Int. J. Nonlinear Anal. Appl. 13 (Special issue for ICDACT-2021), 1–15
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.6326

Global attractivity results for a class of matrix difference equations

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(Communicated by Reena Jain)

Abstract

In this chapter, we investigate the global attractivity of the recursive sequence $\{\mathcal{U}_n\} \subset \mathcal{P}(N)$ defined by

$$\mathcal{U}_{n+k} = \mathcal{Q} + \frac{1}{k}\sum_{j=0}^{k-1} \mathcal{A}^* \psi(\mathcal{U}_{n+j})\mathcal{A}, n = 1, 2, 3 \ldots,$$

where $\mathcal{P}(N)$ is the set of $N \times N$ Hermitian positive definite matrices, k is a positive integer, $\mathcal{Q}$ is an $N \times N$ Hermitian positive semidefinite matrix, $\mathcal{A}$ is an $N \times N$ nonsingular matrix, $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}$ and $\psi : \mathcal{P}(N) \to \mathcal{P}(N)$ is a continuous. For this, we first introduce $\mathcal{FG}$-Prešić contraction condition for $f : \mathcal{X}^k \to \mathcal{X}$ in metric spaces and study the convergence of the sequence $\{x_n\}$ defined by

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), n = 1, 2, \ldots$$

with the initial values $x_1, \ldots, x_k \in \mathcal{X}$. We furnish our results with some examples throughout the chapter. Finally, we apply these results to obtain matrix difference equations followed by numerical experiments.

Keywords: fixed point approximation, iterative method, matrix difference equation, equilibrium point, global attractivity.
2020 MSC: Primary 47H10; Secondary 54H25, 65Q10, 65Q30

1 Introduction and preliminaries

Consider the k-th order nonlinear difference equation:

$$x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), n = 1, 2, \ldots \tag{1.1}$$

with the initial values $x_1, \ldots, x_k \in \mathcal{X}$, where x_n is the value of x in generation n and where the recursion function f depends on nonlinear combinations of its arguments (f may involve quadratics, exponentials, reciprocals, or powers of the x_n's, and so forth). A solution is again a general formula relating x_n to the generation n and to some initially specified values, e.g., x_0, x_1, and so on.

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Received: August 2021 *Accepted:* January 2022

The study of nonlinear difference equations, which has a significant role in the modelling of various problems that emerge in genetics, psychology, sociology, probability theory, economics, biology, and ecology, amongst other fields of knowledge. The study of difference equations of order greater than one is a topic of significant interest, and numerous writers have contributed to the field via their research. For detail, one can refer [8, 10, 13, 14, 15] and the references that are cited within.

The following are some well-known difference equations that may be found in citations [21, 26], as well as their references.

- the generalized Beddington-Holt stock recruitment model:

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}; x_0, x_1 > 0, n \in \mathbb{N};$$

  where $a \in (0,1)$, $b \in \mathbb{R}_+^*$ and $c, d \in \mathbb{R}_+$ with $c + d > 0$;

- the delay model of a perennial grass:

$$x_{n+1} = ax_n + (b + cx_{n-1})e^{x_n}, n \in \mathbb{N};$$

  where $a, c \in (0,1)$ and $b \in \mathbb{R}_+$;

- the our beetle population model:

$$x_{n+3} = ax_{n+2} + bx_n e^{-(cx_n+2+dx_n)}, n \in \mathbb{N};$$

  where $a, b, c, d \geq 0$ and $c + d > 0$.

In the context of difference equations, a steady-state solution x is defined to be the value that satisfies the relations $x_{n+1} = x_n = x$, so that no change occurs from generation n to generation $n + 1$.

Equation (1.1) can be studied by means of fixed point theory in view of the fact that x in $\mathcal{X}$ is a solution of (1.1) if and only if x is a fixed point of mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ given by

$$\mathcal{T}(x) = f(x, x, \ldots, x) \text{ for all } x \in \mathcal{X}.$$

An interesting and important result as a generalization of Banach fixed point theorem [2, Theorem 1], in this direction, is due to Prešić [19], which can be stated as :

Theorem 1.1. [19] Let $(\mathcal{X}, d)$ be a complete metric space, k a positive integer. If a mapping $f : \mathcal{X}^k \to \mathcal{X}$ satisfies the following contractive condition :

$$d(f(x_1, x_2, \ldots x_k), f(x_2, \ldots, x_k, x_{k+1})) \leq \sum_{i=1}^{k} q_i d(x_i, x_{i+1}),$$

for every $x_1, \ldots x_{k+1} \in \mathcal{X}$, where $q_1, q_2, \ldots q_k$ are non-negative constants such that $q_1 + q_2 + \ldots + q_k < 1$. Then there exists a unique point $\nu^* \in \mathcal{X}$ such that $f(\nu^*, \ldots \nu^*) = \nu^*$. Moreover, for any arbitrary points $x_1 \ldots, x_k \in \mathcal{X}$, the sequence (1.1) converges to ν^*.

It is noted that, for $k = 1$, Theorem 1.1 reduces to the Banach contraction principle[2]. Theorem 1.1 is generalized by Ćirić and Prešić [7] as follows:

Theorem 1.2. [7] Let $(\mathcal{X}, d)$ be a complete metric space, and k a positive integer. If $f : \mathcal{X}^k \to \mathcal{X}$ satisfies the following contractive condition:

$$d(f(x_1, x_2, \ldots x_k), f(x_2, \ldots, x_k, x_{k+1})) \leq q \max\{d(x_i, x_{i+1}), 1 \leq i \leq k\}, \tag{1.2}$$

for any $x_1, x_2, \ldots x_{k+1} \in \mathcal{X}$, where $0 < q < 1$.Then there exists $\nu^* \in \mathcal{X}$ such that $f(\nu^*, \ldots, \nu^*) = \nu^*$. Moreover, for any arbitrary points $x_1, \ldots, x_k \in \mathcal{X}$, the sequence (1.1) is convergent and

$$\lim_{n\to\infty} x_n = f(\lim_{n\to\infty} x_n, \ldots, \lim_{n\to\infty} x_n).$$

If in addition,

$$d(f(\vartheta^*, \ldots, \vartheta^*), f(\nu^*, \ldots, \nu^*)) < d(\vartheta^*, \nu^*)$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$, with $\vartheta^* \neq \nu^*$, then ν^* is the unique point in $\mathcal{X}$ with $f(\nu^*, \ldots, \nu^*) = \nu^*$.

Chen (2009) published a work in which he employed the above conclusions to solve the global asymptotic stability of the equilibrium of a nonlinear difference equation, which can be found at [6].

The following are the convergence findings for Prešić-Kannan operators that were obtained by Păcurar [18]:

Theorem 1.3. [18] Let $(\mathcal{X}, d)$ be a complete metric space, k a positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given mapping. Suppose that there exists a constant $a \in \mathbb{R}$ with $0 < ak(k+1) < 1$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \le a \sum_{i=1}^{k+1} d(x_i, f(x_i, \dots, x_i)), \tag{1.3}$$

holds for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$. Then,

1. f has a unique fixed point $\nu^* \in \mathcal{X}$;
2. for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*.

In the work [1], Abbas et al. extended the previous conclusions using the following idea given by Wardowski [28].

Let $\mathfrak{F}$ be the collection of all mappings $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) $\mathcal{F}$ is strictly increasing,that is, for all $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies that $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$.

(F_2) For every sequence α_n of positive real numbers, $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \mathcal{F}(\alpha_n) = -\infty$ are equivalent.

(F_3) There exists $h \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^h \mathcal{F}(\alpha) = 0$.

Definition 1.4. [1] Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{F} \in \mathfrak{F}$. A mapping $f : \mathcal{X}^k \to \mathcal{X}$ is said to be a Pre*šić* type $\mathcal{F}-$contraction if there exists $\tau > 0$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}) > 0$$

implies that

$$\tau + \mathcal{F}(d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}))) \le \mathcal{F}(\max\{d(x_i, x_{i+1}) : 1 \le i \le k\}) \tag{1.4}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$.

Note that, for $\mathcal{F}(\alpha) = \ln \alpha$, Prešić type $\mathcal{F}$-contraction condition becomes

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \le e^{-\tau} \max\{d(x_i, x_{i+1}) : 1 \le i \le k\} \tag{1.5}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}, f(x_1, \dots, x_k) \ne f(x_2, \dots, x_{k+1})$.
Furthermore, for $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ such that $f(x_1, \dots, x_k) = f(x_2, \dots, x_{k+1})$, the inequality (1.5) also holds, that is, f is a Ćirić-Prešić contraction.

Theorem 1.5. [1] Let $(\mathcal{X}, d)$ be a complete metric space, and $f : \mathcal{X}^k \to \mathcal{X}$ a Prešić type $\mathcal{F}$-contraction, where k is a positive integer. Then for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to $\nu^* \in \mathcal{X}$ and ν^* is a fixed point of f. In addition, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) > 0$$

implies that

$$\tau + \mathcal{F}(d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, ^*))) \le \mathcal{F}(d(\vartheta^*, \nu^*))$$

for all $\nu^*, \vartheta^* \in \mathcal{X}$ with $\nu^* \ne \vartheta^*$, then f has a unique fixed point.

In this regard, researchers have worked for a map or a pair of maps using various forms of Prešić contraction type mappings. Some of these works are referred to in [3, 4, 12, 17, 22, 24, 23, 25] and the sources listed therein. In this paper, we use a slightly modified family of functions (than to $\mathfrak{F}$) by Parvaneh et al. [20] to define a new concept called $\mathcal{FG}$-Prešić contractive mapping and prove basic fixed point results. We illustrate our work with illustrative examples and show the superiority of $\mathcal{FG}$-Prešić contractive mapping over Ciric-Prešić (1.2) and Prešić-Kannan (1.3) mapping. In addition, we use this result to construct global attractivity results for a class of matrix difference equations, and we explore its convergence behaviour with regard to three alternative initializations with graphical representations in MATLAB.

2 $\mathcal{FG}$-Prešić contractive mapping and based results

We begin with the idea attributed to Parvaneh *et al.* [20].

Definition 2.1. [20] The collection of all functions $\mathcal{F} : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

($\mathbb{F}_1$) $\mathcal{F}$ is continuous and strictly increasing;

($\mathbb{F}_2$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\lim_{n\to\infty} \xi_n = 0$ iff $\lim_{n\to\infty} \mathcal{F}(\xi_n) = -\infty$,

will be denoted by $\mathbb{F}$.

The collection of all pairs of mappings $(\mathcal{G}, \beta)$, where $\mathcal{G} : \mathbb{R}_+ \to \mathbb{R}$, $\beta : \mathbb{R}_+ \to [0, 1)$, satisfying:

($\mathbb{F}_3$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \mathcal{G}(\xi_n) \geq 0$ iff $\limsup_{n\to\infty} \xi_n \geq 1$;

($\mathbb{F}_4$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\limsup_{n\to\infty} \beta(\xi_n) = 1$ implies $\lim_{n\to\infty} \xi_n = 0$;

($\mathbb{F}_5$) for each $\{\xi_n\} \subseteq \mathbb{R}_+$, $\sum_{n=1}^{\infty} \mathcal{G}(\beta(\xi_n)) = -\infty$,

will be denoted by $\mathbb{G}_\beta$.

Definition 2.2. Let $f : \mathcal{X}^k \to \mathcal{X}$, where $k \geq 1$ is a positive integer. A point $x^* \in \mathcal{X}$ is called a fixed point of f if $x^* = f(x^*, \dots, x^*)$.

Now, we introduce a notion of $\mathcal{FG}$-Prešić contraction for a map in metric space.

Definition 2.3. Let $(\mathcal{X}, d)$ be a metric space. A mapping $f : \mathcal{X}^k \to \mathcal{X}$ is said to be a $\mathcal{FG}$-Prešić contraction if there exist $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}) > 0$$

implies that

$$\begin{aligned} \mathcal{F}(d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}))) \leq{}& \mathcal{F}(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}) \\ &+ \mathcal{G}(\beta(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\})) \end{aligned} \tag{2.1}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$.

Theorem 2.4. Let $(\mathcal{X}, d)$ be a complete metric space, and $f : \mathcal{X}^k \to \mathcal{X}$ a $\mathcal{FG}$-Prešić-contraction and continuous, where k is a positive integer. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to $\nu^* \in \mathcal{X}$ and ν^* is a fixed point of f. In addition, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) > 0 \text{ implies that}$$

$$\mathcal{F}(d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*))) \leq \mathcal{F}(d(\vartheta^*, \nu^*)) + \mathcal{G}(\beta(d(\vartheta^*, \nu^*)))$$

for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, then f has a unique fixed point.

Proof . Begin by assuming that $x_1, \dots, x_k$ is a random k element in $\mathcal{X}$. Define the sequence $\{x_n\}$ in $\mathcal{X}$ by

$$x_{n+k} = f(x_n, x_{n+1}, \dots, x_{n+k-1}), n = 1, 2, \dots \cdot$$

If for some $n_0 \in \{1, 2, 3, \dots, k\}$, we have $x_{n_0} = x_{n_0+1} = x_{n_0+2} = \ldots = x_{n_0+k} = \nu^*$, then we have

$$x_{n_0+k} = f(x_{n_0}, x_{n_0+1}, \dots, x_{n_0+k-1}) = f(\nu^*, \nu^*, \dots, \nu^*) = \nu^*,$$

that is, ν^* is a fixed point of f and the proof is completed. Therefore, we assume that x_1, x_2,..., x_k, x_{k+1} are not all equal. So let, $x_{n+k} \neq x_{n+k+1}$ for all $n \in \mathbb{N}$. Denote $\gamma_{n+k} = d(x_{n+k}, x_{n+k+1})$ for $n = 1, 2, \dots$ and $\theta = \max\{d(x_1, x_2), d(x_2, x_3), \dots, d(x_k, x_{k+1})\}$, then we have $\gamma_{n+k} > 0$ for all $n \in \mathbb{N}$ and $\theta > 0$.

Now for $n \leq k$, we have the following inequalities:

$$\begin{aligned}\mathcal{F}(\gamma_{k+1}) &= \mathcal{F}(d(x_{k+1}, x_{k+2})) = \mathcal{F}(d(f(x_1, x_2, \ldots, x_k), f(x_2, x_3, \ldots, x_{k+1})))\\ &\leq \mathcal{F}(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}) + \mathcal{G}(\beta(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}))\\ &= \mathcal{F}(\theta) + \mathcal{G}(\beta(\theta))\end{aligned}$$

$$\begin{aligned}\mathcal{F}(\gamma_{k+2}) &= \mathcal{F}(d(x_{k+2}, x_{k+3})) = \mathcal{F}(d(f(x_2, x_3, \ldots, x_{k+1}), f(x_3, x_4, \ldots, x_{k+2})))\\ &\leq \mathcal{F}(\max\{d(x_i, x_{i+1}) : 2 \leq i \leq k+1\}) + \mathcal{G}(\beta(\max\{d(x_i, x_{i+1}) : 2 \leq i \leq k+1\}))\\ &= \mathcal{F}(\theta) + 2\mathcal{G}(\beta(\theta)).\end{aligned}$$

Continuing this process, we get

$$\begin{aligned}\mathcal{F}(\gamma_{k+n}) &= \mathcal{F}(d(x_{n+k}, x_{n+k+1})) = \mathcal{F}(d(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k})))\\ &\leq \mathcal{F}(\max\{d(x_i, x_{i+1}) : n \leq i \leq n+k-1\}) + \mathcal{G}(\beta(\max\{d(x_i, x_{i+1}) : n \leq i \leq n+k-1\}))\\ &= \mathcal{F}(\theta) + \sum_{i=1}^{n} \mathcal{G}(\beta(\theta)).\end{aligned} \tag{2.2}$$

Owing to the properties of $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ and from (2.2), we get $\mathcal{F}(\gamma_{k+n}) \to -\infty$ as $n \to \infty$. Thus, from the property $(\mathbb{F}_2)$, we have $\lim_{n\to\infty} \gamma_{k+n} = 0$, that is,

$$\lim_{n\to\infty} d(x_{n+k}, x_{n+k+1}) = 0. \tag{2.3}$$

Next, we must demonstrate that the sequence $\{x_n\}$ is a Cauchy sequence in $(\mathcal{X}, d)$. Assume the opposite; then there is $\varepsilon > 0$ and two subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ of $\{x_n\}$ such that m_i is the smallest index for which $m_i > n_i > i$ and

$$d(x_{m_i+k}, x_{n_i+k}) \geq \varepsilon. \tag{2.4}$$

This means that $m_i > n_i > i$ and

$$d(x_{m_i-1+k}, x_{n_i+k}) < \varepsilon. \tag{2.5}$$

On the other hand for $1 \leq j \leq k-1$, we get

$$\begin{aligned}\varepsilon \leq d(x_{m_i+j}, x_{n_i+j}) &\leq d(x_{m_i+j}, x_{m_i-1+j}) + d(x_{m_i-1+j}, x_{n_i+j}))\\ &\leq d(x_{m_i+j}, x_{m_i-1+j}) + \varepsilon.\end{aligned}$$

Taking $i \to \infty$ and using (2.3), we get

$$\lim_{i\to\infty} d(x_{m_i+j}, x_{n_i+j}) = \varepsilon, \tag{2.6}$$

and hence

$$\lim_{i\to\infty} d(x_{m_i+j+1}, x_{n_i+j+1}) = \varepsilon. \tag{2.7}$$

Using (2.1), we get

$$\begin{aligned}&\mathcal{F}(\limsup_{i\to\infty} d(x_{m_i+k+1}, x_{n_i+k+1}))\\ &= \mathcal{F}(\limsup_{i\to\infty} d(f(x_{m_i+1}, \ldots, x_{m_i+k}), f(x_{n_i+1}, \ldots, x_{n_i+k}))\\ &\leq \mathcal{F}(\limsup_{i\to\infty} \max\{d(x_{m_i+j}, x_{n_i+j}) : 1 \leq j \leq k\})\\ &\quad + \limsup_{i\to\infty} \mathcal{G}(\beta(\max\{d(x_{m_i+j}, x_{n_i+j}) : 1 \leq j \leq k\})).\end{aligned} \tag{2.8}$$

Making use of (2.3), (2.6) and (2.7) in (2.8), we get

$$\begin{aligned}\mathcal{F}(\varepsilon) &= \mathcal{F}(\limsup_{i\to\infty} d(x_{m_i+k+1}, x_{n_i+k+1}))\\ &\leq \mathcal{F}(\limsup_{i\to\infty} \max\{d(x_{m_i+j}, x_{n_i+j}) : 1 \leq j \leq k\})\\ &\quad + \limsup_{i\to\infty} \mathcal{G}(\beta(\max\{d(x_{m_i+j}, x_{n_i+j}) : 1 \leq j \leq k\}))\\ &= \mathcal{F}(\varepsilon) + \limsup_{i\to\infty} \mathcal{G}(\beta(\max\{d(x_{m_i+j}, x_{n_i+j}) : 1 \leq j \leq k\})),\end{aligned}$$

which implies that $\limsup_{i\to\infty} \mathcal{G}(\beta(d(x_{m_i+j}, x_{n_i+j}))) \geq 0$ for $1 \leq j \leq k$, which gives $\limsup_{i\to\infty} \beta(d(x_{m_i+j}, x_{n_i+j})) \geq 1$, and taking in account that $\beta(\xi) < 1$ for all $\xi \geq 0$, we have $\limsup_{i\to\infty} \beta(d(x_{m_i+j}, x_{n_i+j})) = 1$. Therefore, $\limsup_{i\to\infty} d(x_{m_i+j}, x_{n_i+j}) = 0$, a contradiction. Hence, $\{x_n\}$ is Cauchy sequence in $\mathcal{X}$.

Since $(\mathcal{X}, d)$ is complete, there exists ν^* in $\mathcal{X}$ such that

$$\lim_{n,m\to\infty} d(x_{n+j}, x_{m+j}) = \lim_{n\to\infty} d(x_{n+j}, \nu^*) = 0.$$

Now by the continuity of f, we have

$$\begin{aligned}\nu^* &= \lim_{n\to\infty} x_{n+k} = \lim_{n\to\infty} f(x_n, x_{n+1}, \dots, x_{n+k-1}) \\ &= f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} x_{n+1}, \dots, \lim_{n\to\infty} x_{n+k-1}) \\ &= f(\nu^*, \nu^*, \dots, \nu^*).\end{aligned}$$

Finally, we assert that f has just one fixed point. Indeed, if $\nu^*, \vartheta^* \in \mathcal{X}$ such that $\nu^* = f(\nu^*, \dots, \nu^*)$ and $\vartheta^* = f(\vartheta^*, \dots, \vartheta^*)$ with $\nu^* \neq \vartheta^*$. Thus $d(f(\nu^*, \dots, \nu^*), f(\vartheta^*, \dots, \vartheta^*)) > 0$. We therefore have by presumption

$$\begin{aligned}\mathcal{F}(d(\nu^*, \vartheta^*)) &= \mathcal{F}(d(f(\nu^*, \dots, \nu^*), f(\vartheta^*, \dots, \vartheta^*))) \\ &\leq \mathcal{F}(d(\nu^*, \vartheta^*)) + \mathcal{G}(\beta(d(\nu^*, \vartheta^*)))\end{aligned}$$

which gives $\mathcal{G}(\beta(d(\nu^*, \vartheta^*)) \geq 0$ implies $\beta(d(\nu^*, \vartheta^*) \geq 1$ a contradiction. Therefore $\nu^* = \vartheta^*$. □

3 Consequences

We may get various classes of $\mathcal{FG}$-Prešić contractive conditions in a complete metric space by considering a variety of concrete functions $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ in the condition (2.1) of Theorems 2.4.

Corollary 3.1. Let $(\mathcal{X}, d)$ be a complete metric space, k positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given continuous mapping. Suppose that there exist $\tau > 0$ and $\mathcal{F} \in \mathbb{F}$ such that

$$\tau + \mathcal{F}(d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}))) \leq \mathcal{F}(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}), \tag{3.1}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ with $f(x_1, \dots, x_k) \neq f(x_2, \dots, x_{k+1})$. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*, and ν^* is a fixed point of f. Moreover, if

$$\tau + \mathcal{F}(d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*))) \leq \mathcal{F}(d(\vartheta^*, \nu^*))$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, then ν^* is the unique fixed point of f.

Proof . Put $\mathcal{G}(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$ and $\tau = -\ln \lambda > 0$ in the (2.1) of Theorem 2.4, we have Wardowski-type [28] condition (3.1), that is, Theorem 1.2 due to Abbas et al. [1]. □

Corollary 3.2. Let $(\mathcal{X}, d)$ be a complete metric space, k positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given continuous mapping. Suppose that there exists $\lambda \in (0, 1)$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}))) \leq \lambda(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}), \tag{3.2}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ with $f(x_1, \dots, x_k) \neq f(x_2, \dots, x_{k+1})$. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*, and ν^* is a fixed point of f. Moreover, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) \leq \lambda\, d(\vartheta^*, \nu^*)$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, then ν^* is the unique fixed point of f.

Proof . Put $\mathcal{F}(t) = \mathcal{G}(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0, 1)$ in the (2.1) of Theorem 2.4, we have Banach-type contraction condition (3.2), that is, Theorem 1.2 due to Ćirić and Prešić [7]. □

Corollary 3.3. Let $(\mathcal{X}, d)$ be a complete metric space, k positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given continuous mapping. Suppose that there exists $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1}))) \\ \leq \beta(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}) \max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}, \tag{3.3}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ with $f(x_1, \dots, x_k) \neq f(x_2, \dots, x_{k+1})$. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*, and ν^* is a fixed point of f. Moreover, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) \leq \beta(d(\vartheta^*, \nu^*))d(\vartheta^*, \nu^*)$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, then ν^* is the unique fixed point of f.

Proof . Put $\mathcal{F}(t) = \mathcal{G}(t) = \ln t$ $(t > 0)$ in the (2.1) of Theorem 2.4, we have Geraghty-type [11, 9] condition (3.3). □

Corollary 3.4. Let $(\mathcal{X}, d)$ be a complete metric space, k a positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given continuous mapping. Suppose that there exists $\tau > 0$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \leq \frac{\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}}{(1 + \tau\sqrt{\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}})^2}, \tag{3.4}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ with $f(x_1, \dots, x_k) \neq f(x_2, \dots, x_{k+1})$. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*, that is $\nu^* = f(\nu^*, \dots, \nu^*)$. Moreover, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) \leq \frac{d(\vartheta^*, \nu^*)}{(1 + \tau\sqrt{d(\vartheta^*, \nu^*)})^2},$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, then ν^* is the unique fixed point of f.

Proof . Put $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ and $\beta(t) = \lambda \in (0, 1)$, $\tau = -\ln \lambda > 0$ in the (2.1) of Theorem 2.4, we have new Prešić rational type contraction condition (3.4). □

Corollary 3.5. Let $(\mathcal{X}, d)$ be a complete metric space, k a positive integer and $f : \mathcal{X}^k \to \mathcal{X}$ a given continuous mapping. Suppose that there exist $\tau > 0$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that

$$d(f(x_1, \dots, x_k), f(x_2, \dots, x_{k+1})) \\ \leq \frac{\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}}{[1 - \tau\sqrt{\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}} ln(\beta(\max\{d(x_i, x_{i+1}) : 1 \leq i \leq k\}))]^2}, \tag{3.5}$$

for all $(x_1, \dots, x_{k+1}) \in \mathcal{X}^{k+1}$ with $f(x_1, \dots, x_k) \neq f(x_2, \dots, x_{k+1})$. Then, for any arbitrary points $x_1, \dots, x_k \in \mathcal{X}$, the sequence $\{x_n\}$ defined by (1.1) converges to ν^*, that is $\nu^* = f(\nu^*, \dots, \nu^*)$. Moreover, if

$$d(f(\vartheta^*, \dots, \vartheta^*), f(\nu^*, \dots, \nu^*)) \leq \frac{d(\vartheta^*, \nu^*)}{(1 - \tau\sqrt{d(\vartheta^*, \nu^*)} ln(\beta(d(\vartheta^*, \nu^*))))^2},$$

holds for all $\vartheta^*, \nu^* \in \mathcal{X}$ with $\vartheta^* \neq \nu^*$, where $\lambda \in [0, 1)$, then ν^* is the unique fixed point of f.

Proof . Put $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ in the (2.1) of Theorem 2.4, we have another Prešić rational type contraction condition (3.5). □

Remark 3.6.
1. Theorem 2.4 extends and generalizes [1, Theorem 2.1], [7, Theorem 2.1] and [19, Theorem 1.1].
2. If $k = 1$, Corollary 3.1 reduces to [28, Theorem 2.1].
3. If $k = 1$, Corollary 3.2 reduces to [2, Theorem 1].
4. If $k = 1$, Corollaries 3.3 reduces to the theorem of Boyd and Wong [5].

4 Illustrations

Example 4.1. (Inspired by Abbas et al. [1].) Let $\mathcal{X} = \left\{x_n = \frac{n(n+1)}{2} : n \in \mathbb{N}\right\}$ with the usual metric, $k = 2$. Then $(\mathcal{X}, d)$ is a complete metric space. Define the mapping $f : \mathcal{X}^2 \to \mathcal{X}$ by

$$f(x,y) = \begin{cases} \frac{x_{n-1}+y_{m-1}}{2}, \text{if } x = x_n, y = y_m \text{ for } n, m > 1, \\ \frac{x_1+y_1}{2}, \quad \text{otherwise.} \end{cases}$$

Moreover, take $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$ and $\beta(t) = \lambda \in (0,1)$, $\tau = -\ln \lambda > 0$ (Corollary 3.4). Then it is easy to see that all the conditions of Theorem 2.4 are fulfilled—just the condition (2.1).

For $f(x_i, x_{i+1}) \neq f(x_{i+1}, x_{i+2})$, $i = 1, 2, \ldots$, we consider two cases:

(i) If $x = x_1$, $y = x_2$, then

$$d(f(x_1,x_2), f(x_2,x_3)) \leq \frac{\max\{d(x_1,x_2), d(x_2,x_3)\}}{(1+\tau\sqrt{\max\{d(x_1,x_2), d(x_2,x_3)\}})^2}, \tag{4.1}$$

for all $(x_1, x_2, x_3) \in \mathcal{X}^3$.
Then (4.1) becomes

$$1 \leq \frac{3}{(1+\tau\sqrt{3})^2},$$

true for $\tau > 0$.

(ii) If $x = x_n$, $y = x_{n+1}$ with $n > 1$, we have

$$d(f(x_n,x_{n+1}), f(x_{n+1},x_{n+2})) \leq \frac{\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}}{(1+\tau\sqrt{\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\}})^2}, \tag{4.2}$$

for all $(x_n, x_{n+1}, x_{n+2}) \in \mathcal{X}^3$.
Now

$$\begin{aligned} & d(f(x_n,\ x_{n+1}),\ f(x_{n+1},\ x_{n+2}) \\ &= \frac{1}{2}\left|\left(\frac{(n-1)n}{2} + \frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2}\right)\right| \\ &= \left|\frac{n^2}{2} - \frac{2n^2+4n+2}{4}\right| = n + \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} & \max\{d(x_n,\ x_{n+1}),\ d(x_{n+1},\ x_{n+2})\} \\ &= \max\left\{\left|\frac{n(n+1)}{2} - \frac{(n+1)(n+2)}{2}\right|, \left|\frac{(n+1)(n+2)}{2} - \frac{(n+2)(n+3)}{2}\right|\right\} \\ &= \max\{n+1,\ n+2\} = n+2. \end{aligned}$$

Then (4.2) becomes

$$n + \frac{1}{2} \leq \frac{n+2}{(1+\tau\sqrt{n+2})^2},$$

true for $\tau > 0$.

Thus f is the $\mathcal{FG}$-Prešić contraction on $\mathcal{X}$ and $(1,1)$ is a unique fixed point of f.

Next, we demonstrate that the requirement (1.3) of Theorem 1.3 is not met. To demonstrate this, we compute the following: When $k = 2$, $0 < ak(k+1) < 1$ implies $0 < a < 1/6$. Also

$$\begin{aligned} d(x_n, f(x_n,\ x_n)) &= \frac{1}{2}\left|\frac{n(n+1)}{2} - \frac{1}{2}\left(\frac{(n-1)n}{2} + \frac{(n-1)n}{2}\right)\right| \\ &= n, \end{aligned}$$

$$\begin{aligned} d(x_{n+1}, f(x_{n+1},\ x_{n+1}) &= \frac{1}{2}\left|\frac{(n+1)(n+2)}{2} - \frac{1}{2}\left(\frac{n(n+1)}{2} + \frac{n(n+1)}{2}\right)\right| \\ &= n+1 \end{aligned}$$

and

$$\begin{aligned} &d(x_{n+2}, f(x_{n+2},\ x_{n+2}) \\ &= \frac{1}{2}\left|\frac{(n+2)(n+3)}{2} - \frac{1}{2}\left(\frac{(n+1)(n+2)}{2} + \frac{(n+1)(n+2)}{2}\right)\right| \\ &= n+2. \end{aligned}$$

Then (1.3) implies that

$$\begin{aligned} &\frac{d(f(x_n, x_{n+1}), f(x_{n+1}, x_{n+2}))}{d(x_n, f(x_n,\ x_n) + d(x_{n+1}, f(x_{n+1},\ x_{n+1}) + d(x_{n+2}, f(x_{n+2},\ x_{n+2})} \\ &= \frac{n+\frac{1}{2}}{3(n+3)} \geq \frac{1}{6} \text{for } n > 1. \end{aligned}$$

Thus the condition (1.3) is not true for $n > 1$.

Finally, we demonstrate that the condition (1.2) of Theorem 1.2 is not met when $n > 2$. To demonstrate this, we compute the following: For $n > 2$

$$\begin{aligned} &d(f(x_{n-2},\ x_{n-1}),\ f(x_{n-1},\ x_n) \\ &= \frac{1}{2}\left|\left(\frac{(n-3)(n-2)}{2} + \frac{(n-2)(n-1)}{2}\right) - \left(\frac{(n-2)(n-1)}{2} + \frac{(n-1)n}{2}\right)\right| \\ &= n\ - \frac{3}{2} \end{aligned}$$

and

$$\begin{aligned} &\max\{d(x_{n-2},\ x_{n-1}),\ d(x_{n-1},\ x_n)\} \\ &=\ \max\left\{\left|\frac{(n-2)(n-1)}{2} - \frac{(n-1)n}{2}\right|,\ \left|\frac{n(n-1)}{2} - \frac{n(n+1)}{2}\right|\right\} \\ &=\ \max\{n-1,\ n\}\ = n. \end{aligned}$$

$$\lim_{n\to\infty} \frac{d(f(x_{n-2}, x_{n-1}), f(x_{n-1}, x_n))}{\max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}} = \lim_{n\to\infty} \frac{n-3/2}{n} = 1.$$

Thus

$$d(f(x_{n-2},\ x_{n-1}),\ f(x_{n-1},\ x_n)) \nleq q \max\{d(x_{n-2},\ x_{n-1}),\ d(x_{n-1},\ x_n)\}$$

for $q \in (0,1)$. Hence the condition (1.2) of Theorem 1.1 does not satisfied.

As a result, all of the requirements of Theorem 2.4 are met, and f has a unique fixed point $(1,1)$, that is, $f(1,1) = 1$.

Example 4.2. Let $\mathcal{X} = [0,\infty)$ with the usual metric, $k = 2$ and the operators $f : [0,1]^2 \to [0,1]$ defined by $f(x,y) = \begin{cases} \frac{x-y}{5}, & x \geq y \\ 0, & x < y \end{cases}$.

Let us first determine whether or not f satisfies condition (2.1) for $\mathcal{F}(t) = -\frac{1}{\sqrt{t}}$, $\mathcal{G}(t) = \ln t$ $(t > 0)$, $\beta(t) = \lambda \in (0,1)$, $\tau = -\ln\lambda > 0$. In our specific example, $k = 2$, and as a result, the preceding condition (2.1) becomes

$$d(f(x_1, x_2), f(x_2, x_3)) \leq \frac{\max\{d(x_1, x_2), d(x_2, x_3)\}}{(1+\tau\sqrt{\max\{d(x_1, x_2), d(x_2, x_3)\}})^2}, \tag{4.3}$$

for all $(x_1, x_2, x_3) \in [0,1]^3$.

We will look at four different scenarios:

Case (I): When $x_1 \geq x_2 \geq x_3$.

$$d(f(x_1,x_2), f(x_2,x_3)) = \left|\frac{x_1-x_2}{5} - \frac{x_2-x_3}{5}\right| = \left|\frac{x_1-2x_2+x_3}{5}\right|,$$
$$d(x_1,x_2) = |x_1-x_2|,\ d(x_2,x_3) = |x_2-x_3|.$$

- If $|x_1-x_2| > |x_2-x_3|$, then $\max\{d(x_1,x_2), d(x_2,x_3)\} = |x_1-x_2|$ and (4.3) implies that
$$\left|\frac{x_1-2x_2+x_3}{5}\right| \leq \frac{|x_1-x_2|}{(1+\tau\sqrt{|x_1-x_2|})^2}.$$
- If $|x_1-x_2| < |x_2-x_3|$, then $\max\{d(x_1,x_2), d(x_2,x_3)\} = |x_2-x_3|$ and (4.3) implies that
$$\left|\frac{x_1-2x_2+x_3}{5}\right| \leq \frac{|x_2-x_3|}{(1+\tau\sqrt{|x_2-x_3|})^2}.$$

For $\tau > 0$, both of the instances listed above are correct.

Case (II): When $x_1 \geq x_2 < x_3$.

$$d(f(x_1,x_2), f(x_2,x_3)) = \left|\frac{x_1-x_2}{5} - 0\right| = \frac{|x_1-x_2|}{5},$$
$$d(x_1,x_2) = |x_1-x_2|, d(x_2,x_3) = |x_2-x_3|.$$

- If $|x_1-x_2| > |x_2-x_3| \neq 0$, then $\max\{d(x_1,x_2), d(x_2,x_3)\} = |x_1-x_2|$ and (4.3) implies that
$$\frac{|x_1-x_2|}{5} \leq \frac{|x_1-x_2|}{(1+\tau\sqrt{|x_1-x_2|})^2},$$
that is,
$$\tau \leq \frac{\sqrt{5}-1}{\sqrt{|x_1-x_2|}}.$$
- If $|x_1-x_2| < |x_2-x_3| \neq 0$, then $\max\{d(x_1,x_2), d(x_2,x_3)\} = |x_2-x_3|$ and (4.3) implies that
$$\frac{|x_1-x_2|}{5} \leq \frac{|x_2-x_3|}{(1+\tau\sqrt{|x_2-x_3|})^2}.$$

Both of the above examples are valid when $\tau > 0$.

Case (III): When $x_1 < x_2 \geq x_3$.
Similar to Case (II).

Case (IV): When $x_1 < x_2 < x_3$.
Then $d(f(x_1,x_2), f(x_2,x_3)) = 0$, and (4.3) is obviously true.

As a result, all of the criteria of Theorem 2.4 are met, and f has a unique fixed point at the coordinates $(0,0)$.

5 Application

Specifically, in this part, we explore the global attractivity of the recursive sequence $\{\mathcal{U}_n\} \subset \mathcal{P}(N)$ formed by the formula
$$\mathcal{U}_{n+k} = \mathcal{Q} + \frac{1}{k}\sum_{j=0}^{k-1} \mathcal{A}^*\psi(\mathcal{U}_{n+j})\mathcal{A}, n = 1,2,3\ldots, \tag{5.1}$$
where $\mathcal{P}(N)$ is the set of $N \times N$ Hermitian positive definite matrices, k is a positive integer, $\mathcal{Q}$ is an $N \times N$ Hermitian positive semidefinite matrix, $\mathcal{A}$ is an $N \times N$ nonsingular matrix, $\mathcal{A}^*$ is the conjugate transpose of $\mathcal{A}$ and $\psi : \mathcal{P}(N) \to \mathcal{P}(N)$ is a continuous.

To finish this, we need to consider the following ideas.

Definition 5.1. Let k be a positive integer, Δ a nonempty set and $f : \Delta^k \to \Delta$. For given $\nu_1, \nu_2, \dots, \nu_k \in \Delta$, consider the recursive sequence $\{\nu_n\} \subset \Delta$ defined by

$$\nu_{n+k} = f(\nu_n, \nu_{n+1}, \dots, \nu_{n+k-1}), n = 1, 2, \dots, \tag{5.2}$$

The equilibrium point $\overline{\nu}$ of the equation (5.2) is the point that satisfies the condition:

$$\overline{\nu} = f(\overline{\nu}, \dots, \overline{\nu}).$$

Definition 5.2. Let (Δ, d) be a metric space and $\overline{\nu}$ an equilibrium point of equation (5.2). The equilibrium point $\overline{\nu}$ is called a global attractor if for all $\nu_1, \nu_2, \dots, \nu_k \in \Delta$, we have $d(\nu_n, \overline{\nu}) \to 0$ as $n \to \infty$.

We denote by $\mathcal{P}(N)$(for $N \geq 2$), the open convex cone of all $N \times N$ Hermitian positive definite matrices. We endow $\mathcal{P}(N)$ with the Thompson metric defined by

$$\mathcal{A}, \mathcal{B} \in \mathcal{P}(N), d(\mathcal{A}, \mathcal{B}) = \max\{\ln \Delta(\mathcal{A}/\mathcal{B}), \ln \Delta(\mathcal{B}/\mathcal{A})\},$$

where

$$\Delta(\mathcal{A}/\mathcal{B}) = \inf\{\theta > 0 : \mathcal{A} \leq \theta\mathcal{B}\} = \theta^+(\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}),$$

the maximal eigenvalue of $\mathcal{B}^{-1/2}\mathcal{A}\mathcal{B}^{-1/2}$. Here $\mathcal{C} \leq \mathcal{D}$ ($\mathcal{C} < \mathcal{D}$) means that $\mathcal{D} - \mathcal{C}$ is positive semidefinite and positive definite respectively. From Nussbaum [16], $\mathcal{P}(N)$ is a complete metric space with respect to the Thompson metric d and $d(\mathcal{A}, \mathcal{B}) = \| \ln(\mathcal{A}^{-1/2}\mathcal{B}\mathcal{A}^{-1/2})\|$, where $\|.\|$ stands for the spectral norm. The Thompson metric occurs on every open normal convex cone of a real Banach space [16, 27]; specifically, the open convex cone of positive definite operators of a Hilbert space. Now we will briefly discuss the Thompson metric's beautiful characteristics. It is invariant by matrix inversion and congruence transformations, which means it does not change. Moreover,

$$d(\mathcal{A}, \mathcal{B}) = d(\mathcal{A}^{-1}, \mathcal{B}^{-1}) = d(\mathcal{W}^*\mathcal{A}\mathcal{W}, \mathcal{W}^*\mathcal{B}\mathcal{W}), \tag{5.3}$$

for any nonsingular matrix $\mathcal{W}$. The other useful result is the nonpositive curvature property of the Thompson metric

$$d(\mathcal{C}^p, \mathcal{D}^p) \leq p\ d(\mathcal{C}, \mathcal{D}), p \in [0, 1]. \tag{5.4}$$

According to (5.3) and (5.4), we have

$$d(\mathcal{W}^*\mathcal{C}^p\mathcal{W}, \mathcal{W}^*\mathcal{D}^p\mathcal{W}) \leq |p| d(\mathcal{C}, \mathcal{D}), p \in [-1, 1]. \tag{5.5}$$

Lemma 5.3. For any $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathcal{P}(N)$,

$$d(\mathcal{A} + \mathcal{B}, \mathcal{C} + \mathcal{D}) \leq \max\{d(\mathcal{A}, \mathcal{C}), d(\mathcal{B}, \mathcal{D})\}.$$

Furthermore, for all positive semidefinite $\mathcal{A}$ and $\mathcal{B}, \mathcal{C} \in \mathcal{P}(N)$

$$d(\mathcal{A} + \mathcal{B}, \mathcal{A} + \mathcal{C}) \leq d(\mathcal{B}, \mathcal{C}).$$

Theorem 5.4. Consider the problem described by (5.1). Assume that $\psi : \mathcal{P}(N) \to \mathcal{P}(N)$ be an $\mathcal{FG}$-contraction mapping with respect to the Thompson metric d, that is, for all $\mathcal{K}, \mathcal{L} \in \mathcal{P}(N)$, there exist $\mathcal{F} \in \mathbb{F}$ and $(\mathcal{G}, \beta) \in \mathbb{G}_\beta$ such that $d(\psi(\mathcal{K}), \psi(\mathcal{L})) > 0$ implies that

$$\mathcal{F}(d(\psi(\mathcal{K}), \psi(\mathcal{L}))) \leq \mathcal{F}(d(\mathcal{K}, \mathcal{L})) + \mathcal{G}(\beta(d(\mathcal{K}, \mathcal{L}))). \tag{5.6}$$

Then the equation (5.1) has a unique equilibrium point $\overline{\mathcal{U}} \in \mathcal{P}(N)$ and thus, $\overline{\mathcal{U}}$ is a global attractor.

Proof . Define a mapping $f : \mathcal{P}(N)^k \to \mathcal{P}(N)$ by

$$f(\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_k) = \mathcal{Q} + \frac{1}{k}[\mathcal{A}^*\psi(\mathcal{U}_1)\mathcal{A} + \mathcal{A}^*\psi(\mathcal{U}_2)\mathcal{A} + \dots + \mathcal{A}^*\psi(\mathcal{U}_k)\mathcal{A}],$$

for all $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_k \in \mathcal{P}(N)$.

Let $\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_{k+1}\in\mathcal{P}(N)$. Owing Lemma 5.3, we have

$$\begin{aligned}
&d(f(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_k), f(\mathcal{U}_2,\mathcal{U}_3,\dots,\mathcal{U}_{k+1})\\
&= d\left(\mathcal{Q}+\frac{1}{k}\sum_{i=1}^{k}\mathcal{A}^*\psi(\mathcal{U}_i)\mathcal{A}, \mathcal{Q}+\frac{1}{k}\sum_{j=2}^{k+1}\mathcal{A}^*\psi(\mathcal{U}_j)\mathcal{A}\right)\\
&\le d\left(\frac{1}{k}\sum_{i=1}^{k}\mathcal{A}^*\psi(\mathcal{U}_i)\mathcal{A}, \frac{1}{k}\sum_{j=2}^{k+1}\mathcal{A}^*\psi(\mathcal{U}_j)\mathcal{A}\right)\\
&= d\left(\sum_{i=1}^{k}\left(\frac{1}{\sqrt{k}}\mathcal{A}\right)^*\psi(\mathcal{U}_i)\left(\frac{1}{\sqrt{k}}\mathcal{A}\right), \sum_{j=2}^{k+1}\left(\frac{1}{\sqrt{k}}\mathcal{A}\right)^*\psi(\mathcal{U}_j)\left(\frac{1}{\sqrt{k}}\mathcal{A}\right)\right).
\end{aligned}$$

Denote $\mathcal{V}=\frac{1}{\sqrt{k}}\mathcal{A}$. Then, using Lemma 5.3, we have

$$\begin{aligned}
&d(f(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_k), f(\mathcal{U}_2,\mathcal{U}_3,\dots,\mathcal{U}_{k+1}))\\
&\le d\left(\sum_{i=1}^{k}\mathcal{V}^*\psi(\mathcal{U}_i)\mathcal{V}, \sum_{j=2}^{k+1}\mathcal{V}^*\psi(\mathcal{U}_j)\mathcal{V}\right)\\
&= d\begin{pmatrix} \mathcal{V}^*\psi(\mathcal{U}_1)\mathcal{V}+\mathcal{V}^*\psi(\mathcal{U}_2)\mathcal{V}+\dots+V^*\psi(\mathcal{U}_k)V,\\ V^*\psi(\mathcal{U}_2)\mathcal{V}+\mathcal{V}^*\psi(\mathcal{U}_3)\mathcal{V}+\dots+\mathcal{V}^*\psi(\mathcal{U}_{k+1})\mathcal{V}\end{pmatrix}\\
&\le \max\left\{\begin{array}{c} d(\mathcal{V}^*\psi(\mathcal{U}_1)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_2)\mathcal{V}), d(\mathcal{V}^*\psi(\mathcal{U}_2)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_3)\mathcal{V}),\dots,\\ d(\mathcal{V}^*\psi(\mathcal{U}_k)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_{k+1})\mathcal{V})\end{array}\right\}\\
&= \max\{d(\mathcal{V}^*\psi(\mathcal{U}_i)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_{i+1})\mathcal{V}) : i=1,2,\dots,k\}.
\end{aligned}$$

As $\mathcal{A}$ is nonsingular, the matrix $\mathcal{V}$ is also nonsingular. Using property (5.3), for all $i=1,2,\dots,k$, we have

$$d(\mathcal{V}^*\psi(\mathcal{U}_i)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_{i+1})\mathcal{V}) = d(\psi(\mathcal{U}_i),\psi(\mathcal{U}_{i+1})).$$

By virtue of $\mathcal{FG}$-contraction of ψ, for all $i=1,2,\dots,k$, we have

$$\begin{aligned}
&\mathcal{F}(d(\mathcal{V}^*\psi(\mathcal{U}_i)\mathcal{V},\mathcal{V}^*\psi(\mathcal{U}_{i+1})\mathcal{V})) = \mathcal{F}(d(\psi(\mathcal{U}_i),\psi(\mathcal{U}_{i+1})))\\
&\le \mathcal{F}(d(\mathcal{U}_i,\mathcal{U}_{i+1})) + \mathcal{G}(\beta(d(\mathcal{U}_i,\mathcal{U}_{i+1}))),
\end{aligned}$$

that is,

$$\begin{aligned}
&\mathcal{F}(d(f(\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_k), f(\mathcal{U}_2,\mathcal{U}_3,\dots,\mathcal{U}_{k+1}))\\
&\le \mathcal{F}(\max\{d(\mathcal{U}_i,\mathcal{U}_{i+1}) : i=1,2,\dots,k\}) + \mathcal{G}(\beta(\max\{d(\mathcal{U}_i,\mathcal{U}_{i+1}) : i=1,2,\dots,k\}))
\end{aligned}$$

for all $\mathcal{U}_1,\mathcal{U}_2,\dots,\mathcal{U}_{k+1}\in\mathcal{P}(N)$.

Following Theorem 2.4, there exists the existence of a global attractor equilibrium point $\overline{\mathcal{U}}\in\mathcal{P}(N)$. To see the uniqueness of $\overline{\mathcal{U}}\in\mathcal{P}(N)$, let $\overline{\mathcal{W}}\in\mathcal{P}(N)$ be another equilibrium point such that $d(f(\mathcal{U},\mathcal{U},\dots,\mathcal{U})\neq f(\mathcal{W},\mathcal{W},\dots,\mathcal{W}))$. Then we have

$$\begin{aligned}
\mathcal{F}(d(f(\mathcal{U},\mathcal{U},\dots,\mathcal{U}), f(\mathcal{W},\mathcal{W},\dots,\mathcal{W}))) &= \mathcal{F}(d(\mathcal{Q}+\mathcal{A}^*\psi(\mathcal{U})\mathcal{A}, \mathcal{Q}+\mathcal{A}^*\psi(\mathcal{W})\mathcal{A}))\\
&\le \mathcal{F}(d(\mathcal{A}^*\psi(U)\mathcal{A},\mathcal{A}^*\psi(W)\mathcal{A}))\\
&= \mathcal{F}(d(\psi(\mathcal{U}),\psi(\mathcal{W})))\\
&\le \mathcal{F}(d(\mathcal{U},\mathcal{W})) + \mathcal{G}(\beta(d(\mathcal{U},\mathcal{W}))).
\end{aligned}$$

Applying Theorem 2.4, it confirm that the equilibrium point $\overline{\mathcal{U}}\in\mathcal{P}(N)$ is unique. □

Example 5.5. We construct an example with given $\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3\in\mathcal{P}(N)$, and sequence of matrices given by the equation for $0<p<1$

$$\mathcal{X}_{n+3} = \mathcal{Q}+\frac{1}{3}(\mathcal{A}^*\mathcal{X}_n^p\mathcal{A}+\mathcal{A}^*\mathcal{X}_{n+1}^p\mathcal{A}+\mathcal{A}^*\mathcal{X}_{n+2}^p\mathcal{A}), n=1,2,\cdot$$

The residual error $Error(k) = \|\mathcal{X}_k - (\mathcal{Q} + \mathcal{A}^*\mathcal{X}_k^p\mathcal{A})\|$. In our example $\mathcal{A}^*$ represents the conjugate transpose of the matrix $\mathcal{A}$. We conducted this experiment using the high-level machine language MATLAB 2020b (available online), and the following is our system configuration: macOS Mojave version 10.14.6 CPU @1.6 GHz intel core i5 8GB.

We take $\tau = 0.1$; tolerance or error $= 1 \times 10^{-10}$; Matrix Dimension= 5.

$$\mathcal{A} = \begin{bmatrix} 0.4173 & 0.4893 & 0.7803 & 0.1320 & 0.2348 \\ 0.0497 & 0.3377 & 0.3897 & 0.9421 & 0.3532 \\ 0.9027 & 0.9001 & 0.2417 & 0.9561 & 0.8212 \\ 0.9448 & 0.3692 & 0.4039 & 0.5752 & 0.0154 \\ 0.4909 & 0.1112 & 0.0965 & 0.0598 & 0.0430 \end{bmatrix}, \mathcal{Q} = \begin{bmatrix} 0.0109 & 0.0070 & 0.0132 & 0.0097 & 0.0035 \\ 0.0070 & 0.0128 & 0.0163 & 0.0088 & 0.0017 \\ 0.0132 & 0.0163 & 0.0327 & 0.0185 & 0.0066 \\ 0.0097 & 0.0088 & 0.0185 & 0.0152 & 0.0058 \\ 0.0035 & 0.0017 & 0.0066 & 0.0058 & 0.0027 \end{bmatrix},$$

$$\mathcal{X}_1 = \begin{bmatrix} 0.5002 & 0.0002 & 0.0003 & 0.0002 & 0.0001 \\ 0.0002 & 0.5002 & 0.0003 & 0.0002 & 0.0001 \\ 0.0003 & 0.0003 & 0.5006 & 0.0004 & 0.0001 \\ 0.0002 & 0.0002 & 0.0004 & 0.5003 & 0.0001 \\ 0.0001 & 0.0001 & 0.0001 & 0.0001 & 0.5000 \end{bmatrix}, \mathcal{X}_2 = \begin{bmatrix} 1.4397 & 0.4054 & 0.4182 & 0.3372 & 0.3752 \\ 0.4054 & 1.5017 & 0.5212 & 0.5059 & 0.4256 \\ 0.4182 & 0.5212 & 1.6597 & 0.5775 & 0.5052 \\ 0.3372 & 0.5059 & 0.5775 & 1.6408 & 0.4433 \\ 0.3752 & 0.4256 & 0.5052 & 0.4433 & 1.5043 \end{bmatrix},$$

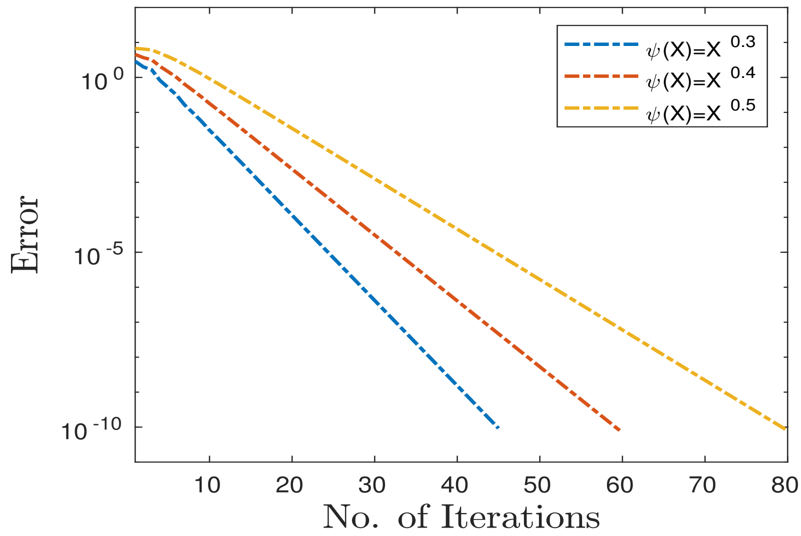
$$\mathcal{X}_3 = \begin{bmatrix} 0.4916 & 0.4868 & 0.4348 & 0.3592 & 0.4397 \\ 0.4868 & 0.6350 & 0.4556 & 0.3462 & 0.6118 \\ 0.4348 & 0.4556 & 0.4658 & 0.3339 & 0.4896 \\ 0.3592 & 0.3462 & 0.3339 & 0.2702 & 0.3378 \\ 0.4397 & 0.6118 & 0.4896 & 0.3378 & 0.7297 \end{bmatrix},$$

with $det(\mathcal{A}) = -0.0384$, $min(eig(\mathcal{Q})) = 3.0962e-05$, $min(eig(\mathcal{X}_1)) = 0.4999$, $min(eig(\mathcal{X}_2)) = 1.0177$,$min(eig(\mathcal{X}_3)) = 7.2571e-04$ which ensures the basic requirement of our matrices such as non-singularity of $\mathcal{A}$, and positive definiteness of the remaining matrices. We have encountered three different functions with same matrix accessories. In Table 1, we arranged all the experimental data including number of iteration(Iter. No.), Error(i) $= \|\mathcal{X}_i - (\mathcal{Q} + \mathcal{A}^*\mathcal{X}_i^p\mathcal{A})\|$ with $\|.\|$ as spectral norm, i.e.; largest singular value and CPU Time (T). The Equilibrium point $\bar{\mathcal{X}}$ with its minimum eigenvalue λ (to confirm its positive definiteness) also shown in fifth and sixth column of the Table 1. Figure 1 shows convergence behavior of the experiment.

Table 1. Analysis for three different values of $\psi(\mathcal{X})$.

$\psi(\mathcal{X})$	Iter.No.	Error	T	Equilibrium Point, $\bar{\mathcal{X}}$	λ
$\mathcal{X}^{0.3}$	45	9.2047	0.020424	$\begin{bmatrix} 3.4288 & 2.6496 & 2.3034 & 3.1573 & 1.6802 \\ 2.6496 & 2.2644 & 1.9031 & 2.6298 & 1.5339 \\ 2.3034 & 1.9031 & 1.8975 & 2.1742 & 1.1714 \\ 3.1573 & 2.6298 & 2.1742 & 3.2807 & 1.8020 \\ 1.6802 & 1.5339 & 1.1714 & 1.8020 & 1.1296 \end{bmatrix}$	0.0063
$\mathcal{X}^{0.4}$	60	6.9374	0.023718	$\begin{bmatrix} 4.8826 & 3.8595 & 3.4740 & 4.5963 & 2.4499 \\ 3.8595 & 3.2199 & 2.8339 & 3.7712 & 2.1193 \\ 3.4740 & 2.8339 & 2.7313 & 3.2979 & 1.7686 \\ 4.5963 & 3.7712 & 3.2979 & 4.5846 & 2.4960 \\ 2.4499 & 2.1193 & 1.7686 & 2.4960 & 1.4682 \end{bmatrix}$	0.0056
$\mathcal{X}^{0.5}$	80	7.9138	0.029695	$\begin{bmatrix} 8.2630 & 6.6148 & 6.0688 & 7.8598 & 4.1987 \\ 6.6148 & 5.4178 & 4.9171 & 6.3866 & 3.4914 \\ 6.0688 & 4.9171 & 4.6343 & 5.7874 & 3.0984 \\ 7.8598 & 6.3866 & 5.7874 & 7.6479 & 4.1257 \\ 4.1987 & 3.4914 & 3.0984 & 4.1257 & 2.3044 \end{bmatrix}$	0.0050

Figure 1. Convergence behaviour for three different initial values.



Conclusions

In this work, we have introduced a new $\mathcal{FG}$-Prešić contraction condition for $f : \mathcal{X}^k \to \mathcal{X}$ in metric spaces and study the convergence of the sequence $\{x_n\}$ defined by $x_{n+k} = f(x_n, x_{n+1}, \ldots, x_{n+k-1}), n = 1, 2, \ldots$. We have supplied sufficient instances to confirm the fixed-point conclusions as well as the significance of related work. This finding was used to explore the global attractivity of the recursive sequence $\{\mathcal{U}_n\} \subset \mathcal{P}(N)$ using a graphical depiction of convergence analysis.

Acknowledgment

Authors are thankful to the Science and Engineering Research Board, India, for providing funds under the project - CRG/2018/000615.

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