Int. J. Nonlinear Anal. Appl. **1** (2010) No.2, 1-10
ISSN: 2008-6822 (electronic)
http://www.ijnaa.com

ISOMORPHISMS IN UNITAL C^*-ALGEBRAS

C. PARK[1*] AND TH. M. RASSIAS[2]

Dedicated to the 70th Anniversary of S.M.Ulam's Problem for Approximate Homomorphisms

Abstract. It is shown that every almost linear bijection $h : A \to B$ of a unital C^*-algebra A onto a unital C^*-algebra B is a C^*-algebra isomorphism when $h(3^n uy) = h(3^n u)h(y)$ for all unitaries $u \in A$, all $y \in A$, and all $n \in \mathbb{Z}$, and that almost linear continuous bijection $h : A \to B$ of a unital C^*-algebra A of real rank zero onto a unital C^*-algebra B is a C^*-algebra isomorphism when $h(3^n uy) = h(3^n u)h(y)$ for all $u \in \{v \in A \mid v = v^*, \|v\| = 1, v \text{ is invertible}\}$, all $y \in A$, and all $n \in \mathbb{Z}$.

Assume that X and Y are left normed modules over a unital C^*-algebra A. It is shown that every surjective isometry $T : X \to Y$, satisfying $T(0) = 0$ and $T(ux) = uT(x)$ for all $x \in X$ and all unitaries $u \in A$, is an A-linear isomorphism. This is applied to investigate C^*-algebra isomorphisms in unital C^*-algebras.

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [11] introduced the following inequality that is called Cauchy–Rassias inequality: Assume that there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that

$$\|f(x + y) - f(x) - f(y)\| \le \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [11] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. The above inequality has provided a a lot of influence in the development of what is called *Hyers-Ulam-Rassias stability* or *generalized Hyers-Ulam stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was

Date: Received: January 2010; Revised: May 2010.

2000 *Mathematics Subject Classification.* Primary 39B52,39B82; Secondary 47B48, 46L05.

Key words and phrases. generalized Hyers-Ulam stability, C^*-algebra isomorphism, real rank zero, isometry.

*: Corresponding author.

studied by a number of mathematicians. Jun and Lee [5] proved the following: Denote by $\varphi : X \times X \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X$. Suppose that $f : X \to Y$ is a mapping satisfying $f(0) = 0$ and

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)| \le \frac{1}{3}(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x))$$

for all $x \in X$. C. Park and W. Park [10] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra. Various functional equations have been investigated by several authors ([13]–[16], [18]).

Throughout this paper, let A be a unital C^*-algebra with norm $|| \cdot ||$ and unit e, and B a unital C^*-algebra with norm $\| \cdot \|$. Let $U(A)$ be the set of unitary elements in A, $A_{sa} = \{x \in A \mid x = x^*\}$, and $I_1(A_{sa}) = \{v \in A_{sa} \mid \|v\| = 1, v \text{ is invertible}\}$.

In Section 2, we prove that every almost linear bijection $h : A \to B$ is a C^*-algebra isomorphism when $h(3^n uy) = h(3^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n \in \mathbb{Z}$, and that for a unital C^*-algebra A of real rank zero (see [3]), every almost linear continuous bijection $h : A \to B$ is a C^*-algebra isomorphism when $h(3^n uy) = h(3^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n \in \mathbb{Z}$. In Section 3, we moreover prove that every surjective isometry, satisfying some conditions, is a C^*-algebra isomorphism.

2. C^*-algebra isomorphisms in unital C^*-algebras

We investigate C^*-algebra isomorphisms in unital C^*-algebras.

Theorem 2.1. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : A \times A \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty, \tag{2.1}$$

$$\|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| \le \varphi(x, y), \tag{2.2}$$

$$\|h(3^n u^*) - h(3^n u)^*\| \le \varphi(3^n u, 3^n u) \tag{2.3}$$

for all $\mu \in S^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$, all $u \in U(A)$, all $n \in \mathbb{Z}$, and all $x, y \in A$. Assume that

$$\lim_{n\to\infty} \frac{h(3^n e)}{3^n} \tag{2.4}$$

is invertible. Then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1 \in S^1$. It follows from the Jun and Lee's theorem [5] that there exists a unique additive mapping $H : A \to B$ such that

$$\|h(x) - H(x)\| \leq \frac{1}{3}\big(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x)\big) \tag{2.5}$$

for all $x \in A$. The additive mapping $H : A \to B$ is given by

$$H(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x)$$

for all $x \in A$.

By the assumption, for each $\mu \in S^1$,

$$\frac{1}{3^n}\|2h(\frac{3^n \mu x}{2}) - \mu h(3^n x)\| \leq \frac{1}{3^n}\varphi(3^n x, 0),$$

which tends to zero as $n \to \infty$ for all $x \in A$. Hence

$$2H(\frac{\mu x}{2}) = \lim_{n\to\infty} \frac{2h(\frac{3^n \mu x}{2})}{3^n} = \lim_{n\to\infty} \frac{\mu h(3^n x)}{3^n} = \mu H(x)$$

for all $\mu \in S^1$ and all $x \in A$. Since $H : A \to B$ is additive,

$$H(\mu x) = 2H(\frac{\mu x}{2}) = \mu H(x) \tag{2.6}$$

for all $\mu \in S^1$ and all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then, we have $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [6], there exist three elements $\mu_1, \mu_2, \mu_3 \in S^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. So by (2.6)

$$\begin{aligned} H(\lambda x) &= H(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot H(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}H(3\frac{\lambda}{M}x) \\ &= \frac{M}{3}H(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(H(\mu_1 x) + H(\mu_2 x) + H(\mu_3 x)) \\ &= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)H(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}H(x) \\ &= \lambda H(x) \end{aligned}$$

for all $x \in A$. Hence

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in A$. And $H(0x) = 0 = 0H(x)$ for all $x \in A$. So the unique additive mapping $H : A \to B$ is a $\mathbb{C}$-linear mapping.

By (2.1) and (2.3), we get

$$\begin{aligned} H(u^*) &= \lim_{n\to\infty} \frac{h(3^n u^*)}{3^n} = \lim_{n\to\infty} \frac{h(3^n u)^*}{3^n} = (\lim_{n\to\infty} \frac{h(3^n u)}{3^n})^* \\ &= H(u)^* \end{aligned}$$

for all $u \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$),

$$\begin{aligned} H(x^*) &= H(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} H(u_j)^* = (\sum_{j=1}^m \lambda_j H(u_j))^* \\ &= H(\sum_{j=1}^m \lambda_j u_j)^* = H(x)^* \end{aligned}$$

for all $x \in A$.

Since $h(3^n uy) = h(3^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n \in \mathbb{Z}$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n uy) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n u)h(y) = H(u)h(y) \tag{2.7}$$

for all $u \in U(A)$ and all $y \in A$. By the additivity of H and (2.7),

$$3^n H(uy) = H(3^n uy) = H(u(3^n y)) = H(u)h(3^n y)$$

for all $u \in U(A)$ and all $y \in A$. Hence

$$H(uy) = \frac{1}{3^n} H(u)h(3^n y) = H(u)\frac{1}{3^n} h(3^n y) \tag{2.8}$$

for all $u \in U(A)$ and all $y \in A$. Taking the limit in (2.8) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{2.9}$$

for all $u \in U(A)$ and all $y \in A$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}$, $u_j \in U(A)$), it follows from (2.9) that

$$\begin{aligned} H(xy) &= H(\sum_{j=1}^m \lambda_j u_j y) = \sum_{j=1}^m \lambda_j H(u_j y) = \sum_{j=1}^m \lambda_j H(u_j)H(y) \\ &= H(\sum_{j=1}^m \lambda_j u_j)H(y) = H(x)H(y) \end{aligned}$$

for all $x, y \in A$.

By (2.7) and (2.9),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in A$. Since $\lim_{n\to\infty} \frac{h(3^n e)}{3^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in A$.

Therefore, the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism. □

Corollary 2.2. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| &\leq \theta(||x||^p + ||y||^p), \\ \|h(3^n u^*) - h(3^n u)^*\| &\leq 2 \cdot 3^{np}\theta \end{aligned}$$

for all $\mu \in S^1$, all $u \in U(A)$, all $n \in \mathbb{Z}$, and all $x, y \in A$. Assume that $\lim_{n\to\infty} \frac{h(3^n e)}{3^n}$ is invertible. Then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 2.1. □

Theorem 2.3. *Let $h : A \to B$ be a bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in U(A)$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : A \times A \to [0, \infty)$ satisfying (2.1), (2.3), and (2.4) such that*

$$\|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| \le \varphi(x, y) \tag{2.10}$$

for $\mu = 1, i$, and all $x, y \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (2.10). By the same reasoning as in the proof of Theorem 2.1, there exists a unique additive mapping $H : A \to B$ satisfying (2.5). By the same reasoning as in the proof of [11], the additive mapping $H : A \to B$ is $\mathbb{R}$-linear.

Put $\mu = i$ and $y = 0$ in (2.10). By the same method as in the proof of Theorem 2.1, one can obtain that

$$H(ix) = 2H(\frac{ix}{2}) = \lim_{n\to\infty} \frac{2h(\frac{3^n ix}{2})}{3^n} = \lim_{n\to\infty} \frac{ih(3^n x)}{3^n} = iH(x)$$

for all $x \in A$.

For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(sx + itx) = sH(x) + tH(ix) = sH(x) + itH(x) \\ &= (s + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the additive mapping $H : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

From now on, assume that A is a unital C^*-algebra of real rank zero, where "real rank zero" means that the set of invertible self-adjoint elements is dense in the set of self-adjoint elements (see [3]).

Now we investigate continuous C^*-algebra isomorphisms in unital C^*-algebras.

Theorem 2.4. *Let $h : A \to B$ be a continuous bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : A \times A \to [0, \infty)$ satisfying (2.1), (2.2), (2.3) and (2.4). Then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.1, there exists a unique $\mathbb{C}$-linear involution $H : A \to B$ satisfying (2.5).

Since $h(3^n uy) = h(3^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n \in \mathbb{Z}$,

$$H(uy) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n uy) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n u)h(y) = H(u)h(y) \tag{2.11}$$

for all $u \in I_1(A_{sa})$ and all $y \in A$. By the additivity of H and (2.11),

$$3^n H(uy) = H(3^n uy) = H(u(3^n y)) = H(u)h(3^n y)$$

for all $u \in I_1(A_{sa})$ and all $y \in A$. Hence

$$H(uy) = \frac{1}{3^n} H(u)h(3^n y) = H(u)\frac{1}{3^n} h(3^n y) \tag{2.12}$$

for all $u \in I_1(A_{sa})$ and all $y \in A$. Taking the limit in (2.12) as $n \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{2.13}$$

for all $u \in I_1(A_{sa})$ and all $y \in A$.

By (2.11) and (2.13),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in A$. Since $\lim_{n\to\infty} \frac{h(3^n e)}{3^n} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in A$. So $H : A \to B$ is continuous. But by the assumption that A has real rank zero, it is easy to show that $I_1(A_{sa})$ is dense in $\{x \in A_{sa} \mid ||x|| = 1\}$. So for each $w \in \{z \in A_{sa} \mid ||z|| = 1\}$, there is a sequence $\{\kappa_j\}$ such that $\kappa_j \to w$ as $j \to \infty$ and $\kappa_j \in I_1(A_{sa})$. Since $H : A \to B$ is continuous, it follows from (2.13) that

$$\begin{aligned} H(wy) &= H(\lim_{j\to\infty} \kappa_j y) = \lim_{j\to\infty} H(\kappa_j y) = \lim_{j\to\infty} H(\kappa_j)H(y) \\ &= H(\lim_{j\to\infty} \kappa_j)H(y) = H(w)H(y) \end{aligned} \tag{2.14}$$

for all $w \in \{z \in A_{sa} \mid ||z|| = 1\}$ and all $y \in A$.

For each $x \in A$, $x = \frac{x+x^*}{2} + i\frac{x-x^*}{2i}$, where $x_1 := \frac{x+x^*}{2}$ and $x_2 := \frac{x-x^*}{2i}$ are self-adjoint.

First, consider the case that $x_1 \neq 0, x_2 \neq 0$. Since $H : A \to B$ is $\mathbb{C}$-linear, it follows from (2.14) that

$$\begin{aligned} H(xy) &= H(x_1 y + ix_2 y) = H(||x_1||\frac{x_1}{||x_1||}y + i||x_2||\frac{x_2}{||x_2||}y) \\ &= ||x_1||H(\frac{x_1}{||x_1||}y) + i||x_2||H(\frac{x_2}{||x_2||}y) \\ &= ||x_1||H(\frac{x_1}{||x_1||})H(y) + i||x_2||H(\frac{x_2}{||x_2||})H(y) \\ &= \{H(||x_1||\frac{x_1}{||x_1||}) + iH(||x_2||\frac{x_2}{||x_2||})\}H(y) = H(x_1 + ix_2)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $y \in A$.

Next, consider the case that $x_1 \neq 0, x_2 = 0$. Since $H : A \to B$ is $\mathbb{C}$-linear, it follows from (2.14) that

$$\begin{aligned} H(xy) &= H(x_1 y) = H(||x_1||\frac{x_1}{||x_1||}y) = ||x_1||H(\frac{x_1}{||x_1||}y) \\ &= ||x_1||H(\frac{x_1}{||x_1||})H(y) = H(||x_1||\frac{x_1}{||x_1||})H(y) = H(x_1)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $y \in A$.

Finally, consider the case that $x_1 = 0, x_2 \neq 0$. Since $H : A \to B$ is $\mathbb{C}$-linear, it follows from (2.14) that

$$\begin{aligned} H(xy) &= H(ix_2y) = H(i||x_2||\frac{x_2}{||x_2||}y) = i||x_2||H(\frac{x_2}{||x_2||}y) \\ &= i||x_2||H(\frac{x_2}{||x_2||})H(y) = H(i||x_2||\frac{x_2}{||x_2||})H(y) = H(ix_2)H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $y \in A$. Hence

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Therefore, the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism. □

Corollary 2.5. *Let $h : A \to B$ be a continuous bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| &\leq \theta(||x||^p + ||y||^p), \\ \|h(3^n u^*) - h(3^n u)^*\| &\leq 2 \cdot 3^{np}\theta \end{aligned}$$

for all $\mu \in S^1$, all $u \in I_1(A_{sa})$, all $n \in \mathbb{Z}$ and all $x, y \in A$. Assume that $\lim_{n\to\infty} \frac{h(3^n e)}{3^n}$ is invertible. Then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x, y) = \theta(||x||^p + ||y||^p)$ to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 2.4. □

Theorem 2.6. *Let $h : A \to B$ be a continuous bijective mapping satisfying $h(0) = 0$ and $h(3^n uy) = h(3^n u)h(y)$ for all $u \in I_1(A_{sa})$, all $y \in A$, and all $n \in \mathbb{Z}$, for which there exists a function $\varphi : A \times A \to [0, \infty)$ satisfying (2.1), (2.3), (2.4), and (2.10). Then the bijective mapping $h : A \to B$ is a C^*-algebra isomorphism.*

Proof. By the same reasoning as in the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (2.5).

The rest of the proof is the same as in the proofs of Theorems 2.1 and 2.4. □

3. On the Mazur-Ulam theorem in modules over C^*-algebras

Surjective isometries between normed vector spaces have been investigated by several authors ([1], [2], [8], [9], [12], [17]). We apply the results to investigate C^*-algebra isomorphisms in unital C^*-algebras.

Lemma 3.1. ([4]) *If T is an isometry from a normed vector space X onto a normed vector space Y, then*

$$\begin{aligned} &T(x + y) = T(x) + T(y) - T(0), \\ &T(rx) = rT(x) + (1 - r)T(0), \quad \forall r \in \mathbb{R}. \end{aligned}$$

Corollary 3.2. ([4]) *If T is an isometry from a normed vector space X onto a normed vector space Y and if $T(0) = 0$, then T is $\mathbb{R}$-linear*

Theorem 3.3. *Let X and Y be left normed modules over a unital C^*-algebra A. If $T : X \to Y$ is a surjective isometry with $T(0) = 0$ and $T(ux) = uT(x)$ for all $u \in U(A)$ and all $x \in X$, then $T : X \to Y$ is an A-linear isomorphism.*

Proof. By Corollary 3.2, $T : X \to Y$ is a $\mathbb{R}$-linear.

Since $i \in U(A)$, $T(ix) = iT(x)$ for all $x \in X$. For each $\lambda \in \mathbb{C}$, $\lambda = \lambda_1 + i\,\lambda_2 \quad (\lambda_1, \lambda_2 \in \mathbb{R})$. So

$$\begin{aligned} T(\lambda x) &= T(\lambda_1\, x + i\,\lambda_2\, x) = T(\lambda_1\, x) + T(i\,\lambda_2\, x) = \lambda_1 T(x) + iT(\lambda_2\, x) \\ &= (\lambda_1 + i\,\lambda_2)T(x) = \lambda T(x) \end{aligned}$$

for all $x \in X$.

Since each $a \in A$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^n \lambda_j u_j \quad (\lambda_j \in \mathbb{C},\, u_j \in U(A))$,

$$T(ax) = T(\sum_{j=1}^n \lambda_j u_j x) = \sum_{j=1}^n \lambda_j T(u_j x) = \sum_{j=1}^n \lambda_j u_j T(x) = aT(x)$$

for all $x \in X$. So

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in A$ and all $x, y \in X$, as desired. □

Now we investigate C^*-algebra isomorphisms in unital C^*-algebras.

Theorem 3.4. *If $T : A \to B$ is a surjective isometry with $T(0) = 0$, $T(iu) = iT(u)$, $T(u^*) = T(u)^*$, and $T(uv) = T(u)T(v)$ for all $u, v \in U(A)$, then $T : A \to B$ is a C^*-algebra isomorphism.*

Proof. By the same reasoning as in the proof of Theorem 3.3, $T : A \to B$ is $\mathbb{R}$-linear and

$$T(\lambda u) = \lambda T(u)$$

for all $\lambda \in \mathbb{C}$ and all $u \in U(A)$.

Since each $a \in A$ is a finite linear combination of unitary elements, i.e., $a = \sum_{j=1}^n \lambda_j u_j \quad (\lambda_j \in \mathbb{C},\, u_j \in U(A))$,

$$\begin{aligned} T(\lambda a) &= T(\sum_{j=1}^n \lambda\lambda_j u_j) = \sum_{j=1}^n \lambda\lambda_j T(u_j) = \lambda(\sum_{j=1}^n \lambda_j T(u_j)) \\ &= \lambda T(\sum_{j=1}^n \lambda_j u_j) = \lambda T(a) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $a \in A$. So $T : A \to B$ is $\mathbb{C}$-linear. Furthermore,

$$\begin{aligned} T(a^*) &= T(\sum_{j=1}^n \overline{\lambda_j} u_j^*) = \sum_{j=1}^n \overline{\lambda_j} T(u_j^*) = \sum_{j=1}^n \overline{\lambda_j} T(u_j)^* \\ &= T(\sum_{j=1}^n \lambda_j u_j)^* = T(a)^* \end{aligned}$$

for all $a \in A$. And

$$\begin{aligned} T(av) &= T(\sum_{j=1}^{n} \lambda_j u_j v) = \sum_{j=1}^{n} \lambda_j T(u_j v) = \sum_{j=1}^{n} \lambda_j T(u_j) T(v) \\ &= T(\sum_{j=1}^{n} \lambda_j u_j) T(v) = T(a)T(v) \end{aligned}$$

for all $a \in A$ and all $v \in U(A)$. Since each $b \in A$ is a finite linear combination of unitary elements, i.e., $b = \sum_{j=1}^{m} \nu_j v_j \quad (\nu_j \in \mathbb{C},\, v_j \in U(A))$,

$$\begin{aligned} T(ab) &= T(\sum_{j=1}^{m} \nu_j a v_j) = \sum_{j=1}^{m} \nu_j T(a v_j) = \sum_{j=1}^{m} \nu_j T(a) T(v_j) \\ &= T(a) T(\sum_{j=1}^{m} \nu_j v_j) = T(a)T(b) \end{aligned}$$

for all $a, b \in A$. So $T : A \to B$ is multiplicative.

Therefore, $T : A \to B$ is a C^*-algebra isomorphism. □

References

1. J. Baker, *Isometries in normed spaces*, Amer. Math. Monthly **78** (1971), 655–658.
2. J. Bourgain, *Real isomorphic complex Banach spaces need not be complex isomorphic*, Proc. Amer. Math. Soc. **96** (1986), 221–226.
3. L. Brown and G. Pedersen, *C^*-algebras of real rank zero*, J. Funct. Anal. **99** (1991), 131–149.
4. R.J. Fleming and J.E. Jamison, *Isometries on Banach Spaces: Function Spaces*, Monographs and Surveys in Pure and Applied Mathematics Vol. **129**, Chapman & Hall/CRC, Boca Raton, London, New York and Washington D.C., 2003.
5. K. Jun and Y. Lee, *A generalization of the Hyers-Ulam-Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
6. R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators*, Math. Scand. **57** (1985), 249–266.
7. R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras: Elementary Theory*, Academic Press, New York, 1983.
8. N. Kalton, *An elementary example of a Banach space not isomorphic to its complex conjugate*, Canad. Math. Bull. **38** (1995), 218–222.
9. S. Mazur and S. Ulam, *Sur les transformation d'espaces vectoriels normé*, C.R. Acad. Sci. Paris **194** (1932), 946–948.
10. C. Park and W. Park, *On the Jensen's equation in Banach modules*, Taiwanese J. Math. **6** (2002), 523–531.
11. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
12. Th.M. Rassias, *Properties of isometic mappings*, J. Math. Anal. Appl. **235** (1997), 108–121.
13. Th.M. Rassias, *On the stability of the quadratic functional equation and its applications*, Studia Univ. Babes-Bolyai **XLIII** (1998), 89–124.
14. Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. **246** (2000), 352–378.

15. Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
16. Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
17. Th.M. Rassias and P. Šemrl, *On the Mazur-Ulam theorem and the Aleksandrov problem for unit distance preserving mapping*, Proc. Amer. Math. Soc. **118** (1993), 919–925.
18. Th.M. Rassias and K. Shibata, *Variational problem of some quadratic functionals in complex analysis*, J. Math. Anal. Appl. **228** (1998), 234–253.

[1] Department of Mathematics, Hanyang University,, Seoul 133-791, Republic of Korea, .

E-mail address: baak@@hanyang.ac.kr

[2]Department of Mathematics, National Technical University of Athens,, Zografou Campus, 15780 Athens, Greece

E-mail address: trassias@@math.ntua.gr