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Some sandwich theorems for meromorphic univalent functions defined by Hadamard product of integral operators

Waggas Galib Atshan[a], Suad Hassan Mahdy[b,*]

[a]*Department of Mathematics, College of Science, University of Al-Qadisiyah, Al-Diwaniyah, Iraq*
[b]*Department of Mathematics, College of Education for Girls, University of Kufa, Najaf, Iraq*

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Abstract

In the present paper, we obtain some subordination and superordination results, involving the operator T^a for functions of the form $f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k$, which are meromorphic univalent in the punctured open unit disk these results are applied to obtain sandwich results.

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1. Introduction

Let $\sum$ denote the class of functions of the form:

$$f(z) = z^{-1} + \sum_{k=1}^{\infty} a_k z^k \tag{1.1}$$

which are meromorphic univalent in the punctured open unit disk $U^* = \{z : z \in \mathbb{C},\ 0 < |Z| < 1\}$.

Let R be the linear space of all analytic functions in U. For a positive integer number n and $a \in \mathbb{C}$, we let,

$$R[a, n] = \{f \in R : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\}.$$

*Corresponding author
Email addresses: waggashnd@gmail.com, waggas.galib@qu.edu.iq (Waggas Galib Atshan), sadhsn380@gmail.com (Suad Hassan Mahdy)

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For f and g analytic functions in R, we say that f is subordinate to g in U and write $f(z) \prec g(z)$, if there exists Schwars function μ, which is analytic in U with $\mu(0) = 0$ and $|\mu(z)| < 1$ $(z \in U)$, such that $f(z) = g(\mu(z)),\ (z \in U)$.

Furthermore, if the function g is univalent in U, we have the following equivalence relationship (c f., e.g. [12, 15, 16]),

$$f(z) \prec g(z) \leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U),\ (z \in U).$$

Definition 1.1. *[15] Let $\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be univalent in U. If $p(z)$ is analytic in U and satisfies the second-order differential aubordination:*

$$\phi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{1.2}$$

then $p(z)$ is called a solution of the differential subordination (1.2)*, and the univalent function $q(z)$ is called a dominant of the solution of the differential subordination* (1.2)*, or more simply dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying* (1.2)*. A univalent dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(Z)$ for all dominant $q(z)$ of* (1.2) *is said to be the best dominant is unique up to a relation of U.*

Definition 1.2. *([15]also see [13]) Let $\phi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$ and let $h(z)$ be analytic in U. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and if P satisfies the second-order differential superordination,*

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z), z) \tag{1.3}$$

then $p(z)$ is called a solution of the differential superordination (1.3)*. An analytic function $q(z)$ which is called a subordinant of the solutions of the differential superordination* (1.3) *or more simply a subordinant, if $q \prec p$ for all p satisfying* (1.3)*. A univalent subordinant $\tilde{q}(z)$ that satisfies $q \prec \tilde{q}$ for all subordinants q of* (1.3) *is said to be the best subordinant.*

Several authors [1, 2, 9, 13, 15, 17] obtained sufficient conditions on the functions h, p and ϕ for which the following implication holds

$$\phi(p(z), zp'(z), z^2p''(z); z).$$

Then

$$q(z) \prec p(z). \tag{1.4}$$

Using the results (see [3, 4, 5, 6, 10, 11, 16]) to obtain sufficient conditions for normalized analytic function to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$. Also, several authors (see [1, 3, 5, 6, 7, 8, 13]) derived some differential subordination and superordination results with some sandwich theorems.

Let $f \in \sum$ is given by (1.1) *and $g \in \sum$, defined by*

$$g(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k,\ z \in U^*.$$

The convolution (or Hadamard product) of the functions f and g denoted by $f * g$ is defined by

$$(f * g)(z) = z^{-1} + \sum_{k=1}^{\infty} b_k z^k, \ \ z \in U^*.$$

Lashin [14] found several properties of integeral operator:

$$P_\beta^\alpha : \sum \to \sum,$$

which defined as follows:

$$\begin{aligned} P_\beta^\alpha f(z) &= \frac{\beta^\alpha}{\lceil(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^\beta (\log \frac{z}{t})^{\alpha-1} f(t) dt (\alpha > 0, \beta > 0, z \in U^*) \\ &= z^{-1} + \sum_{k=1}^{\infty} (\frac{\beta}{k+\beta+1})^\alpha a_k z^k, z \in U^*, (\alpha, \beta > 0; z \in U^*). \end{aligned}$$

Atshan, Battor and Abaas [7] found some sandwich theorems for meromorphic univalent functions defined by the integral operator

$$R^\eta : \sum \to \sum,$$

which defined as follows:

$$R^\eta f(z) = \left(\frac{\lambda+\tau-1}{\gamma+\delta-r}\right) z^{-1-\frac{\lambda+\tau-1}{\gamma+\delta-r}} \int_0^z t^{\left(\frac{\lambda+\tau-1}{\gamma+\delta-r}\right)} f(t) dt, \tag{1.5}$$

$$(\lambda > 1, \gamma > 1, \delta > 0, \eta > 0, \tau > 0, 0 < r < 1; z \in U^*),$$

such that

$$R^\eta f(z) = z^{-1} + \sum_{k=1}^{\infty} \left(\frac{\lambda+\tau-1}{\lambda+\tau-1+(k+1)(\gamma+\delta-r)}\right)^\eta a_k z^k. \tag{1.6}$$

Define the convolution (or Hadamard product) $T^{\alpha,\eta} f(z)$ *of the operators* $P_\beta^\alpha f(z)$ *and* $R^\eta f(z)$ *as followes:*

$$T_{\beta,\delta,\tau,r}^{\alpha,\eta\lambda,\gamma} f(z) = z^{-1} + \sum_{k=1}^{\infty} \left[\frac{\beta}{k+\beta+1}\right]^\alpha \left[\frac{\lambda+\tau-1}{\lambda+\tau-1+(k+1)(\gamma+\delta-r)}\right]^\eta a_k z^k. \tag{1.7}$$

In our paper, we will denote to the Hadamard product operator $T_{\beta,\delta,\tau,r}^{\alpha,\eta\lambda,\gamma} f(z)$ *by* $T^a f(z)$*. From* (1.7)*, we note that*

$$z(T^{a+1} f(z))' = \beta T^\alpha f(z) - (\beta+1) T^{\alpha+1} f(z). \tag{1.8}$$

The main object of this idea is to find sufficient conditions for certain analytic functions f *in* U^* *satisfy:*

$$q_1(z) \prec \left(\frac{(1-\rho) z T^a f(z) + \rho z T^{a+1} f(z)}{\rho}\right)^\mu \prec q_2(z),$$

$$q_1(z) \prec (z T^{a+1} f(z))^\mu \prec q_2(z).$$

2. Preliminaries

Lemma 2.1. *[16] Let q be a convex univalent function in U and let $\alpha \in \mathbb{C},\ \Psi \in \mathbb{C}\backslash\{0\}$ with $q(0) = 1$,*

$$Re\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\{0, -Re\{\frac{\alpha}{\Psi}\}\}.$$

If p is analytic in U and

$$\alpha p(z) + \Psi zp'(z) \prec \alpha q(z) + \Psi zq'(z), \tag{2.1}$$

then $p \prec q$, and q is best dominant of (2.1).

Lemma 2.2. *[4] Let q be univalent in the unit disk U and let θ and ϕ be analytic in the domain D containing $q(U)$ with $\phi(w) \neq 1$, when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z) \text{ and } h(z) = \theta(q(z)) + Q(z).$$

Suppose that

- *$Q(z)$ is starlike univalent in U,*
- *$Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in U$.*

If p is analytic in U, with $p(0) = q(0); p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.2}$$

then $p \prec q$ and q the best dominant of (2.2).

Lemma 2.3. *[7] Let q be a convex univalent in U and let $\Psi \in \mathbb{C}$. Further assume that $Re(\Psi) > 0$. If $p \in H[q(0), 1] \cap Q$ and $p(z) + \Psi zp'(z)$ is univalent in U, then*

$$q(z) + \Psi zq'(z) \prec p(z) + \Psi zp'(z), \tag{2.3}$$

which implies then $q \prec p$ and q best subordinant of (2.3).

Lemma 2.4. *[16] Let q be a convex univalent in U and let θ and ϕ be analytic in adomain D contaning $q(U)$, suppose that*

- *$Re\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ for $z \in U$,*
- *$Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.*

If $p \in H[q(0), 1] \cap Q$, with $p(U) \subset D$, $\theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{2.4}$$

then $q \prec p$ and q is the best subordinant of (2.4).

3. Differential Subordination Results

Here, we introduce some differential subordination results by using Hadamard product operator.

Theorem 3.1. *Let q be univalent in unit disk U with $q(0) = 1$, $q'(z) \neq 0$ $\forall z \in U$. Let μ, $\rho \in \mathbb{C}^*$, $s, t \in \mathbb{C}$ and $f \in \sum$. Suppose that f and q satisfy the conditions*

$$\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho} \neq 0,$$

and

$$Re\left(\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1\right) > 0. \tag{3.1}$$

If

$$\left[1 + \mu st\frac{(1-\rho)T^{\alpha-1}f(z) + (2\rho-1)T^{\alpha}f(z) - \rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)}\right] \prec 1 + tz\frac{q'(z)}{q(z)}, \tag{3.2}$$

then

$$\left[\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right]^{\mu} \prec q(z), \tag{3.3}$$

and q is the best dominant of (3.3).

Proof . Define the function p by

$$p(z) = \left[\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right]^{\mu}, \tag{3.4}$$

then the function p is analytic in U and $q(0) = 1$, and differentiating (3.4) with respect to z, we get

$$z\frac{p'(z)}{p(z)} = \mu\left(\frac{(1-\rho)z(T^{\alpha}f(z))' + \rho z(T^{\alpha+1}f(z))'}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)} + 1\right). \tag{3.5}$$

Now, in view of (1.8), we obtain the following equation

$$z\frac{p'(z)}{p(z)} = \mu s\left(\frac{(1-\rho)T^{\alpha-1}f(z) + (2\rho-1)T^{\alpha}f(z) - \rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)}\right).$$

Therefore,

$$z\frac{p'(z)}{p(z)} = S\left(\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right)\left[\frac{(1-\rho)T^{\alpha-1}f(z) + (2\rho-1)T^{\alpha}f(z) - \rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)}\right].$$

The subordination (3.2) from hypothesis becomes

$$p(z) + s\mu zp'(z) \prec q(z) + s\mu zq'(z)$$

by setting $\theta(w) = 1$ and $\phi(w) = \frac{\gamma}{w}$, it can easily observed that $\theta(w)$ is analytic in $\mathbb{C}$ and $\phi(w) \neq 0$ is an analytic in $\mathbb{C}^*$. Moreover, we let

$$Q(z) = zq'(z)\phi(z) = \frac{tzq'(z)}{q(z)}, \tag{3.6}$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \frac{tzq'(z)}{q(z)}. \tag{3.7}$$

We find that $Q(z)$ is starlike univalent in U, and from (3.1)

$$Re\left\{\frac{zh'(z)}{Q(z)}\right\} = Re\left\{1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0, \tag{3.8}$$

and by using Lemma 2.2, we deduce the subordination (3.3) implies $p(z) \prec q(z)$ and the function q is the best dominant of (3.3). □

Corollary 3.2. *Putting* $q(z) = \left(\frac{1+z}{1-z}\right)^{\rho}$, $0 < \rho \leq 1$ *and* $f \in \sum$ *satisfies the following subordination condition:*

$$1 - \mu s\left(1 - \frac{(T^{\alpha+1}f(z))'}{T^{\alpha+1}f(z)}\right) \prec 1 + 2\rho\frac{tz}{1-z^2}, \tag{3.9}$$

then

$$(zT^{\alpha+1}f(z))^{\mu} \prec \left(\frac{1+z}{1-z}\right)^{\rho} \tag{3.10}$$

and $\left(\frac{1+z}{1-z}\right)^{\rho}$ *is the best dominant.*

Theorem 3.3. *Let q be a convex univalent function in U with $q(0) = 1$, Let $Y > 0$, $\mu \in \mathbb{C}^*$, ℓ, $\varepsilon \in \mathbb{C}$, $f \in \sum$ and suppose f and q satisfy the following conditions:*

$$zT^{\alpha+1}f(z) \neq 0 \tag{3.11}$$

and

$$Re\left\{1 + \frac{6\varepsilon\ell q^2(z)}{Y} + \frac{2\ell^2 q(z)}{Y} + \frac{4\varepsilon^2 q^3(z)}{Y} + z\frac{q''(z)}{q'(z)}\right\} > 0. \tag{3.12}$$

If

$$\phi(z) \prec (\varepsilon q^2(z) + \ell q(z))^2 + Yzq'(z), \tag{3.13}$$

where

$$\phi(z) = (zT^{\alpha+1}f(z))^{\mu}\left[\varepsilon^2(zT^{\alpha+1}f(z))^{3\mu} + 2\varepsilon\ell(zT^{\alpha+1}f(z))^{2\mu} + \ell^2(zT^{\alpha+1}f(z))^{\mu} + Y\mu s\left(\frac{T^{\alpha}f(z)}{T^{\alpha+1}f(z)} - 1\right)\right], \tag{3.14}$$

then

$$(zT^{\alpha+1}f(z))^{\mu} \prec q(z), \tag{3.15}$$

and q is the best dominant of (3.15)

Proof . Let

$$p(z) = (zT^{\alpha+1}f(z))^{\mu}. \tag{3.16}$$

According to (3.11) the function $p(z)$ is analytic in U with $p(0) = 1$. A simple computation shows that

$$\begin{aligned}\phi(z) &= (zT^{\alpha+1}f(z))^{\mu}\left[\varepsilon^2(zT^{\alpha+1}f(z))^{3\mu} + 2\varepsilon\ell(zT^{\alpha+1}f(z))^{2\mu} + \ell^2(zT^{\alpha+1}f(z))^{\mu} + Y\mu s\left(\frac{T^{\alpha}f(z)}{T^{\alpha+1}f(z)} - 1\right)\right] \\ &= (\varepsilon p^2(z) + \ell p(z))^2 + Yzp'(z),\end{aligned} \tag{3.17}$$

to prove our result by Lemma 2.2. Consider in this lemma $\theta(w) = (\varepsilon w^2 + \ell w)^2$, and $\phi(w) = Y$, then θ is analytic in $\mathbb{C}$ and ϕ is analytic in $\mathbb{C}^*$. Also, if we let

$$Q(z) = zq'(z)\phi(q(z)) = Yzq'(z), \tag{3.18}$$

and

$$h(z) = \theta(q(z)) + Q(z) = (\varepsilon(q^2(z) + \ell q(z))^2 + Yzq'(z), \tag{3.19}$$

then the assumptation q is convex would yield Q is starlike function in U. From (3.12), we have

$$Re\left(\frac{zh'(z)}{Q(z)}\right) = Re\left\{1 + \frac{6\varepsilon\ell}{Y}q^2(z) + \frac{2\ell^2}{Y}q(z) + \frac{4\varepsilon^2}{Y}q^3(z) + Z\frac{q(z)''(z)}{q'(z)}\right\} > 0, \tag{3.20}$$

by using Lemma 2.2, we deduce the subordination (3.13) implies that $p(z) \prec q(z)$, and the function q is the best dominant. □

Corollary 3.4. *Let* $q(z) = \frac{1+AZ}{1-AZ}$ *with* $A \in (-1, 0) \cup (0, 1),\ \varepsilon, \ell > 0,\ Y \in (0, 1)$

$$Re\left(\frac{6\varepsilon\ell}{Y}\left(\frac{1+AZ}{1-AZ}\right)^2 + \frac{2\ell^2}{Y}\left(\frac{1+AZ}{1-AZ}\right) + \frac{4\varepsilon^2}{Y}\left(\frac{1+AZ}{1-AZ}\right)^3 + \frac{1+AZ}{1-AZ}\right) > 0. \tag{3.21}$$

If $f \in \sum$ *satisfies the subordination*

$$\phi(z) \prec \left(\varepsilon\left(\frac{1+AZ}{1-AZ}\right)^2 + \ell\frac{1+AZ}{1-AZ}\right)^2 + Yz\frac{2Az}{(1-Az)^2}, \tag{3.22}$$

where

$$\phi(z) = (zT^{\alpha+1}f(z))^{\mu}[\varepsilon^2(zT^{\alpha+1}f(z))^{3\mu} + 2\varepsilon\ell(zT^{\alpha+1}f(z))^{2\mu} + \ell^2(zT^{\alpha+1}f(z))^{\mu} + Y\mu s\left(\frac{T^{\alpha}f(z)}{T^{\alpha+1}f(z)} - 1\right), \tag{3.23}$$

then

$$(zT^{\alpha+1}f(z))^{\mu} \prec \frac{1+AZ}{1-AZ}, \tag{3.24}$$

and q *is the best dominant.*

4. Differential Superordination Results

Theorem 4.1. *Let* q *be convex univalent function in* U *with* $q(0) = 1,\ s \in \mathbb{C},\ Re\{s\} > 0,\ \mu, \rho \in \mathbb{C}^*,\ \gamma \in \mathbb{C}^*,\ z \in U$, *if* $f \in \sum$. *Suppose that*

$$\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho} \neq 0, \tag{4.1}$$

and

$$\left(\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right)^{\mu} \in H[q(0), 1] \cap Q. \tag{4.2}$$

If the function

$$K(z) = \left[\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right]^{\mu}\left(1 + \mu s\frac{(1-\rho)T^{\alpha-1}f(z) + (2\rho-1)T^{\alpha}f(z) - \rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)}\right), \tag{4.3}$$

is univalent and the following superordination condition:

$$q(z) + \mu s q'(z) \prec K(z), \tag{4.4}$$

holds, then

$$q(z) \prec \left[\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right]^{\mu}, \tag{4.5}$$

and $q(z)$ is the best subordinant.

Proof . Consider a function $p(z)$ by

$$p(z) = \left[\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right]^{\mu}. \tag{4.6}$$

Then the function p is analytic in U and $q(0) = 1$, and differentiating (4.6) with respect to z, we get

$$z\frac{p'(z)}{p(z)} = \mu\left(\frac{(1-\rho)z(T^{\alpha}f(z))' + \rho z(T^{\alpha+1}f(z))'}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)} + 1\right). \tag{4.7}$$

After some computations and using (1.8), from (4.7), we obtain

$$\left(\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right)^{\mu} + \mu s\left(\frac{(1-\rho)zT^{\alpha}f(z) + \rho zT^{\alpha+1}f(z)}{\rho}\right)^{\mu}\left(\frac{(1-\rho)T^{\alpha-1}f(z) + (2\rho-1)T^{\alpha}f(z) - \rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z) + \rho T^{\alpha+1}f(z)}\right) = p(z) + \mu s z p'(z)$$

and now, by using Lemma 2.4 we get the desired result. □

Corollary 4.2. *taking* $q(z) = \left(\frac{1+z}{1-z}\right)^{b},\ 0 < b \le 1,\ \rho = 1.$
If $f \in \sum$ *is satisfies* $(zT^{\alpha+1}f(z))^{\mu} \in H[q(0),1] \cap Q$ *and*
$[zT^{\alpha+1}f(z)]^{\mu}\left(1 + \mu s\frac{(1-\rho)T^{\alpha-1}f(z)+(2\rho-1)T^{\alpha}f(z)-\rho T^{\alpha+1}f(z)}{(1-\rho)T^{\alpha}f(z)+\rho T^{\alpha+1}f(z)}\right)$ *is univalent in U and*

$$\left(\frac{1+z}{1-z}\right)^{b}[1 + \mu s\frac{2bz}{1-z^2}] \prec (zT^{\alpha+1}f(z))^{\mu}\left[1 + \mu s\frac{z(T^{\alpha+1}f(z))'}{T^{\alpha+1}f(z)}\right], \tag{4.8}$$

then

$$\left(\frac{1+z}{1-z}\right)^{b} < (zT^{\alpha+1}f(z))^{\mu}. \tag{4.9}$$

Theorem 4.3. *Let q be convex univalent in U with $q(0) = 1$, let $Y > 0$, $\mu \in \mathbb{C}^*$, $\varepsilon, \ell \in \mathbb{C}$, and $f \in \sum$. Suppose that*

$$Re\left\{\left(\frac{6\varepsilon\ell}{Y}q^2(z) + \frac{2\ell^2}{Y}q(z) + \frac{4\varepsilon^2}{Y}q^3(z)\right)q'(z)\right\} > 0, \tag{4.10}$$

and f satisfies the next conditions

$$zT^{\alpha+1}f(z) \neq 0, \quad (\alpha > 0,\ z \in U), \tag{4.11}$$

and

$$(zT^{\alpha+1}f(z))^{\mu} \in H[q(z),1] \cap Q, \tag{4.12}$$

also, if the function $\phi(z)$ defined by (3.14) is univalent in U and the following superordination condition

$$(\varepsilon q^2(z) + \ell q(z))^2 + Y z q'(z) \prec \phi(z), \tag{4.13}$$

holds, then

$$q(z) \prec (zT^{\alpha+1} f(z))^{\mu}, \tag{4.14}$$

and q is the best subordination.

Proof . Let the function $g(z)$ defined as:

$$g(z) = (zT^{\alpha+1} f(z))^{\mu}, \tag{4.15}$$

after some computations, we get

$$(\varepsilon g^2(z) + \ell g(z))^2 + Y z g'(z) = \phi(z), \tag{4.16}$$

this implies

$$(\varepsilon q^2(z) + \ell q(z))^2 + Y z q'(z) \prec (\varepsilon g^2(z) + \ell g(z))^2 + Y z g'(z). \tag{4.17}$$

By setting

$$\theta(w) = (\varepsilon w^2 + \ell w)^2, \text{ and } \varphi(w) = Y.$$

It can be easily observe that $\theta(w)$ is analytic in $\mathbb{C}$, and $\varphi(w) \neq 0$ is an analytic in U^*. Also, we obtain

$$Re\left\{\frac{\theta'(q(z))}{\varphi(z)}\right\} = Re(q'(z))\left(\frac{4\varepsilon^2}{Y} q^3(z) + \frac{6\varepsilon\ell}{Y} q^2(z) + \frac{2\ell^2}{Y} q(z)\right) > 0. \tag{4.18}$$

Therefore by Lemma 2.3, we have

$$q(z) \prec (zT^{\alpha+1} f(z))^{\mu}. \tag{4.19}$$

□

Corollary 4.4. *Let $q(z) = e^{dz}$, $|d| \leq 1$, $Re\left\{\left(\frac{6\varepsilon\ell}{Y} e^{2dz} + \frac{2\ell^2}{Y} e^{dz} + \frac{4\varepsilon^2}{Y} e^{3dz}\right) de^{dz}\right\} > 0$ and $f \in \sum$, such that*

$$(zT^{\alpha+1} f(z))^{\mu} \in H[q(0), 1] \cap Q.$$

If function $\phi(z)$ defined by (3.14) is univalent in U and satisfied the following superordination condition

$$(\varepsilon e^{2dz} + \ell e^{dz})^2 + Y dz e^{dz} \prec \phi, \tag{4.20}$$

then

$$e^{dz} \prec (zT^{\alpha+1} f(z))^{\mu} \tag{4.21}$$

and e^{dz} is the best subordinant.

5. Sandwich Results

Combining Theorem 3.3 with Theorem 4.3, we obtain the following sandwich Theorem.

Theorem 5.1. *Let q_1, q_2 be convex univalent in U, $q_1(0) = 1$ and satisfies (3.13) with $q_2(0) = 1$ and satisfies (4.13), respectively. If $f \in \sum$ and suppose that f satisfies the next conditions:*

$$(zT^{\alpha+1}f(z))^{\mu} \in H[q(0), 1] \cap Q, \text{ and } zT^{\alpha+1}f(z) \neq 0.$$

If $\phi(z) = (zT^{\alpha+1})^{\mu}[\delta^2(zT^{\alpha+1})^{3\mu} + 2\delta\eta(zT^{\alpha+1})^{2\mu} + \eta^2(zT^{\alpha+1})^{\mu} + Y\mu\beta\left(\frac{T_{\beta}^{\alpha,\eta}f(z)}{T_{\beta}^{\alpha+1,\eta}f(z)} - 1\right)$, is univalent in U, then

$$(\varepsilon q_1^2(z) + \ell q_1(z))^2 Y z q_1'(z) \prec \phi(z) \prec (\varepsilon q_2^2(z) + \ell q_2(z))^2 Y z q_2'(z), \tag{5.1}$$

then

$$q_1(z) \prec (zT^{\alpha+1}f(z))^{\mu} \prec q_2(z),$$

and q_1 and q_2 are the best subordinant and the best dominant respectively.

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