 = [0, 0.7] \times \{1\} \times \{0.7\} \times [0, 1).$$

Example 4.2. Consider the following equation

$$T_D(0.6, x_1) \vee T_D(0.7, n_D(x_1)) \vee T_D(0.9, x_2) \vee T_D(0.5, n_D(x_2)) \vee T_D(0.7, x_3) \vee T_D(0.9, n_D(x_3))$$

$$\vee T_D(0.5, x_4) \vee T_D(0.5, n_D(x_4)) = 0.7$$

. By Theorem 3.3, this equation is solvable. Thus its feasible domain can be found. We have $S^1 = \{3\}$, $K^1 = \emptyset$, $S_1 = \{1, 4\}$, $S_2 = \{3\}$, $S_3 = \{2\}$, $S_4 = \emptyset$. Hence, we have:

$$D^{(1)} = \prod_{j=1}^{4} D_j = [0, 1] \times [0, 1) \times \{1\} \times [0, 1],$$

Similarly, it is concluded that $S^2 = \{1\}$, $K^2 = \emptyset$, $S_1 = \{1, 4\}$, $S_2 = \{3\}$, $S_3 = \{2\}$, $S_4 = \emptyset$. Consequently, we create set $D^{(2)}$ as follows:

$$D^{(2)} = \bigcup_{k \in S^2} D^k = D^1 = \prod_{j=1}^{4} D_j^1 = [0, 1) \times [0, 1) \times \{1\} \times [0, 1].$$

Hence, we have $D = D^{(1)} \cup D^{(2)}$.

5 Minimal Solutions for Bipolar Max-T_D FRE with the n_D Negation

Consider equation (3.1) and let this equation be solvable. With regard to the discussed points about the feasible domain in the previous section, we obtain minimal solutions for two cases $b = 0$ and $b \neq 0$.
Case 1: At first, let $b = 0$. By the structure of the feasible domain in this case, equation (3.1) has a minimal solution. The vector $\check{x} = [\check{x}_j]_{j \in J}$ is the minimal solution where

$$\check{x}_j = \begin{cases} 0 & a_j^- = 0, \\ 1 & a_j^- \neq 0, \end{cases}$$

Case 2: Secondly, let $b \neq 0$, By the structure of the feasible domain in this case, we put $M = M_1 \cup M_2 \cup M_3$ where M_1, M_2, and M_3 are the sets of minimal solution(s) corresponding to conditions 1, 2, and 3 in Theorem 3.3, respectively, and we will obtain each of the sets M_1, M_2, and M_3 based on the conditions.
1-1 If condition 1 is satisfied and index set $K^1 \neq \emptyset$, then there exists a minimal solution as $\check{x}^1 = [\check{x}_j]_{j \in J}$ corresponding to it, where

$$\check{x}_j = \begin{cases} 0 & j \notin S_2, \\ 1 & j \in S_2, \end{cases}$$

and we put $M_1 = \{\check{x}^1\}$.
1-2 If condition 1 is satisfied and index set $K^1 = \emptyset$, then the minimal solutions are vectors in the form $\check{x}^{1k} = [\check{x}_j^k]_{j \in J}$ with $k \in S^1$, where

$$\check{x}_j^k = \begin{cases} 1 & j \in S_2 \text{ or } j = k, \\ 0 & otherwise, \end{cases}$$

and we set $M_1 = \{\check{x}^{1k} | k \in S^1\}$. With regard to the assumptions in this case, we have, $|M_1| = |S^1|$.
2-1 If condition 2 is satisfied and index set $K^2 \neq \emptyset$, then there exists a minimal solution as $\check{x}^2 = [\check{x}_j]_{j \in J}$, where

$$\check{x}_j = \begin{cases} 0 & j \notin S_2, \\ 1 & j \in S_2, \end{cases}$$

and we put $M_2 = \{\check{x}^2\}$.
2-2 If condition 2 is satisfied and index set $K^2 = \emptyset$, then the minimal solutions are as $\check{x}^{2k} = [\check{x}_j^k]_{j \in J}$ with $k \in S^2$, where

$$\check{x}_j^k = \begin{cases} 0 & j \notin S_2 \text{ or } j = k, \\ 1 & Otherwise, \end{cases}$$

and we let $M_2 = \{\check{x}^{2k} | k \in S^2\}$. In this case, we have $|M_2| = |S^2|$.
3 If condition 3 is satisfied, then the minimal solutions are as $\check{x}^{3k} = [\check{x}_j^k]_{j \in J}$ with $k \in S_4$, where

$$\check{x}_j^k = \begin{cases} 1 & j \in S_2, \\ b & j = k, \\ 0 & Otherwise, \end{cases}$$

and $M_3 = \{\check{x}^{2k} | k \in S_4\}$ is set. In this case, we have $|M_2| = |S_4|$.

Example 5.1. Consider the equation in Example (4.1). With regard to the structure of the feasible domain of this equation, we have: $M_1 = M_2 = \emptyset$ and $M_3 = \{\check{x}^{31}, \check{x}^{33}\}$ where $\check{x}^{31} = (0.7, 1, 0, 0)$ and $\check{x}^{33} = (0, 1, 0.7, 0)$.

6 Maximal Solutions for Bipolar Max-T_D FRE with n_D Negation

By the structure of feasible domain discussed in Section 4, maximal solutions for two cases $b = 0$ and $b \neq 0$ are determined.
Case 1: Suppose that $b = 0$. If there exists $j \in J$ such that $a_j^+ \in (0, 1)$, then there is no maximal solution. Otherwise, when $a_j^+ = 0$ or $a_j^+ = 1$, there exist a maximal solution as $\hat{x}^1 = [\hat{x}_j]_{j \in J}$, where

$$\hat{x}_j = \begin{cases} 0 & \text{if } \ a_j^+ = 1, \\ 1 & \text{if } \ a_j^+ = 0. \end{cases}$$

Case 2: Now, let $b \neq 0$. With regard to the structure of its feasible domain in Section 4, if $S_3 \neq \emptyset$ then there is no maximal solution. Otherwise, we put $N = N_1 \cup N_2 \cup N_3$ where N_1, N_2, and N_3 are the sets of maximal solutions corresponding to the conditions 1, 2, and 3 in Theorem 3.3, respectively, and we will obtain each of the sets N_1, N_2, and N_3 based on the conditions.
1-1 If condition 1 is satisfied and index set $K^1 \neq \emptyset$, then there exists a maximal solution as $\hat{x}^1 = [\hat{x}_j]_{j \in J}$, where

$$\hat{x}_j = \begin{cases} b & j \in S_4, \\ 1 & j \notin S_4, \end{cases}$$

and we put $N_1 = \{\hat{x}^1\}$.
1-2 If condition 1 is satisfied and index set $K^1 = \emptyset$, then the maximal solutions are as $\hat{x}^{1k} = [\hat{x}_j^k]_{j \in J}$ with $k \in S^1$, where

$$\hat{x}_j^k = \begin{cases} b & j \in S_4 \ \text{ or } \ j = k, \\ 1 & Otherwise, \end{cases}$$

and we put $N_1 = \{\hat{x}^{1k} | k \in S^1\}$. Moreover, we have $|N_1| = |S^1|$.
2-1 If condition 2 is satisfied and index set $K^2 \neq \emptyset$, then there exists a maximal solution as $\hat{x}^2 = [\hat{x}_j]_{j \in J}$, where

$$\hat{x}_j = \begin{cases} b & j \in S_4, \\ 1 & j \notin S_4, \end{cases}$$

and we set $N_2 = \{\hat{x}^2\}$.
2-2 If condition 2 holds and index set $K^2 = \emptyset$, then there is no maximal solution.
3 If condition 3 is satisfied then the vectors $\hat{x}^{3k} = [\hat{x}_j^k]_{j \in J}$ are maximal solutions with $k \in S_4$, where

$$\hat{x}_j^k = \begin{cases} b & j \in S_4 \ \text{ or } \ j = k, \\ 1 & Otherwise, \end{cases}$$

and $N_3 = \{\hat{x}^{2k} | k \in S_4\}$ is put. In this case, we have $|N_3| = |S_4|$.

Example 6.1. Consider the equation mentioned in Example (4.1). Since index set $S_3 \neq \emptyset$, this equation has no any maximal solutions.

7 Bipolar Max-T_D FRE System with the n_D Negation

Let $a^+_{ij}, a^-_{ij}, b_i \in [0,1]$, for all $i \in I = \{1, 2, \dots, n\}$ and $j \in J = \{1, \dots, m\}$. The aim of bipolar max-T_D FRE system is to find the set of vectors $x \in [0,1]^m$ such that

$$\bigvee_{j=1}^{m} T_D(a^+_{ij}, x_j) \vee T_D(a^-_{ij}, n_D(x_j)) = b_i, \qquad i \in I. \tag{7.1}$$

First of all, we intend to present some conditions for the solvability of system (7.1).

Lemma 7.1. If in the i^{th} equation $a^+_{ij} > b_i$ and $a^-_{ij} > b_i$ for each $i \in I$ and $j \in J$, then system (7.1) is not solvable.

Proof. By contradiction, suppose that system (7.1) be solvable and $x = (x_j)_{j\in J}$ is a solution for this system. With regard to Definitions 2.2 and 2.4, we discuss the two following cases:

1. If $x_j = 1$, we have:
$$T_D(a^+_{ij}, 1) \vee T_D(a^-_{ij}, n_D(1)) = a^+_{ij} \vee 0 = a^+_{ij} > b_i,$$
2. If $x_j \neq 1$, we have:
$$T_D(a^+_{ij}, x_j) \vee T_D(a^-_{ij}, n_D(x_j)) = T_D(a^+_{ij}, x_j) \vee a^-_{ij} \geq a^-_{ij} > b_i,$$

Therefore, in each two cases, we conclude that vector x is not a solution. Hence, system (7.1) has no solution. According to Lemma 7.1, If system (7.1) is solvable, then for each $i \in I$ and $j \in J$, $a^+_{ij} \leq b_i$ or $a^-_{ij} \leq b_i$.

Theorem 7.2. Consider the system FREs (7.1). Let $b \in (0,1]$ and the system be solvable. Then
(a) for each $i \in I$, there exists $j \in J$ such that at least one of the following conditions holds:

1. $a^+_{ij} = b_i$ and $a^+_{hj} \leq b_h$ for each $h \in I$ or
2. $a^-_{ij} = b_i$ and $a^-_{hj} \leq b_h$ for each $h \in I$ or
3. $a^+_{ij} = 1$ and for $b_i \neq 1$, we have ($a^-_{hj} \leq b_h$ for each $h \in I$, and $b_i \leq b_k$ for each $k \in K = \{k \in I | a^+_{kj} = 1\}$) and also for $b_i = 1$, we have ($a^+_{hj} \leq b_h$ for each $h \in I$).

(b) If for each $i \in I$, there exists $j \in J$ such that there is not satisfied in none of the above conditions, in this case $a^+_{i_1 j} \leq b_{i_1}$ or $a^-_{i_2 j} \leq b_{i_2}$, where $a^+_{i_1 j} = \max_{i\in I}\{a^+_{ij}\}$ and $a^-_{i_2 j} = \max_{i\in I}\{a^-_{ij}\}$.

Proof. Suppose system (7.1) is solvable. Therefore, for each $i \in I$ there exist $j \in J$ such that

$$T_D(a^+_{ij}, x_j) \vee T_D(a^-_{ij}, n_D(x_j)) = b_i, \tag{7.2}$$

and for each $j' \in J$ and $j' \neq j$, we have:

$$T_D(a^+_{ij'}, x_{j'}) \vee T_D(a^-_{ij'}, n_D(x_{j'})) \leq b_i, \tag{7.3}$$

With regard to equality (7.2), we have two cases $T_D(a^+_{ij}, x_j) = b_i$ or $T_D(a^-_{ij}, n_D(x_j)) = b_i$. If $T_D(a^+_{ij}, x_j) = b_i$ then we have ($a^+_{ij} = b_i$ and $x_j = 1$) or ($a^+_{ij} = 1$ and $x_j = b_i$) and if $T_D(a^-_{ij}, n_D(x_j)) = b_i$ then we have ($a^-_{ij} = b_i$ and $n_D(x_j) = 1$). Now, with regard to the following cases and inequality (7.3), we discuss the following four cases to show that one of conditions 1, 2, 3 and statement (b) are satisfied.

1. If for $j' \in J$, there exists $i \in I$ such that $a^+_{ij'} = b_i$ and $x_{j'} = 1$, then for each $h \in I$, we have:
$$T_D(a^+_{hj'}, 1) \vee T_D(a^-_{hj'}, n_D(1)) = a^+_{hj'} \vee 0 = a^+_{hj'} \leq b_h,$$
Therefore condition 1 is satisfied.
2. If for $j' \in J$, there exists $i \in I$ such that $a^-_{ij'} = b_i$ and $n_D(x_{j'}) = 1$, then for each $h \in I$, we have:
$$T_D(a^+_{hj'}, x_{j'}) \vee T_D(a^-_{hj'}, n_D(x_{j'})) = T_D(a^+_{hj'}, x_{j'}) \vee a^-_{hj'} = \begin{cases} 0 \vee a^-_{hj'} & \text{if } a^+_{hj'} \neq 1, \\ x_{j'} \vee a^-_{hj'} & \text{if } a^+_{hj'} = 1, \end{cases}$$
which both of them are less than b_h. Since system (7.1) is solvable, in above equality, we have $a^-_{hj'} \leq b_h$. Thus condition 2 is satisfied.

3. If for $j' \in J$, there exists $i \in I$ such that $a^+_{ij'} = 1$ and $x_{j'} = b_i$, then for each $h \in I$, we have:

$$T_D(a^+_{hj'}, b_i) \vee T_D(a^-_{hj'}, n_D(b_i)) \le b_h,$$

According to this inequality, we have two cases $T_D(a^+_{hj'}, b_i) \le b_h$ and $T_D(a^-_{hj'}, n_D(b_i)) \le b_h$.
In the case $T_D(a^+_{hj'}, b_i) \le b_h$, if $a^+_{hj'} = 1$, then we have $T_D(a^+_{hj'}, b_i) = b_i \le b_h$. Therefore, we must have $b_i \le b_k$ for each $k \in K = \{k \in |a^+_{kj'} = 1\}$. If $a^+_{hj'} \neq 1$ and $b_i = 1$, then $T_D(a^+_{hj'}, b_i) = a^+_{hj'} \le b_h$ holds and if $a^+_{hj'} \neq 1$ and $b_i \neq 1$, then $T_D(a^+_{hj'}, b_i) = 0 \le b_h$ which it is always true.
In the case $T_D(a^-_{hj'}, n_D(b_i)) \le b_h$, if $b_i = 1$, we have $T_D(a^-_{hj'}, n_D(b_i)) = 0 \le b_h$ that it is always true, and if $b_i \neq 1$, we have $T_D(a^-_{hj'}, n_D(b_i)) = a_{hj'} \le b_h$. Thus condition 3 is satisfied.

4. If for $j' \in J$, there does not exist $i \in I$ such that $a^+_{ij'} = b_i$, $a^-_{ij'} = b_i$ or $a^+_{ij'} = 1$, we show that (b) holds. If there exist $i_1, i_2 \in I$ such that $a^+_{i_1j'} = \max_{i\in I}\{a^+_{ij'}\} > b_{i_1}$ and $a^-_{i_2j'} = \max_{i\in I}\{a^-_{ij'}\} > b_{i_2}$, then we will obtain a contradiction.
Since system (7.1) is solvable and with regard to Definition 2.4, we consider two states $x_{j'} = 1$ and $x_{j'} \neq 1$. If $x_{j'} = 1$, then

$$T_D(a^+_{i_1j'}, 1) \vee T_D(a^-_{i_1j'}, n_D(1)) = a^+_{i_1j'} \vee 0 = a^+_{i_1j'} > b_{i_1},$$

and if $x_{j'} \neq 1$ then

$$T_D(a^+_{i_2j'}, x_{j'}) \vee T_D(a^-_{i_2j'}, n_D(x_{j'})) = T_D(a^+_{i_2j'}, x_{j'}) \vee a^-_{i_2j'} \ge a^-_{i_2j'} > b_{i_2},$$

Two above inequalities contradict with inequality (7.3). Thus, we can ensure that statement (b) holds.

Theorem 7.3. Consider system FREs (7.1) and let $b \in (0, 1]$. This system is solvable if **(a)** there exist three index sets J_1, J_2 and $J_3 \subseteq J$ with $J_1 \cap J_2 = \emptyset$ and $J_1 \cap J_3 = \emptyset$ such that at least one of the following three conditions, holds for each $i \in I$:

1. there exists $j \in J_1$ such that $a^+_{ij} = b_i$ and $a^+_{hj} \le b_h$ for each $h \in I$,
2. there exists $j \in J_2$ such that $a^-_{ij} = b_i$ and $a^-_{hj} \le b_h$ for each $h \in I$,
3. there exists $j \in J_3$ such that $a^+_{ij} = 1$ and for $b_i \neq 1$, we have ($a^-_{hj} \le b_h$ for each $h \in I$, and $b_i \le b_k$ for each $k \in K = \{k \in I | a^+_{kj} = 1\}$) and also for $b_i = 1$, we have ($a^+_{hj} \le b_h$ for each $h \in I$),

and **(b)** for each $i \in I$, there exists $j \in J$ such that there is not satisfied in none of the above conditions, in this case $a^+_{i_1j} \le b_{i_1}$ or $a^-_{i_2j} \le b_{i_2}$, where $a^+_{i_1j} = \max_{i\in I}\{a^+_{ij}\}$ and $a^-_{i_2j} = \max_{i\in I}\{a^-_{ij}\}$.

Proof. Under the assumptions of theorem, we show that system (7.1) is solvable. To do this, we construct vector $x = (x_j)_{j\in J}$ as follows:

$$\forall j \in J: \quad x_j = \begin{cases} 1 & \text{if } j \in J_1 \text{ or } (a^+_{hj} \le b_h \text{ and } a^-_{i_2j} = \max_{h\in I}\{a^-_{hj}\} > b_{i_2}), \\ 0 & \text{if } j \in J_2 \text{ or } (a^+_{hj} \le b_h \text{ and } a^-_{hj} \le b_h) \text{ or } (a^+_{i_1j} = \max_{h\in I}\{a^+_{hj}\} > b_h \text{ and } a^-_{hj} \le b_h), \\ b_i & \text{if } j \in J_3 \text{ or } j \in J_2 \cap J_3, \end{cases}$$

Now, it is shown that vector x is a solution for system (7.1). We firstly show for each $i \in I$, there exists $j \in J$ such that

$$T_D(a^+_{ij}, x_j) \vee T_D(a^-_{ij}, n_D(x_j)) = b_i, \tag{7.4}$$

If $j \in J_1$, then $x_j = 1$. With regard to condition 1, we have:

$$T_D(a^+_{ij}, 1) \vee T_D(a^-_{ij}, n_D(1)) = a^+_{ij} \vee 0 = a^+_{ij} = b_i, \tag{7.5}$$

If $j \in J_2$, then $x_j = 0$. By condition 2, we have:

$$T_D(a^+_{ij}, 0) \vee T_D(a^-_{ij}, n_D(0)) = 0 \vee a^-_{ij} = a^-_{ij} = b_i, \tag{7.6}$$

If $j \in J_3$ or $j \in J_2 \cap J_3$, then $x_j = b_i$. Conditions 2 and 3 leads us to the following result:

$$T_D(1, b_i) \vee T_D(a^-_{ij}, n_D(b_i)) = b_i \vee T_D(a^-_{ij}, n_D(b_i)) = b_i. \tag{7.7}$$

Thus, with regard to relations (7.5), (7.6), and (7.7), relation (7.4) is satisfied. Now, we will show for each $j' \in J$ and $j' \neq j$, we have:

$$T_D(a^+_{ij'}, x_{j'}) \vee T_D(a^-_{ij'}, n_D(x_{j'})) \leq b_i, \quad i \in I, \tag{7.8}$$

Its reason is as follows:

1. If $j' \in J_1$, then $x_{j'} = 1$. By condition 1, it is concluded that

$$T_D(a^+_{hj'}, 1) \vee T_D(a^-_{hj'}, n_D(1)) = a^+_{hj'} \vee 0 = a^+_{hj'} \leq b_h, \quad h \in I. \tag{7.9}$$

2. If $j' \in J_2$, then $x_{j'} = 0$. With regard to condition 2, we have:

$$T_D(a^+_{hj'}, 0) \vee T_D(a^-_{hj'}, n_D(0)) = 0 \vee a^-_{hj'} = a^-_{hj'} \leq b_h, \quad h \in I. \tag{7.10}$$

3. If $j' \in J_3$ or $j' \in J_2 \cap J_3$, then $x_{j'} = b_i$. Conditions 2 and 3 lead us to the following results:

$$T_D(a^+_{hj'}, b_i) \vee T_D(a^-_{hj'}, n_D(b_i)) = \begin{cases} b_i \vee T_D(a^-_{hj'}, n_D(b_i)) & \text{if } a^+_{hj'} = 1, \\ a^+_{hj'} \vee 0 = a^+_{hj'} & \text{if } a^+_{hj'} \neq 1 \text{ and } b_i = 1, \\ 0 \vee a^-_{hj'} = a^-_{hj'} & \text{if } a^+_{hj'} \neq 1 \text{ and } b_i \neq 1, \end{cases} \quad h \in I, \tag{7.11}$$

   which in each of the three cases, the statement is less than or equal to b_h.

4. If for $j' \in J \setminus (J_1 \cup J_2 \cup J_3)$, and for each $h \in I$, ($a^+_{hj'} \leq b_h$ and $a^-_{hj'} \leq b_h$) or ($a^+_{i_1j'} = \max_{h \in I}\{a^+_{hj'}\} > b_h$ and $a^-_{hj'} \leq b_h$), then $x_{j'} = 0$. So, we have:

$$T_D(a^+_{hj'}, 0) \vee T_D(a^-_{hj'}, n_D(0)) = 0 \vee a^-_{hj'} = a^-_{hj'} \leq b_h, \quad h \in I, \tag{7.12}$$

5. If for $j' \in J \setminus (J_1 \cup J_2 \cup J_3)$, and for each $h \in I$, $a^+_{hj'} \leq b_h$ and $a^-_{i_2j'} = \max_{h \in I}\{a^-_{hj'}\} > b_{i_2}$, then $x_{j'} = 1$. Therefore, we have:

$$T_D(a^+_{hj'}, 0) \vee T_D(a^-_{hj'}, n_D(0)) = 0 \vee a^-_{hj'} = a^-_{hj'} \leq b_h, \quad h \in I. \tag{7.13}$$

With regard to relations (7.9)-(7.13), relation (7.8) is satisfied.

Example 7.4. Consider the following system of bipolar max-T_D FREs $A^+ \circ x \vee A^- \circ \neg x = b$, where

$$A^+ = \begin{pmatrix} 0.3 & 0.57 & 0.59 & 0.6 & 0.75 \\ 1 & 0.49 & 0.78 & 1 & 0.52 \\ 1 & 0.39 & 0.6 & 0.6 & 0.26 \\ 0.7 & 0.58 & 0.41 & 0.8 & 0.43 \\ 0.17 & 0.27 & 0.45 & 0.33 & 0.29 \end{pmatrix}, \quad A^- = \begin{pmatrix} 0.6 & 0.3 & 0.43 & 0.56 & 0.07 \\ 0.65 & 0.5 & 0.48 & 0.43 & 0.22 \\ 0.35 & 0.63 & 0.51 & 0.68 & 0.26 \\ 0.43 & 0.5 & 0.39 & 0.53 & 0.56 \\ 0.27 & 0.2 & 0.45 & 0.38 & 0.44 \end{pmatrix},$$

and $b = (0.6, 0.8, 0.7, 0.58, 0.45)^T$. With regard to conditions of Theorem 7.3, this system is solvable. We can put $J_1 = \{2\}$, $J_2 = \{1, 3\}$ and $J_3 = \{1, 4\}$. Therefore $x = (0.7, 1, 0, 0.8, 0)$ is a solution for this system. Also we can put $J_1 = \{2, 3\}, J_2 = \{1\}$ and $J_3 = \{1, 4\}$. Hence $x = (0.7, 1, 1, 0.8, 0)$ is another solution for system.

8 Conclusions

We studied the solvability of system of bipolar max-drastic product fuzzy relation equations with a specific drastic negation of n_D. First of all, we provided the necessary and sufficient conditions for consistency in a particular case when the system contains an equation. The solution set of the solvable equation was determined in two cases under certain conditions. Also, specifying the consistency conditions of the equation, the conditions related to existence of maximal and minimal solutions were determined. Then, we obtained its set of minimal and maximal solutions in each case according to the conditions. Finally, we presented the necessary conditions and the sufficient conditions for solvability of the system of bipolar max-drastic product FREs with the n_D negation. The systems using max-drastic product with the n_D negation have not yet been investigated. Hence, the determination of necessary conditions and sufficient conditions for checking consistency of the systems could be one of the main challenges which this paper overcame it. Recently, the systems of bipolar max t-norm FREs have been investigated by minimum t-norm [10, 13], the product t-norm [7], and the lukasiewicz t-norm [27] with the standard negation. The systems with the product t-norm and the n'_D negation have been considered in [6, 8]. Furthermore, the authors in [24] showed that the solutions of the system of bipolar max-Archimedean FREs with the standard negation correspond to irredundant coverings of the covering problem. The t-norms and negations used in the references [6, 7, 8, 10, 13, 24, 27] are completely different with the drastic product t-norm and the n_D negation. Therefore, we cannot apply the results and methods obtained in [6, 7, 8, 10, 13, 24, 27] for the proposed system in this paper. Hence, the existing methods in [6, 7, 8, 10, 13, 24, 27] cannot be used for checking the consistency of the proposed system and the determination of the solution set and the minimal and maximal solutions of one of its equations, in a general case. These points are the advantages of this research compared to the existing methods in [6, 7, 8, 10, 13, 24, 27].

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