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Selberg and refinement type inequalities on semi-Hilbertian spaces

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Abstract

In this paper, we will study a type and refinement of Selberg type inequalities on semi-Hilbertian spaces, which is a simultaneous extension of the Bombieri type inequality in a semi-Hilbertian space. As applications, we give some examples of the Selberg inequality and its refinement on semi-Hilbertian spaces.

Keywords: Selberg's inequality, Semi inner products, operators algebra
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1 Introduction and preliminaries

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\ ,\ \rangle$. By $\mathcal{B}(\mathcal{H})$ we denote the algebra of all linear bounded operators from $\mathcal{H}$ to $\mathcal{H}$ and by $\mathcal{B}(\mathcal{H})^+$ the cone of positive (semi-definite) operators of $\mathcal{B}(\mathcal{H})$. Also, for $T \in \mathcal{B}(\mathcal{H})$, the range and the null space of T are denoted by $\mathcal{R}(T)$ and $\mathcal{N}(T)$, respectively.

Any $A \in \mathcal{B}(\mathcal{H})^+$ defines a positive semi-definite sesquilinear form as follows

$$\langle\ ,\ \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}, \ \ \langle x, y\rangle_A = \langle Ax, y\rangle .$$

By $\|\cdot\|_A$ we denote the semi-norm induced by $\langle x, y\rangle_A$, i.e., $\|x\|_A = \langle x, x\rangle_A^{\frac{1}{2}}$. Observe that $\|x\|_A = 0$ if and only if $x \in \mathcal{N}(A)$. Then $\|\cdot\|_A$ is a norm if and only if A is an injective operator. Moreover, $\|\cdot\|_A$ induces a semi-norm on a certain subset of $\mathcal{B}(\mathcal{H})$, namely, on the subset of all $T \in \mathcal{B}(\mathcal{H})$ for which there exists a constant $c > 0$ such that $\|Tx\|_A \le c\|x\|_A$ for all $x \in \mathcal{H}$. For these operators it holds

$$\|T\|_A = \sup_{x\in\overline{\mathcal{R}(A)}, x\neq 0} \frac{\|Tx\|_A}{\|x\|_A} < \infty.$$

For more details refer [1].

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The inequality of Selberg

$$\sum_{i=1}^{n} \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle|} \leq \|x\|^2, \quad x, y_1, \ldots, y_n \in \mathcal{H},\ y_i \neq 0, \quad 1 \leq i \leq n, \tag{1.1}$$

is originating from analytic theory of numbers [12]. It was discovered by A. Selberg around 1949, on account of the arguments of the distribution of primes [3, 5, 9, 10, 12, 13].

In 1971, Bombieri [2] showed the following inequality: If x, y_1, …, y_n are nonzero vectors in $\mathcal{H}$, then

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \|x\|^2 \max_{1\leq i\leq n} \sum_{j=1}^{n} |\langle y_j, y_i\rangle|.$$

Since that time it has interested many mathematicians who gave it many proofs, many extensions and refinements, see [2, 4, 8, 6, 11]. Moreover, in 1998, M. Fujii and R. Nakamoto [7] obtained in a Hilbert space, the following refinement for previous inequalities,

$$|\langle y, x\rangle|^2 + \sum_{i=1}^{n} \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle|} \|y\|^2 \leq \|x\|^2 \|y\|^2, \quad x, y, y_1, \ldots, y_n \in \mathcal{H},\ y_i \neq 0, \quad 1 \leq i \leq n \tag{1.2}$$

with the condition that $\langle y, y_i\rangle = 0$.

The purpose of this work is to show selberg's inequality and its refinement in semi-Hilbertian space. As an application, we give an extension of (1.2) in semi-Hilbertian space.

2 Main results

We start our work by presenting the Selberg inequality in semi-Hilbertian space.

Theorem 2.1. Let $\mathcal{H}$ be a Hilbert space and y_j be a vector such that $y_j \notin \mathcal{N}(A)$ for all $j = 1, \ldots, n$. If $x \in \mathcal{H}$ then

$$\sum_{i=1}^{n} \frac{|\langle y_i, x\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \leq \|x\|_A^2. \tag{2.1}$$

The equality in (2.1) holds if $x - \sum_{i=1}^{n} a_i y_i \in \mathcal{N}(A)$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$,

$$\begin{cases} \text{(C1)} \quad \langle y_i, y_j\rangle_A = 0 \\ \textit{or} \\ \text{(C2)} \quad \langle a_i y_i, a_j y_j\rangle_A = |\langle a_i y_i, a_j y_j\rangle_A| \text{ and } |a_i| = |a_j|. \end{cases}$$

Proof .

$$\begin{aligned} 0 \leq \quad & \left\| x - \sum_{i}^{n} a_i y_i \right\|_A^2 = \|x\|_A^2 - 2\mathrm{Re} \sum_{i=1}^{n} \overline{a_i} \langle x, y_i\rangle_A + \sum_{i,j}^{n} a_i \overline{a_j} \langle y_i, y_j\rangle_A \\ = \quad & \|x\|_A^2 - 2\mathrm{Re} \sum_{i=1}^{n} \overline{a_i} \langle x, y_i\rangle_A + \sum_{i,j}^{n} \mathrm{Re}(a_i \overline{a_j} \langle y_i, y_j\rangle_A) \\ \leq \quad & \|x\|_A^2 - 2\mathrm{Re} \sum_{i=1}^{n} \overline{a_i} \langle x, y_i\rangle_A + \frac{1}{2} \sum_{i,j}^{n} |a_i|^2 + |a_j|^2 |\langle y_i, y_j\rangle_A| \\ = \quad & \|x\|_A^2 - 2\mathrm{Re} \sum_{i=1}^{n} \overline{a_i} \langle x, y_i\rangle_A + \sum_{i=1}^{n} \left(|a_i|^2 \sum_{j=1}^{n} |\langle y_i, y_j\rangle_A| \right). \end{aligned}$$

If we put $a_i = \frac{\langle x, y_i\rangle_A}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|}$, then, we have the desired result.

The equality in (2.1) holds if the following (2.2) and (2.3),

$$x - \sum_{i=1}^{n} a_i y_i \in \mathcal{N}(A) \tag{2.2}$$

$$\sum_{i,j}^{n} a_i \overline{a_j} \langle y_i, y_j \rangle_A = \frac{1}{2} \sum_{i,j} (|a_i|^2 + |a_j|^2) | \langle y_i, y_j \rangle_A |. \tag{2.3}$$

The condition (2.3) is equivalent to the following (2.4)

$$\sum_{i,j=1}^{n} 2\mathrm{Re}\{a_i \overline{a_j} \langle y_i, y_j \rangle_A\} = \frac{1}{2} \sum_{i,j=1}^{n} \left(|a_i|^2 + |a_j|^2\right) | \langle y_i, y_j \rangle_A |. \tag{2.4}$$

On the other hand, the following inequality (2.5) is always valid for all i and j,

$$2\mathrm{Re}\{\langle a_i y_i, a_j y_j \rangle_A\} \leq 2|a_i||a_j| | \langle y_i, y_j \rangle_A | \leq \left(|a_i|^2 + |a_j|^2\right) | \langle y_i, y_j \rangle_A |. \tag{2.5}$$

So (2.3) is equivalent to the following (2.6) or (2.7) for arbitrary i and j because comparing (2.3) with (2.4)

$$\langle y_i, y_j \rangle_A = 0 \quad \text{for} \quad i \neq j \tag{2.6}$$

$$\langle a_1 y_i, a_j y_j \rangle_A = | \langle a_i y_i, a_j y_j \rangle_A | \quad \text{and} \quad |a_i| = |a_j|. \tag{2.7}$$

Whence the proof of Theorem is complete. □

Example 2.2. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(i) In $\mathbb{R}^4$, $x = \begin{bmatrix} 1 \\ 2 \\ c \\ d \end{bmatrix}$, $y_1 = \begin{bmatrix} 1 \\ 0 \\ r \\ s \end{bmatrix}$, $y_2 = \begin{bmatrix} 0 \\ 1 \\ l \\ m \end{bmatrix}$, where c, d, r, s, l and m are arbitrary scalars in $\mathbb{R}$. We have $\mathcal{N}(A) = vect\{(0,0,1,0)^t, (0,0,0,1)^t\}$, $x - y_1 - 2y_2 \in \mathcal{N}(A)$ and we have (C1) since $y_1 \perp_A y_2$ (i.e., $\langle y_1, y_2 \rangle_A = 0$). By Selberg's inequality we have equality in (2.1).

(ii) In $\mathbb{R}^4$, $x = \begin{bmatrix} 1 \\ 3 \\ c \\ d \end{bmatrix}$, $y_1 = \begin{bmatrix} 1 \\ 2 \\ r \\ s \end{bmatrix}$, $y_2 = \begin{bmatrix} 1 \\ 1 \\ l \\ m \end{bmatrix}$, where c, d, r, s, l and m are arbitrary scalars in $\mathbb{R}$. We have $\mathcal{N}(A) = vect\{(0,0,1,0)^t, (0,0,0,1)^t\}$, $x - y_1 - y_2 \in \mathcal{N}(A)$ and we have (C2). By Selberg's inequality we have equality in (2.1).

(iii) In $\mathbb{R}^4$, if $x - ay_1 - by_2 \in \mathcal{N}(A)$, $y_1 = \begin{bmatrix} 1 \\ 2 \\ r \\ s \end{bmatrix}$, and $y_2 = \begin{bmatrix} 1 \\ 1 \\ l \\ m \end{bmatrix}$, where a, b, r, s, l and m are arbitrary scalars in $\mathbb{R}$, then the Selberg inequality can be written as

$$\frac{\left(\langle ay_1, y_1 \rangle_A + \langle by_2, y_1 \rangle_A \right)^2}{| \langle y_1, y_1 \rangle_A | + | \langle y_1, y_2 \rangle_A |} + \frac{\left(\langle ay_1, y_2 \rangle_A + \langle by_2, y_2 \rangle_A \right)^2}{| \langle y_2, y_1 \rangle_A | + | \langle y_2, y_2 \rangle_A |} \leq a^2 \langle y_1, y_1 \rangle_A + 2ab \langle y_1, y_2 \rangle_A + b^2 \langle y_2, y_2 \rangle_A$$

i.e.,

$$\frac{(5a + 3b)^2}{8} + \frac{(3a + 2b)^2}{5} \leq 5a^2 + 6ab + 2b^2.$$

If $a = b$, then we have equality.

Theorem 2.3. Let $\mathcal{H}$ be a Hilbert space, A be an injective positive bounded operator and $y_1, \ldots, y_n$ be non zero vectors in $\mathcal{H}$. If $x \in \mathcal{H}$ then

$$\sum_{i=1}^{n} \frac{|\langle y_i, x\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \leq \|x\|_A^2 . \tag{2.8}$$

The equality in (2.8) holds if and only if $x = \sum_{i=1}^{n} a_i y_i$ for some complex scalars $a_1, a_2, \ldots, a_n$ such that for arbitrary $i \neq j$,

$$\begin{cases} \langle y_i, y_j\rangle_A = 0 \\ or \\ \langle a_i y_i, a_j y_j\rangle_A = |\langle a_i y_i, a_j y_j\rangle_A| \text{ and } |a_i| = |a_j|. \end{cases}$$

In the following corollary, we give the Bombieri type inequality on semi-Hilbertian spaces.

Corollary 2.4. Let $\mathcal{H}$ be a Hilbert space and $y_1, \ldots, y_n$ not in $\mathcal{N}(A)$. If $x \in \mathcal{H}$ then

$$\sum_{i=1}^{n} |\langle y_i, x\rangle_A|^2 \leq \|x\|_A^2 \left\{ \max_{1\leq i\leq n} \sum_{j=1}^{n} |\langle y_i, y_j\rangle_A| \right\}.$$

As a corollary, we have the following Boas-Bellman type inequality on semi-Hilbertian spaces.

Corollary 2.5. Let $\mathcal{H}$ be a Hilbert space and $y_1, \ldots, y_n$ not in $\mathcal{N}(A)$. If $x \in \mathcal{H}$ then

$$\sum_{i=1}^{n} |\langle y_i, x\rangle_A|^2 \leq \|x\|_A^2 \left\{ \max_{1\leq i\leq n} ||y_i||_A^2 + (n-1) \max_{i\neq k} |\langle y_i, y_j\rangle_A| \right\}.$$

With the following theorem we gave a refinement of Selberg inequality on semi-Hilbertian spaces.

Theorem 2.6. Let $\mathcal{H}$ be a Hilbert space, $y_1 \ldots y_n$ be vectors such that $y_j \notin \mathcal{N}(A)$ for all $j = 1, \ldots, n$ and y be a vector such that $\langle y, y_j\rangle_A = 0$ for $j = 1, \ldots, n$. If $x \in \mathcal{H}$ then

$$|\langle y, x\rangle_A|^2 + \sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \|y\|_A^2 \leq \|x\|_A^2 \|y\|_A^2 . \tag{2.9}$$

Proof . $u = x - \sum_{i=1}^{n} a_i y_i$. Then we have

$$\begin{aligned} \|u\|_A^2 &= \left\| x - \sum_{i=1}^{n} a_i y_i \right\|_A^2 \\ &\leq \|x\|_A^2 - 2\text{Re} \sum_{i=1}^{n} \overline{a}_i \langle x, y_i\rangle_A + \sum_{i=1}^{n} \left(|a_i|^2 \sum_{j=1}^{n} |\langle y_i, y_j\rangle_A| \right) \\ &= \|x\|_A^2 - \sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|}. \end{aligned}$$

Hence it follows that

$$\begin{aligned} \|y\|_A^2 \left(\|x\|_A^2 - \sum_{i=1} \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \right) &\geq \|y\|_A^2 \|u\|_A^2 \geq |\langle y, u\rangle_A|^2 \\ &= \left| \left\langle y, x - \sum_{i=1}^{n} \frac{\langle x, y_i\rangle_A}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \right\rangle_A \right|^2 \\ &= |\langle y, x\rangle_A|^2. \end{aligned}$$

□

Theorem 2.7. Let $\mathcal{H}$ be a Hilbert space, A be an injective bounded positive operator and $y, y_1 \dots y_n$ be non zero vectors in $\mathcal{H}$ such that $\langle y, y_j\rangle_A = 0$ for $j = 1, \dots, n$. If $x \in \mathcal{H}$ then

$$|\langle y, x\rangle_A|^2 + \sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \|y\|_A^2 \leq \|x\|_A^2 \|y\|_A^2.$$

The next theorem give an extension of the inequlity (2.9). For this, we will need the following lemma.

Lemma 2.8. Let $\mathbb{R}$ denote the set of real numbers. If $f : [0, \infty) \to \mathbb{R}$ is a derivable convex function on $[0, \infty)$ and $f(0) = 0$, then

$$f(x - y) \leq f(x) - f(y) \tag{2.10}$$

for all $x, y \in [0, \infty)$ and $x \geq y \geq 0$.

Proof . Assume that $f : [0, \infty) \to \mathbb{R}$ is a convex function with $f(0) = 0$. Let $\varphi : [a, \infty) \to \mathbb{R}$ be a function defined by $\varphi(x) = f(x-a)-f(x)+f(a)$ for all $x \in [a, \infty)$ and $a \geq 0$ is fixed. It is clear that $\varphi(a) = 0$ and $\varphi'(x) = f'(x-a)-f'(x)$. As f is a convex function, then f' is non-decreasing on $[0, \infty)$. So, $\varphi'(x) \leq 0$ for all $x \in [a, \infty)$, i.e., φ is non-increasing on $[a, \infty)$, this implies that $\varphi(x) \leq \varphi(a) = 0$ for all $x \in [a, \infty)$. The proof of the Lemma is complete. □

Theorem 2.9. Let $\mathcal{H}$ be a Hilbert space, $y_1, \dots, y_n$ be vectors such that $y_j \notin \mathcal{N}(A)$ for all $j = 1, \dots, n$ and y a vector such that $\langle y, y_j\rangle_A = 0$ for $j = 1, \dots, n$. If the function $f : [0, \infty) \to \mathbb{R}$ is derivable and non-decreasing convex on $[0, \infty)$ with $f(0) = 0$, then

$$f\big(|\langle y, x\rangle_A|^2\big) + f\left(\sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \|y\|_A^2\right) \leq f\big(\|x\|_A^2 \, \|y\|_A^2\big)$$

for all $x \in \mathcal{H}$.

We end the paper with two examples of applications of Theorem 2.9.

Example 2.10. Let $\mathcal{H}$ be a Hilbert space, $y_1, \dots, y_n$ be vectors in $\mathcal{H}$ such that $y_j \notin \mathcal{N}(A)$ for all $j = 1, \dots, n$ and y be a vector such that $\langle y, y_j\rangle_A = 0$ for $j = 1, \dots, n$. Then

$$|\langle y, x\rangle_A|^{2p} + \left(\sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|}\right)^p \|y\|^{2p} \leq \|x\|_A^{2p} \, \|y\|_A^{2p}$$

for all $x \in \mathcal{H}$ and $p \geq 1$.

Proof . It suffice to take $f(x) = x^p$ for all $x \in [0, \infty)$ and $p \geq 1$ in Theorem 2.9. □

Example 2.11. Let $\mathcal{H}$ be a Hilbert space, $y_1, \dots, y_n$ be vectors in $\mathcal{H}$ such that $y_j \notin \mathcal{N}(A)$ for all $j = 1, \dots, n$ and y be a vector such that $\langle y, y_j\rangle_A = 0$ for $j = 1, \dots, n$. Then

$$\exp\big(|\langle y, x\rangle_A|^2\big) + \exp\left(\sum_{i=1}^{n} \frac{|\langle x, y_i\rangle_A|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle_A|} \|y\|^2\right) \leq \exp\big(\|x\|_A^2 \, \|y\|_A^2\big) + 1$$

for all $x \in \mathcal{H}$.

Proof . It suffice to take $f(x) = \exp(x) - 1$ for all $x \in [0, \infty)$ in Theorem 2.9. □

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