Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1703–1711
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2022.25780.3126

Ergodic properties of pseudo-differential operators on compact Lie groups

Zahra Faghih, Mohammad Bagher Ghaemi*

School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran

(Communicated by Reza Saadati)

Abstract

Let $\mathbb{G}$ be a compact Lie group. This article shows that a contraction pseudo-differential operator A_τ on $L^p(\mathbb{G})$ has a Dominated Ergodic Estimate (DEE), and is trigonometrically well-bounded. Then we express ergodic generalization of the Vector-Valued M. Riesz theorem for invertible contraction pseudo-differential operator A_τ on $L^p(\mathbb{G})$. For this purpose, we show that A_τ is a Lamperti operator. Then we find a formula for its symbols τ. According to this formula, a representation for the symbol of adjoint and products is given.

Keywords: Pseudo-differential operators, Lamperti operator, Dominated Ergodic Estimate, trigonometrically well-bounded, M. Riesz theorem, Adjoints
2010 MSC: 47G30, 47A35, 46E30, 22E30

1 Introduction

The theory of pseudo-differential operators(abbreviated ΨDO) is essential in modern analysis and Mathematical Physics. ΨDO are a powerful and natural tool for studying partial differential operators. Some properties of ΨDO on the compact Lie group, like the study on the adjoint, boundedness, compactness and nuclearity, are investigated in [6], [8], [7]. First, some definitions and notions from [12] are recalled.

Suppose $\mathbb{G}$ is a compact Lie group with the unit element $1_{\mathbb{G}}$, and with $\widehat{\mathbb{G}}$ the unitary dual, consisting of the equivalence classes $[\pi]$ of the continuous irreducible unitary representations $\pi : \mathbb{G} \longrightarrow \mathbb{C}^{d_\pi \times d_\pi}$ of dimension d_π. The Fourier coefficient at π is defined by

$$\widehat{f}(\pi) := \int_{\mathbb{G}} f(x)\pi(x)^* dx \in \mathbb{C}^{d_\pi \times d_\pi}, \qquad (f \in C^\infty(\mathbb{G}))$$

where the integral is always taken w.r.t. the Haar measure on $\mathbb{G}$.
If τ be a function taking values in $\mathbb{C}^{d_\pi \times d_\pi}$, the ΨDO A_τ on $L^p(\mathbb{G})$, $p \geq 1$, defined as

$$\begin{aligned}(A_\tau f)(x) &= \sum_{[\pi]\in\widehat{\mathbb{G}}} d_\pi Tr(\pi(x)\tau(x,\pi)\widehat{f}(\pi)) \\ &= \sum_{[\pi]\in\widehat{\mathbb{G}}} \sum_{l,k=1}^{d_\pi} d_\pi (\pi(x)\tau(x,\pi))_{kl} (\widehat{f}(\pi))_{lk}.\end{aligned}$$

*Corresponding author
Email addresses: zahra_faghih@MathDep.iust.ac.ir (Zahra Faghih), mghaemi@iust.ac.ir (Mohammad Bagher Ghaemi)

Received: January 2022 *Accepted:* March 2022

Function τ is called the symbol of the ΨDO A_τ .

Our first aim in this paper is to show that the contraction ΨDO, $A_\tau : L^p(\mathbb{G}) \longrightarrow L^p(\mathbb{G}), 1 < p < \infty$, are Lamperti operator, then it is proved the contraction ΨDO, A_τ on $L^p(\mathbb{G})$ have a DEE with constant $\dfrac{p}{p-1}$, and is trigonometrically well-bounded. We know that to prove the pointwise ergodic convergence of a contraction U on an L^p-space it is enough to prove a Dominated Ergodic Estimate (DEE) for U (see e.g. [13]). The DEE for general positive L^p contractions for long was an open problem finally proved by Akcoglu [1] in 1974. We find a new display for the symbol of A_τ, its adjoint and the symbol of the products of the two ΨDOs on $L^p(\mathbb{G})$. In Sect.2, we introduce the concept of Lamperti operators on L^p-space. Then, we introduce the concept of DEE for L^p operators and prove that the contraction ΨDO, A_τ on $L^p(\mathbb{G})$ have a DEE and is trigonometrically well-bounded. Then we express ergodic generalization of the Vector-Valued M. Riesz theorem for invertible contraction pseudo-differential operator A_τ on $L^p(\mathbb{G})$. Finally, we give a formula for its symbols τ. In Sect.3, the symbol will be determined. In Sect.4, we will determine the symbol of the products.

2 Lamperti operators and Ergodic properties

2.1 Lamperti operators

Definition 2.1. A Lamperti operator is an order bounded and disjointness preserving operator $T : E \to F$ between Banach lattices.

Definition 2.2. A linear operator on a Banach space of functions is said to separate supports if it maps functions with disjoint supports to the same.

Definition 2.3. Suppose that $(\Omega, \mathcal{M}, \mu)$ is an arbitrary measure space, and $1 \leq p < \infty$. A linear mapping $T : L^p(\mu) \to L^p(\mu)$ is said to be separation-preserving provided that whenever $f \in L^p(\mu)$, $g \in L^p(\mu)$, and $fg = 0\ \mu - a.e.$ on Ω, the pointwise product $(Tf)(Tg)$ vanishes $\mu - a.e$ on Ω. Equivalently T is separation-preserving operator if it be separate supports.

Theorem 2.4. ([14], Theorem 2.5) Suppose that T is separate supports bounded linear operator on an L^p-space , $1 \leqslant p < \infty$, then T is a Lamperti operator.

So on L^p-space, Lamperti operator and separation-preserving operator are equivalent. L^p isometries, $1 \leqslant p < \infty, p \neq 2$, and positive L^2 isometries are Lamperti operators. The idea goes back to Banach [2]. In the following, we have the following characterization of Lamperti operators.

Theorem 2.5. ([9], Theorem3.1). A bounded linear operator U on an L^p-space, $1 \leqslant p < \infty$, separates supports if and only if there exists a positive linear operator $\mid U \mid$ on L^p such that

$$\mid Uf \mid = \mid U \mid\mid f \mid \quad \text{for every } f \in L^p \tag{2.1}$$

$\mid U \mid$ is called the linear modulus of U (see [4]).

Definition 2.6. Let (Ω, Σ, μ) and (Y, Δ, ν) be measure spaces. we call $\Phi : \Sigma \longrightarrow \Delta$ a regular (σ-)homomorphism modulo nullsets if for all $B \in \Sigma$ and (in-) finite disjoint sequences $(B_n)_n$ in Σ holds:

1. $\Phi(\Omega \setminus B) = \Phi(Y) \setminus \Phi(B)$,
2. $\Phi(\bigcup_n B_n) = \bigcup_n \Phi(B_n)$,
3. $\nu(\Phi(B)) = 0\ if\ \mu(B) = 0$.

In the following, we introduce a mappaing with unique properties related to a regular(σ-) homomorphism on sets of measurable functions. Equivalence considering both measurable functions that are almost equal everywhere, we consider the resulting mapping on the set of equivalence classes.

Theorem 2.7. (cf.[10]) Let (Ω, Σ, μ) be a measure space and Φ be a (σ-) homomorphism. There is a unique linear operator Φ^* on the space of measurable functions, such that:

i) $\Phi^* 1_E = 1_{\Phi E}$, for every $E \in \Sigma$,

ii) for every sequence of measurable functions like $g, g_1, g_2, ...$ if $g_n \longrightarrow g \quad \mu \ a.e$ then $\Phi^* g_n \longrightarrow \Phi^* g \quad \mu \ a.e$ when $n \to \infty$.

Theorem 2.8. ([9] Theorem 4.1) Let (Ω, Σ, μ) be a σ-finite measure space and $1 \leq p < \infty$. Let T be a Lamperti operator on $L^p(\mu)$ and Φ , σ-homomorphism associated of T and Φ^* be linear operator associated of Φ.Then there is a unique $h = \sum_{n=1}^{\infty} T1_{E_i}$, where $\{E_i : i \geqslant 1\}$ is a countable decomposition of Ω into the subset of finite measure, with supp $h = \Phi\Omega$, and

$$Tf(x) = h(x)\Phi^* f(x) \quad \text{for all } f \in L^p. \tag{2.2}$$

2.2 Ergodic properties and Mean-bounded

Let (Ω, Σ, μ) be a σ-finite measure space and $L^p = L^p(\Omega, \Sigma, \mu)$, $1 \leqslant p \leqslant \infty$. The indicator function of a set E is denoted 1_E. The support of a function g is the set $suppg = \{x : g(x) \neq 0\}$. The maximal operator $M(T) \equiv M$ of an L^p operator T is defined by $Mf = \sup_{n \geq 1} \mid T_n f \mid$, where $T_n = n^{-1} \sum_{i=0}^{n-1} T^i$. The truncated maximal operator M_N, N a positive integer , is defined similarly with the sup taken over $n = 1, ..., N$. T is said to have a Dominated Ergodic Estimate (DEE) with (finite) constant C if

$$\parallel Mf \parallel \leqslant C \parallel f \parallel \quad \text{for all } f \in L^p. \tag{2.3}$$

This will be the case if 2.3 holds for all M_N with the same C.

Definition 2.9. Let (Ω, Σ, μ) be a measure space. A bounded invertible linear operator $T : L^p(\mu) \to L^p(\mu)$ is said to be mean-bounded if

$$\sup_{n \geq 0} \parallel \frac{1}{2n+1} \sum_{j=-n}^{n} T^j \parallel < \infty$$

Theorem 2.10. ([3], Theorem 3.2) Suppose that (Ω, Σ, μ) is a σ-finite measure space, $1 < p < \infty$, and T is a bounded, invertible, separation-preserving linear mapping of $L^p(\mu)$ onto $L^p(\mu)$. The following assertions are equivalent.
(i) There is a real constant $C > 0$ such that for any $f \in L^p(\mu)$,

$$\int_\Omega |Mf|^p d\mu \leqslant C \int_\Omega |f|^p d\mu,$$

where M is the maximal operator defined on $L^p(\mu)$. Equivalence T have Dominated Ergodic Estimate (DEE).
(ii) $\mid T \mid$, the linear modulus of T, is mean-bounded, that is

$$\sup_{n \geqslant 0} \parallel \frac{1}{2n+1} \sum_{j=-n}^{n} \mid T \mid^j \parallel < \infty.$$

Let $B(Y)$ denote the Banach algebra of all bounded linear mappings of a Banach space Y into itself. F(.), the spectral family in Y, is a projection-valued function, mapping the real line $\mathbb{R}$ into $B(Y)$. If there is a compact interval $[a, b]$ such that $F(\lambda) = 0$ for $\lambda < a$ and $F(\lambda) = I$ for $\lambda \geqslant b$, then we say that F (.) is concentrated on $[a, b]$. Corresponding to any spectral family F (.) of projections in Y , a RiemannStieltjes notion of spectral integration with respect to F (.) can be defined as follows. For convenience, we suppose here that F (.) is concentrated on a compact interval $K = [a, b]$ of $\mathbb{R}$. Given a bounded function $f : K \to \mathbb{C}$ for each partition $\mathcal{P} = (\lambda_0, \lambda_1, ..., \lambda_n)$ of K we put

$$\mathcal{S}(\mathcal{P}, f, F) = \Sigma_{k=1}^{n} f(\lambda_k) \left\{F(\lambda_k) - F(\lambda_{k-1})\right\}.$$

If the net $\{\mathcal{S}(\mathcal{P}, f, F)\}$ converges in the strong operator topology of $B(Y)$ as $\mathcal{P}$ increases through the partitions of K directed by inclusion, then the strong limit is called the spectral integral of f with respect to F $(\cdot)$, and denoted by $\int_{[a,b]} f dF$. We then further define $\int_{[a,b]}^{\oplus} f dF$ by writing

$$\int_{[a,b]}^{\oplus} f dF = f(a)F(a) + \int_{[a,b]} f dF.$$

Definition 2.11. An operator $U \in B(Y)$ is said to be trigonometrically well-bounded provided there is a spectral family E($\cdot$) in Y concentrated on $[0, 2\pi]$ such that $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} dE(\lambda)$. In this case it is always possible to arrange matters so that we also have $E((2\pi)^-) = I$. With this additional property, the spectral family E($\cdot$) is uniquely determined by U, and called the spectral decomposition of U. Note that in this event, $\sigma(U)$, the spectrum of U, must be a subset of $[0, 2\pi]$.

Theorem 2.12. ([3], Theorem 4.2) Suppose that (Ω, Σ, μ) is a σ-finite measure space, $1 < p < \infty$, and T is a bounded, invertible, separation-preserving linear mapping of $L^p(\mu)$ onto $L^p(\mu)$ such that the linear modulus of T, be mean-bounded, that is

$$\sup_{n\geqslant 0} \| \frac{1}{2n+1} \sum_{j=-n}^{n} | T |^j \| < \infty$$

Then T is trigonometrically well-bounded.

Theorem 2.13. (ergodic generalization of the Vector-Valued M. Riesz theorem)([3], Theorem 6.7) Let T satisfy the hypotheses of Theorem 2.12, and let E($\cdot$) be the spectral decomposition of T . Then there is a real constant $K > 0$ such that

$$\left\| \left\{ \sum_{i=1}^{\infty} | E(b_i) f_i |^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)} \leqslant K \left\| \left\{ \sum_{i=1}^{\infty} | f_i |^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mu)}$$

for all sequences $\{b_i\}_{i=1}^{\infty} \subseteq [0, 2\pi)$, and all sequences $\{f_i\}_{i=1}^{\infty} \subseteq L^p(\mu)$.

Theorem 2.14. ([9], Theorem 5.1) Suppose that S be a Lamperti contraction on $L^p, 1 < p < \infty$. Then S has a DEE with constant $\dfrac{p}{p-1}$.

In the following, we give the main result. We show that the contraction ΨDO on $L^p(\mathbb{G})$ has a DEE.

Theorem 2.15. Suppose that $\tau(x, \pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ and $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ be contraction ΨDO for $1 < p < \infty$. Then A_τ has a DEE with constant $\dfrac{p}{p-1}$.

Proof . We show that the contraction ΨDO, A_τ is Lamperti operator for $1 \leq p < \infty$. For this purpose, we first show that $A_\tau(\mathbb{G})$ is Lamperti operator on indicator function. By definition of A_τ, we have

$$(A_\tau 1_E)(x) = \sum_{[\pi]\in\widehat{\mathbb{G}}} d_\pi Tr(\pi(x)\tau(x, \pi)\widehat{1_E}(\pi))$$

such that

$$\widehat{1_E}(\pi) = \int_{\mathbb{G}} 1_E(x)\pi(x)^* dx = \int_E \pi(x)^* dx$$

Therefore, $A_\tau 1_E(x) = \sum_{[\pi]\in\widehat{\mathbb{G}}} d_\pi Tr(\pi(x)\tau(x, \pi)\widehat{1_E}(\pi))$ where $x \in E$ and $A_\tau 1_E(x) = 0$ otherwise. So $(\text{supp}A_\tau 1_E) \subseteq E$ and the same way $(\text{supp}A_\tau 1_F) \subseteq F$. So $(\text{supp}A_\tau 1_E) \cap (\text{supp}A_\tau 1_F) = \emptyset$ and thus the desired result is obtaind. Now, we show that the assumptions of Theorem 2.5 is hold for $A_\tau(\mathbb{G})$. That means $A_\tau(\mathbb{G})$ is Lamperti operator on $L^p(\mathbb{G})$. For this, let $g \in L^p(\mathbb{G}) \cap L^+$ we define

$$| A_\tau | (g) =| A_\tau g | \tag{2.4}$$

and

$$| A_\tau | (g) =| A_\tau | (g^+)- | A_\tau | (g^-) \tag{2.5}$$

where $g \in L^p(\mathbb{G})$ is a real function and $g = g^+ - g^-$. For every $g \in L^p(\mathbb{G})$, we define

$$| A_\tau | (g) =| A_\tau | (g_r) + i | A_\tau | (g_i) \tag{2.6}$$

where $g \in L^p(\mathbb{G})$ and $g = g_r + ig_i$. $\mid A_\tau \mid: L^p(\mathbb{G}) \longrightarrow L^p(\mathbb{G})$ is well-defined . Because A_τ is Lamperti operator on indicator function and as in steps 1, 2 and 3, proof of theorem 3 in [5], prove that $\mid A_\tau \mid$ is linear and by definition 2.4 $\mid A_\tau \mid$ is positive. In the next step, we show that for every $g \in L^p(\mathbb{G})$,

$$\mid A_\tau g \mid = \mid A_\tau \mid g \mid\mid \quad \text{almost everywhere.} \tag{2.7}$$

If $t = \sum_{i=1}^n \beta_i 1_{E_i}$ is simple and integrable function, then $E_i \cap E_j = \emptyset$ for every $1 \le i \le n$,$1 \le j \le n$, $i \ne j$. Because A_τ is Lamperti operator on indicator function so

$$(A_\tau 1_{E_i})(A_\tau 1_{E_j}) = 0 \quad ,$$

So by lemma 1 in [5] ,

$$\mid \sum_{i=1}^n \beta_i A_\tau 1_{E_i} \mid = \sum_{i=1}^n \mid \beta_i A_\tau 1_{E_i} \mid$$

and

$$\mid \sum_{i=1}^n \mid \beta_i \mid A_\tau 1_{E_i} \mid = \sum_{i=1}^n \mid\mid \beta_i \mid A_\tau 1_{E_i} \mid$$

But for every $1 \le i \le n$,

$$\mid\mid \beta_i \mid A_\tau 1_{E_i} \mid = \mid \beta_i \mid\mid A_\tau 1_{E_i} \mid = \mid \beta_i A_\tau 1_{E_i} \mid$$

So

$$\mid \sum_{i=1}^n \beta_i A_\tau 1_{E_i} \mid = \mid \sum_{i=1}^n \mid \beta_i \mid A_\tau 1_{E_i} \mid$$

By lemma 1 in [5] , $\mid t \mid = \sum_{i=1}^n \mid \beta_i \mid 1_{E_i}$. So

$$\begin{aligned} \mid A_\tau s \mid &= \mid \sum_{i=1}^n \beta_i A_\tau 1_{E_i} \mid = \mid \sum_{i=1}^n \mid \beta_i \mid A_\tau 1_{E_i} \mid \\ &= \mid A_\tau (\sum_{i=1}^n \mid \beta_i \mid 1_{E_i}) \mid = \mid A_\tau \mid s \mid\mid \end{aligned} \tag{2.8}$$

By [11] (Theorem 13.3) , there is $\{t_n\}_{n=1}^\infty$ of simple and integrable function where

$$\parallel g - t_n \parallel_{L^p(\mu)} \to 0 \quad as \quad n \to \infty. \tag{2.9}$$

A_τ is bounded and by (2.8):

$$\begin{aligned} \parallel\mid A_\tau g \mid - \mid A_\tau \mid g \mid\mid\parallel_{L^p(\mu)} &\le \parallel\mid A_\tau g \mid - \mid A_\tau t_n \mid\parallel_{L^p(\mu)} + \parallel\mid A_\tau \mid t_n \mid\mid - \mid A_\tau \mid g \mid\mid\parallel_{L^p(\mu)} \\ &\le \parallel A_\tau g - A_\tau t_n \parallel_{L^p(\mu)} + \parallel A_\tau \mid t_n \mid - A_\tau \mid g \mid\parallel_{L^p(\mu)} \\ &\le \parallel A_\tau \parallel\parallel g - t_n \parallel_{L^p(\mu)} + \parallel A_\tau \parallel\parallel\mid t_n \mid - \mid g \mid\parallel_{L^p(\mu)} \\ &\le 2 \parallel A_\tau \parallel\parallel g - t_n \parallel_{L^p(\mu)} \end{aligned}$$

Now we have the equality by (2.9) . In the following, we show that for every $f \in L^p(\mathbb{G})$,

$$\mid A_\tau g \mid = \mid A_\tau \mid\mid g \mid \tag{2.10}$$

$\mid f \mid \ge 0$, so by definition (2.4), $\mid A_\tau \mid (\mid g \mid) = \mid A_\tau(\mid g \mid) \mid$. The desired equality obtained by (2.7). As in steps 7 and 8, proof of theorem 3 in [5], and because A_τ is Lamperti operator on indicator function, prove that $\mid A_\tau \mid$ is bounded and unique. So by Theorem 2.5, the contraction linear operator A_τ on $L^p(\mathbb{G})$ is a Lamperti operator. So A_τ is a contraction Lamperti operator. Now the proof is completed by Theorem 2.14. □

Corollary 2.16. Let $\tau(x, \pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ and $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ be a bounded ΨDO, then A_τ is Lamperti operator for $1 \le p < \infty$.

Corollary 2.17. Suppose that $\tau(x,\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ and $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ be invertible contraction ΨDO for $1 < p < \infty$. Then A_τ is trigonometrically well-bounded.

Proof . By theorem 2.15 A_τ is a separation-preserving (Lamperti) operator with DEE, so according to Theorem 2.10, $\mid A_\tau \mid$ is mean-bounded. Therefore the hypotheses of Theorem 2.12 are established, so A_τ is trigonometrically well-bounded. □

Corollary 2.18. Suppose that $\tau(x,\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ and $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ be invertible contraction ΨDO for $1 < p < \infty$, and E(.) be the spectral decomposition of A_τ. Then there is a real constant $K > 0$ such that

$$\left\| \left\{ \sum_{i=1}^{\infty} \mid E(\lambda_i) f_i \mid^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{G})} \leqslant K \left\| \left\{ \sum_{i=1}^{\infty} \mid f_i \mid^2 \right\}^{\frac{1}{2}} \right\|_{L^p(\mathbb{G})}$$

for all sequences $\{\lambda_i\}_{i=1}^{\infty} \subseteq [0, 2\pi)$, and all sequences $\{f_i\}_{i=1}^{\infty} \subseteq L^p(\mathbb{G})$.

Proof . By corollary 2.17, A_τ satisfy the hypotheses of Theorem 2.12, so according to Theorem 2.13, inequality is obtained. □

The following Theorem give the formula for the symbol of the $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$.

Theorem 2.19. Let $\tau(x,\pi) \in \mathbb{C}^{d_\pi \times d_\pi}$ and $A_\tau : L^p(\mathbb{G}) \to L^p(\mathbb{G})$ be bounded, then there exist a unique measurable function h on $\mathbb{G}$ such that:

$$\tau(x,\pi) = (\pi(x))^* h(x)(\Phi^* \pi)(x). \tag{2.11}$$

Proof . According to Corollary2.16, A_τ is a Lamperti operator. So by Theorem 2.8, we have

$$(A_\tau g)(x) = h(x)(\Phi^* g)(x), \quad g \in L^p(\mathbb{G})$$

By definition of A_τ, we have

$$\begin{aligned} (A_\tau g)(x) &= \sum_{[\pi] \in \widehat{\mathbb{G}}} d_\pi Tr(\pi(x)\tau(x,\pi)\widehat{g}(\pi)) \\ &= \sum_{\eta \in \mathbb{G}} \sum_{i,j=1}^{d_\eta} d_\eta (\eta(x)\tau(x,\eta))_{ij} \widehat{g}(\eta)_{ji} \\ &= \int_{\mathbb{G}} \sum_{\eta \in \widehat{\mathbb{G}}} \sum_{i,j=1}^{d_\eta} d_\eta ((\eta(x)\tau(x,\eta))_{ij} \overline{\eta(y)_{ij}} g(y) d\mu(y) \end{aligned} \tag{2.12}$$

for all $x \in \mathbb{G}$. Let $\pi \in \widehat{\mathbb{G}}$ is fixed. Then the function g on $\mathbb{G}$ for $1 \leq m, n \leq d_\pi$ is definde by

$$g(y) = \pi(y)_{nm}, \quad y \in \mathbb{G}$$

We have

$$\int_{\mathbb{G}} \pi(y)_{nm} \overline{\eta(y)_{ij}} d\mu(y) = \frac{1}{d_\pi}$$

if and only if $\pi = \eta$ and $n = i$ and $m = j$, and is zero o.w, it follows from 2.12

$$(\pi(x)\tau(x,\pi))_{nm} = h(x)(\Phi^* \pi(y)_{nm})(x)$$

Thus,

$$\tau(x,\pi) = (\pi(x))^* h(x)(\Phi^* \pi)(x), \quad (x,\pi) \in \mathbb{G} \times \widehat{\mathbb{G}},$$

where

$$(\Phi^* \pi)(x) = (\Phi^* \pi_{nm})(x), \quad 1 \leq n, m \leq d_\pi.$$

□

3 Adjoints

In the following, we get the symbol of the A_τ^* explicitly.

Theorem 3.1. Let $\tau(x,\pi)\in\mathbb{C}^{d_\pi\times d_\pi}$ such that $A_\tau: L^p(\mathbb{G})\to L^p(\mathbb{G})$ is bounded for $1\leqslant p<\infty$. Then $A_\tau^*: L^{p'}(\mathbb{G})\longrightarrow L^{p'}(\mathbb{G})$ is a Lamperti operator and its symbol γ given by

$$\gamma(x,\eta)=\gamma(x)^*\widehat{h}(\eta)^*A(x),\quad (x,\eta)\in\mathbb{G}\times\widehat{\mathbb{G}}$$

Where,

$$A(x)=\sum_{\rho\in\widehat{\mathbb{G}}}d_\rho tr(\rho(x)(\Phi^*\rho)^*)$$

Proof . Suppose that $f\in L^p(\mathbb{G})$ and $g\in L^{p'}(\mathbb{G})$. Then

$$\int_{\mathbb{G}}(A_\tau f)(x)\overline{g(x)}d\mu(x)=\int_{\mathbb{G}}f(x)\overline{(A_\tau^*g)(x)}d\mu(x)$$

So

$$\begin{aligned}&\int_{\mathbb{G}}\left(\int_{\mathbb{G}}\sum_{\pi\in\widehat{\mathbb{G}}}\sum_{i,j=1}^{d_\pi}d_\pi(\pi(x)\tau(x,\pi))_{ij}\overline{\pi(y)_{ij}}f(y)d\mu(y)\right)\overline{g(x)}d\mu(x)\\ &=\int_{\mathbb{G}}f(x)\overline{\left(\int_{\mathbb{G}}\sum_{\pi\in\widehat{\mathbb{G}}}\sum_{i,j=1}^{d_\pi}d_\pi(\pi(x)\gamma(x,\pi))_{ij}\overline{\pi(y)_{ij}}g(y)d\mu(y)\right)}d\mu(x)\end{aligned}\tag{3.1}$$

In the following, suppose ψ and η be elements in $\widehat{\mathbb{G}}$. Then for $1\le t,m\le d_\psi$ and $1\le n,l\le d_\eta$, we let f and g be functions on $\mathbb{G}$ be defined by

$$f(x)=\psi(x)_{tm},\quad x\in\mathbb{G}$$

and

$$g(x)=\eta(x)_{nl}\ ,\quad x\in\mathbb{G}.$$

Therefore by 3.1,

$$\int_{\mathbb{G}}(\psi(x)\tau(x,\psi)_{tm}\overline{\eta(x)_{nl}}d\mu(x)=\int_{\mathbb{G}}\psi(x)_{tm}\overline{(\eta(x)\gamma(x,\eta))_{nl}}d\mu(x)$$

and we get

$$\overline{\int_{\mathbb{G}}(\psi(x)\tau(x,\psi)_{tm}\overline{\eta(x)_{nl}}d\mu(x)}=\int_{\mathbb{G}}(\eta(x)\gamma(x,\eta))_{nl}\overline{\psi(x)_{tm}}d\mu(x).$$

Thus,

$$\overline{((\psi(.)\tau(.,\psi))_{tm})^\wedge(\eta)_{ln}}=((\eta(.)\gamma(.,\eta))_{nl})^\wedge(\psi)_{mt}.\tag{3.2}$$

By Corollary 2.16, A_τ is a Lamperti operator, so by theorem 2.19, there exists a unique, measurable function $h:\mathbb{G}\to\mathbb{C}$ such that

$$(\psi(y)\tau(y,\psi))_{tm}=h(y)(\Phi^*\psi)_{tm}(y),\quad 1\le m,t\le d_\psi,\quad (y,\psi)\in\mathbb{G}\times\widehat{\mathbb{G}}.$$

So, for all $(x,\eta)\in\mathbb{G}\times\widehat{\mathbb{G}}$,

$$\begin{aligned}((\eta(x)\gamma(x,\eta))_{nl}&=\sum_{\psi\in\widehat{\mathbb{G}}}d_\psi tr\left[\psi(x)((\eta(.)\gamma(.,\eta))_{nl})^\wedge(\psi)\right]\\ &=\sum_{\psi\in\widehat{\mathbb{G}}}\sum_{t,m=1}^{d_\psi}d_\psi\psi(x)_{tm}(((\eta(.)\gamma(.,\eta))_{nl})^\wedge(\psi))_{mt}.\end{aligned}$$

Hence for all $(x,\eta)\in\mathbb{G}\times\widehat{\mathbb{G}}$, we get by 3.2

$$\begin{aligned}((\eta(x)\gamma(x,\eta))_{nl} &= \sum_{\psi\in\widehat{\mathbb{G}}}\sum_{t,m=1}^{d_\psi} d_\psi\psi(x)_{tm}\overline{(((\psi(.)\tau(.,\psi))_{tm})^\wedge(\eta))_{ln}}\\
&= \sum_{\psi\in\widehat{\mathbb{G}}}\sum_{t,m=1}^{d_\psi} d_\psi\psi(x)_{tm}\int_{\mathbb{G}}\overline{\overline{((\psi(y)\tau(y,\psi))_{tm}}\eta(y)_{nl}}d\mu(y)\\
&= \sum_{\psi\in\widehat{\mathbb{G}}}\sum_{t,m=1}^{d_\psi} d_\psi\psi(x)_{tm}\int_{\mathbb{G}}\overline{h(y)}(\Phi^*\psi)^*_{mt}\eta(y)_{nl}d\mu(y)\\
&= \overline{\widehat{h}(\eta)_{ln}}\sum_{\psi\in\widehat{\mathbb{G}}} d_\psi tr(\psi(x)(\Phi^*\psi)^*)\\
&= \overline{\widehat{h}(\eta)_{ln}}A(x)\\
&= \widehat{h}(\eta)^*_{nl}A(x),\end{aligned}$$

for $1\le n,l\le d_\eta$. Thus, for all $(x,\eta)\in\mathbb{G}\times\widehat{\mathbb{G}}$, we get

$$\eta(x)\gamma(x,\eta)=\widehat{h}(\eta)^*A(x)$$

and hence

$$\gamma(x,\eta)=\eta(x)^*\widehat{h}(\eta)^*A(x).$$

□

4 Products

The following theorem shows that the product of two ΨDOs on $L^p(\mathbb{G})$ is a Lamperti ΨDO on $L^p(\mathbb{G})$, for $1\le p<\infty$, and a formula for the symbol of the product of two ΨDOs on $L^p(\mathbb{G})$ is given.

Theorem 4.1. If A_σ and A_τ are the ΨDO on $L^p(\mathbb{G})(p\le 1<\infty)$, then $A_\lambda=A_\tau A_\sigma: L^p(\mathbb{G})\to L^p(\mathbb{G})$ is a Lamperti ΨDO and the symbol λ of $A_\tau A_\sigma$ is given by

$$\lambda(x,\xi)=\xi(x)^*h'(x)(\Phi^*\xi)$$

for all $(x,\xi)\in\mathbb{G}\times\widehat{\mathbb{G}}$, where

$$h'(x)=\sum_{\eta\in\widehat{\mathbb{G}}} tr\left[\eta(x)\tau(x,\eta)\widehat{h}(\eta)\right],\quad x\in\mathbb{G},$$

Proof . Let $f\in L^p(\mathbb{G})$. Then

$$\begin{aligned}(A_\tau A_\sigma f)(x) &= \sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\widehat{A_\sigma f}(\eta)\right]\\
&= \sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\int_{\mathbb{G}}\sum_{\xi\in\widehat{\mathbb{G}}} d_\xi tr\left(\xi(y)\sigma(y,\xi)\widehat{f}(\xi)\right)\eta(y)^*d\mu(y)\right].\end{aligned}$$

By Corollary 2.16 A_σ is a Lamperti operator, now by Theorem 2.19, we have :

$$\xi(y)\sigma(y,\xi)=h(y)(\Phi^*\xi).$$

So,

$$
\begin{aligned}
(A_\tau A_\sigma f)(x) &= \sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\int_{\mathbb{G}}\sum_{\xi\in\widehat{\mathbb{G}}} d_\xi tr\left(h(y)(\Phi^*\xi)\widehat{f}(\xi)\right)\eta(y)^* d\mu(y)\right]\\
&= \sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\sum_{\xi\in\widehat{\mathbb{G}}}\widehat{h}(\eta) d_\xi tr\left(\Phi^*(\xi)\widehat{f}(\xi)\right)\right]\\
&= \sum_{\xi\in\widehat{\mathbb{G}}}\sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\widehat{h}(\eta)\right] tr\left((\Phi^*\xi)\widehat{f}(\xi)\right)\\
&= \sum_{\xi\in\widehat{\mathbb{G}}} d_\xi tr\left(\xi(x)\lambda(x,\xi)\widehat{f}(\xi)\right), \quad x\in\mathbb{G},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda(x,\xi) &= \xi(x)^*\sum_{\eta\in\widehat{\mathbb{G}}} d_\eta tr\left[\eta(x)\tau(x,\eta)\widehat{h}(\eta)\right](\Phi^*\xi)\\
&= \xi(x)^* h'(x)(\Phi^*\xi)
\end{aligned}
$$

for all $(x,\xi)\in\mathbb{G}\times\widehat{\mathbb{G}}$. This completes the proof. □

References

[1] M.A. Akcoglu, *A pointwise ergodic theorem in L_p-spaces*, Canad. J. Math. **27** (1975), no. 5, 1075–1082.

[2] S. Banach, *Théorie des opérations linéaires*, Z. Subwencji funduszu kultury Narodowej, 1979.

[3] E. Berkson and T.A. Gillespie, *Mean-boundedness and Littlewood-Paley for separation-preserving operators*, Trans. Amer. Math. Soc. **349** (1997), no. 3, 1169–1189.

[4] R.V. Chacon and U. Krengel, *Linear modulus of a linear operator*, Proc. Amer. Math. Soc. **15** (1964), no. 4, 553–559.

[5] Z. Faghih and M.B. Ghaemi, *Characterizations of pseudo-differential operators on $\mathbb{S}^1$ based on separation-preserving operators*, J. Pseudo-Diff. Oper. Appl. **12** (2021), no. 1, 1–14.

[6] M.B. Ghaemi and M. Jamalpour Birgani, *L^p-boundedness, compactness of pseudo-differential operators on compact Lie groups*, J. Pseudo-Differ. Oper. Appl. **8** (2017), no. 1, 1–11.

[7] M.B. Ghaemi, M. Jamalpour Birgani and M.W. Wong, *Characterizations of nuclear pseudo-differential operators on $\mathbb{S}^1$ with applications to adjoints and products*, J. Pseudo-Differ. Oper. Appl. **8** (2017), no. 2, 191–201.

[8] M.B. Ghaemi, E. Nabizadeh, M. Jamalpour Birgani and M.K. Kalleji, *A study on the adjoint of pseudo-differential operators on compact lie groups*, Complex Var. Elliptic Equ. **63** (2018), no. 10, 1408–1420.

[9] C.H. Kan, *Ergodic properties of Lamperti operators*, Canad. J. Math. **30** (1978), no. 6, 1206–1214.

[10] J. Lamperti, *On the isometries of certain function-spaces*, Pacific J. Math. **8** (1958), no. 3, 459–466.

[11] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Company, 1987.

[12] M. Ruzhansky and V. Turunen, *Pseudo-differential operators and symmetries: Background analysis and advanced topics*, Vol. 2. Birkhauser and Boston, 2009.

[13] A. Tulcea, *Ergodic properties of isometries in L^p spaces* $1<p<\infty$, Bull. Amer. Math. Soc. **70** (1964), no. 3, 366–371.

[14] A. Wolfgang, *Spectral properties of Lamperti operators*, Indiana Univ. Math. J. **32** (1983), no. 2, 199–215.