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Results on coupled fixed point involving altering distances in metric spaces

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Abstract

The objective of this paper is to demonstrate the results of coupled fixed point that possesses the property of mixed monotone involving altering distance functions in the framework of partially ordered metric spaces. To illustrate our results, we provide an example.

Keywords: Coupled fixed point, partial ordered, complete metric space.
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1 Introduction

The Banach Contraction Principle is the vital result of the analysis. In numerous directions the generalizations of this principle have been obtained. The utilization of a control work that adjusts the distance between two focuses in a metric space have been started by Khan et. al [10] in 1984. Such mappings are called altering distances. In various papers altering distance has been utilized in metric fixed point hypothesis (see [7, 8, 9, 11]).

Bhaskar and Lakshmikantham [3] started the investigation of a coupled fixed point hypothesis in ordered metric spaces and applied their outcomes to demonstrate the existence and uniqueness for a periodic boundary value problem. Numerous specialists have gotten coupled fixed point results for mappings under different contractive conditions in the system of partial metric spaces [1, 2, 4, 5, 6, 12, 13].

The objective of this paper is to demonstrate the results on coupled fixed point that possesses the property of mixed monotone involving altering distance functions in the framework of partially ordered metric spaces. Lastly, we provide an example that satisfies the main theorem.

From the outset, we need the accompanying definitions and results.

Definition 1.1. [10]. A function is said to be An altering distance function $\phi : [0, \infty) \to [0, \infty)$ if:

1. ϕ is non-decreasing and continuous.

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2. $\phi(w) = 0 \iff w = 0.$

Definition 1.2. [3] $R(q,w) = q$ and $R(w,q) = w$ is said to be coupled fixed point of the mapping $R : W \times W \to W$ for an element $(q,w) \in W \times W$.

Definition 1.3. [3] Presuppose $(W, \leq)$ be a partially ordered set and $R : W \times W \to W$. We say that R has the mixed monotone property if $R(q,w)$ is monotone non-increasing in w and is monotone non-decreasing in q, that is, for any $q, w \in W$,

$$q_1, q_2 \in W, q_1 \leq q_2 \implies R(q_1, w) \leq R(q_2, w)$$

and

$$w_1, w_2 \in W, w_1 \leq w_2 \implies R(q, w_1) \geq R(q, w_2).$$

Theorem 1.4. [3]. Presume $(W, \leq, d)$ be a complete metric space. Presuppose a mapping $R : W \times W \to W$ having the property of mixed monotone on W. Presume that there exists a $k \in [0,1)$ and

$$d(R(q,w), R(p,v)) \leq \frac{k}{2}[d(q,p) + d(w,v)] \tag{1.1}$$

$\forall\ p, q, w, v \in W$ for $q \geq p$ and $w \leq v$. Presuppose either R is continuous or W has the subsequent properties:

(i) if a nondecreasing sequence $q_h \to r$, then $q_h \leq q$, $\forall h$,

(ii) if a non-increasing sequence $w_h \to w$, then $w_h \geq w$, $\forall h$.

If there exists $q_0, w_0 \in W$ with $q_0 \leq R(q_0, w_0)$ and $w_0 \geq R(w_0, q_0)$, then R has a coupled fixed point.

2 Main Theorem

Theorem 2.1. Presume $(W, d, \leq)$ be a complete metric space. Suppose a continuous map $R : W \times W \to W$ having the property of mixed monotone on W such that

$$\begin{aligned}\psi(d(R(q,w), R(p,v))) \leq\ & \psi(d(U((q,w),(p,v))) - \phi(d(U((q,w),(p,v))) \\ & + K\ (P((q,w),(p,v)))\end{aligned} \tag{2.1}$$

where

$$\begin{aligned}U((q,w),(p,v)) &= max\{d(q,p), d(w,v), d(R(q,w),q), d(R(p,v),p)\} \\ P((q,w),(p,v)) &= min\{d(q,p), d(w,v), d(R(q,w),q), d(R(q,w),p), d(R(p,v),q)\}\end{aligned}$$

$\forall\ q, w, p, v \in W$ for $w \leq v$ and $q \geq p$, here an altering distance functions are ψ and ϕ and $K \geq 0$. Presuppose that there exists $q_0, w_0 \in W$ then $q_0 \leq R(q_0, w_0)$ and $w_0 \geq R(w_0, q_0)$, thus R posses a coupled fixed point.

Proof . Take $q_0, w_0 \in W$; set $q_1 = R(q_0, w_0)$ and $w_1 = R(w_0, q_0)$. Repeating this process, set $q_{h+1} = R(q_h, w_h)$ and $w_{h+1} = R(w_h, q_h)$. Then by inequality (2.1), we have

$$\begin{aligned}\psi(d(q_{h+1}, q_h)) &= \psi(d(R(q_h, w_h), R(q_{h-1}, w_{h-1}))) \\ &\leq \psi(U((q_h, w_h),(q_{h-1}, w_{h-1}))) - \phi(U((q_h, w_h),(q_{h-1}, w_{h-1}))) \\ &\quad + K\ P((q_h, w_h),(q_{h-1}, w_{h-1})),\end{aligned}$$

and

$$\begin{aligned}\psi(d(w_{h+1}, w_h)) &= \psi(d(R(w_h, q_h), R(w_{h-1}, q_{h-1}))) \\ &\leq \psi(U((w_h, q_h),(w_{h-1}, q_{h-1}))) - \phi(U((w_h, q_h),(w_{h-1}, q_{h-1}))) \\ &\quad + K\ P((w_h, q_h),(w_{h-1}, q_{h-1})),\end{aligned}$$

where,

$$\begin{aligned}U((q_h,w_h),(q_{h-1},w_{h-1})) &= max\{d(q_h,q_{h-1}),d(w_h,w_{h-1}),\\&\qquad d(R(q_h,w_h),q_h),d(R(q_{h-1},w_{h-1}),q_{h-1})\}\\&= max\{d(q_h,q_{h-1}),d(w_h,w_{h-1}),d(q_{h+1},q_h),d(q_h,q_{h-1})\}\\&= max\{d(q_h,q_{h-1}),d(w_h,w_{h-1}),d(q_{h+1},q_h)\}.\end{aligned}$$

also,

$$\begin{aligned}P((q_h,w_h),(q_{h-1},w_{h-1})) &= min\ \{d(q_h,q_{h-1}),d(w_h,w_{h-1}),d(R(q_h,w_h),q_h),d(R(q_h,w_h),q_{h-1}),d(R(q_{h-1},w_{h-1}),q_h)\}\\&= min\ \{d(q_h,q_{h-1}),d(w_h,w_{h-1}),d(q_{h+1},q_h),d(q_{h+1},q_{h-1}),d(q_h,q_h)\}\\&= 0.\end{aligned}$$

Similarly,

$$P((w_h,q_h),(w_{h-1},q_{h-1})) = 0$$

Now, we study the subsequent two cases.
Case I: If

$$U((q_h,w_h),(q_{h-1},w_{h-1})) = max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}.$$

We have

$$\psi(d(q_{h+1},q_h)) \le \psi(max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}) - \phi(max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}), \tag{2.2}$$

and

$$\psi(d(w_{h+1},w_h)) \le \psi(max\{d(w_h,w_{h-1}),d(q_h,q_{h-1})\}) - \phi(max\{d(w_h,w_{h-1}),d(q_h,q_{h-1})\}). \tag{2.3}$$

Case II: If

$$U((q_h,w_h),(q_{h-1},w_{h-1})) = d(q_{h+1},q_h).$$

We profess that

$$U((q_h,w_h),(q_{h-1},w_{h-1})) = d(q_{h+1},q_h) = 0.$$

Now, if $d(q_{h+1},q_h) \neq 0$, then

$$\psi(d(q_{h+1},q_h)) \le \psi(d(q_{h+1},q_h)) - \phi(d(q_{h+1},q_h)) < \psi(d(q_{h+1},q_h)) \ \text{ as } \ \phi \ge 0.$$

This implies

$$d(q_{h+1},q_h) < d(q_{h+1},q_h),$$

which is a contradiction. As, $U((q_h,w_h),(q_{h-1},w_{h-1})) = 0$. Then it is obvious that (2.2) and (2.3) hold. Now, from inequalities (2.2) and (2.3), we have

$$\psi(d(q_{h+1},q_h)) \le \psi(max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}) - \phi(max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}) \tag{2.4}$$

As $\phi \ge 0$.

$$\psi(d(q_{h+1},q_h)) \le \psi(max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}),$$

and as ψ is non-decreasing, we have

$$d(q_{h+1},q_h) \le max\{d(q_h,q_{h-1}),d(w_h,w_{h-1})\}. \tag{2.5}$$

Similarly,

$$\begin{aligned}\psi(d(w_{h+1},w_h)) &\le \psi(max\{d(w_h,w_{h-1}),d(q_h,q_{h-1})\}) - \phi(max\{d(w_h,w_{h-1}),d(q_h,q_{h-1})\})\\&\le \psi(max\{d(w_h,w_{h-1}),d(q_h,q_{h-1})\}),\end{aligned} \tag{2.6}$$

and consequently

$$d(w_{h+1}, w_h) \leq max\{d(w_h, w_{h-1}), d(q_h, q_{h-1})\}, \tag{2.7}$$

by (2.5) and (2.7), we obtain

$$max\{d(q_{h+1}, q_h), d(w_{h+1}, w_h)\} \leq max\{d(q_h, q_{h-1}), d(w_h, w_{h-1})\},$$

and thus, $max\{d(q_{h+1}, q_h), d(w_{h+1}, w_h)\}$ is nonnegative non-increasing sequence. Thus $\exists\, x \geq 0$ thus

$$\lim_{h\to\infty} max\{d(q_{h+1}, q_h), d(w_{h+1}, w_h)\} = x. \tag{2.8}$$

Now, if $\psi : [0, \infty) \to [0, \infty)$ is non-decreasing, $\psi(max(b, a)) = max(\psi(b), \psi(a))$ for $b, a \in [0, \infty)$. Taking this and (2.4) and (2.6), we have

$$\begin{aligned} max\{\psi(d(q_{h+1}, q_h)), \psi(d(w_{h+1}, w_h))\} &= \psi(max\{d(q_{h+1}, q_h), d(w_{h+1}, w_h)\}) \\ &\leq \psi(max\{d(q_h, q_{h-1}), d(w_h, w_{h-1})\}) \\ &\quad - \phi(max\{d(q_h, q_{h-1}), d(w_h, w_{h-1})\}). \end{aligned} \tag{2.9}$$

Letting $h \to \infty$ in (2.9) and (2.8), we have

$$\psi(x) \leq \psi(x) - \phi(x) \leq \psi(x),$$

and thus $\phi(x) = 0$. As ϕ is an altering distance function, $x = 0$. This implies

$$\lim_{h\to\infty} max\{d(q_{h+1}, q_h), d(w_{h+1}, w_h)\} = 0.$$

Thus

$$\lim_{h\to\infty} d(q_{h+1}, q_h) = \lim_{h\to\infty} d(w_{h+1}, w_h) = 0. \tag{2.10}$$

Next, we claim that $\{q_h\}, \{w_h\}$ are Cauchy sequences.
We will validate for each $\varepsilon > 0$, $\exists\, s \in N$, if $h, m \geq s$,

$$max\{d(q_{m(s)}, q_{h(s)}), d(w_{m(s)}, w_{h(s)})\} < \varepsilon.$$

Presuppose the above statement is not true.

Thus, $\exists\, \varepsilon > 0$ for which we can find sequence $\{q_{m(s)}\}, \{q_{h(s)}\}$ with $s < m(s) < h(s)$ such that

$$max\{d(q_{m(s)}, q_{h(s)}), d(w_{m(s)}, w_{h(s)})\} \geq \varepsilon. \tag{2.11}$$

Furthermore, we can take $h(s)$ corresponding to $m(s)$ in a way that $m(s) < h(s)$ is smallest integer and satisfing (2.11). Then

$$max\{d(q_{m(s)}, q_{h(s)-1}), d(w_{m(s)}, w_{h(s)-1})\} < \varepsilon. \tag{2.12}$$

From triangle inequality

$$d(q_{h(s)}, q_{m(s)}) \leq d(q_{h(s)}, q_{h(s)-1}) + d(q_{h(s)-1}, q_{m(s)}). \tag{2.13}$$

Similarly

$$d(w_{h(s)}, w_{m(s)}) \leq d(w_{h(s)}, w_{h(s)-1}) + d(w_{h(s)-1}, w_{m(s)}). \tag{2.14}$$

From (2.13) and (2.14), we have

$$\begin{aligned} max\{d(q_{h(s)}, q_{m(s)}), d(w_{h(s)}, w_{m(s)})\} &\leq max\{d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1})\} \\ &\quad + max\{d(q_{h(s)-1}, q_{m(s)}), d(w_{h(s)-1}, w_{m(s)})\}. \end{aligned} \tag{2.15}$$

From (2.11), (2.12) and (2.15), we get

$$\varepsilon \leq max\{d(q_{h(s)}, q_{m(s)}), d(w_{h(s)}, w_{m(s)})\} \leq max\{d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1})\} + \varepsilon. \tag{2.16}$$

Let $s \to \infty$ in (2.16) and (2.10) we have

$$\lim_{s\to\infty} max\{d(q_{h(s)}, q_{m(s)}), d(w_{h(s)}, w_{m(s)})\} = \varepsilon. \tag{2.17}$$

Again, the triangle inequality, we have

$$d(q_{h(s)-1}, q_{m(s)-1}) \leq d(q_{h(s)-1}, q_{m(s)}) + d(q_{m(s)}, q_{m(s)-1}), \tag{2.18}$$

and

$$d(w_{h(s)-1}, w_{m(s)-1}) \leq d(w_{h(s)-1}, w_{m(s)}) + d(w_{m(s)}, w_{m(s)-1}). \tag{2.19}$$

From (2.18) and (2.19), we have

$$\begin{aligned} max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1})\} &\leq max\{d(q_{h(s)-1}, q_{m(s)}), d(w_{h(s)-1}, w_{m(s)})\} \\ &\quad + max\{d(q_{m(s)}, q_{m(s)-1}), d(w_{m(s)}, w_{m(s)-1})\}. \end{aligned} \tag{2.20}$$

From (2.12), we have

$$max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1})\} \leq max\{d(q_{m(s)}, q_{m(s)-1}), d(w_{m(s)}, w_{m(s)-1})\} + \varepsilon. \tag{2.21}$$

Using the triangle inequality, we have

$$d(q_{h(s)}, q_{m(s)}) \leq d(q_{h(s)}, q_{h(s)-1}) + d(q_{h(s)-1}, q_{m(s)-1}) + d(q_{m(s)-1}, q_{m(s)}), \tag{2.22}$$

and

$$d(w_{h(s)}, w_{m(s)}) \leq d(w_{h(s)}, w_{h(s)-1}) + d(w_{h(s)-1}, w_{m(s)-1}) + d(w_{m(s)-1}), w_{m(s)}). \tag{2.23}$$

From (2.22), (2.23) and (2.11), we get

$$\begin{aligned} \varepsilon \leq\ & max\{d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1})\} \\ !! \quad & + max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1})\} \\ & + max\{d(q_{m(s)-1}, q_{m(s)}), d(w_{m(s)-1}), w_{m(s)})\}. \end{aligned} \tag{2.24}$$

From (2.24) and (2.21), we have

$$\begin{aligned} \varepsilon - max\{d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1})\} - max\{d(q_{m(s)-1}, q_{m(s)}), d(w_{m(s)-1}), w_{m(s)})\} \\ \leq max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1})\} \\ < max\{d(q_{m(s)-1}, q_{m(s)}), d(w_{m(s)-1}, w_{m(s)})\} + \varepsilon. \end{aligned} \tag{2.25}$$

Let $s \to \infty$ in (2.25) and using (2.10), we get

$$\lim_{s\to\infty} max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1})\} = \varepsilon. \tag{2.26}$$

Since $q_{h(s)-1} \geq q_{m(s)-1}$ and $w_{h(s)-1} \leq w_{m(s)-1}$, using the inequality (2.1) we can obtain

$$\begin{aligned} \psi(d(q_{h(s)}, q_{m(s)})) &= \psi(d(R(q_{h(s)-1}, w_{h(s)-1}), d(R(q_{m(s)-1}, w_{m(s)-1}))) \\ &\leq \psi(U((q_{h(s)-1}, w_{h(s)-1}), (q_{m(s)-1}, w_{m(s)-1}))) \\ &\quad - \phi(U((q_{h(s)-1}, w_{h(s)-1}), (q_{m(s)-1}, w_{m(s)-1}))) \\ &\quad + K\ P((q_{h(s)-1}, w_{h(s)-1}), (q_{m(s)-1}, w_{m(s)-1})), \end{aligned} \tag{2.27}$$

where

$$\begin{aligned} U((q_{h(s)-1}, w_{h(s)-1}), (q_{m(s)-1}, w_{m(s)-1})) &= max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}), \\ &\qquad d(R(q_{h(s)-1}, w_{h(s)-1}), q_{h(s)-1}), \\ &\qquad d(R(q_{m(s)-1}, w_{m(s)-1}), q_{m(s)-1})\} \\ &= max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}), \\ &\qquad d(q_{h(s)}, q_{h(s)-1}), d(q_{m(s)}, q_{m(s)-1})\} \end{aligned}$$

and

$$\begin{aligned}P((q_{h(s)-1}, w_{h(s)-1}), (q_{m(s)-1}, w_{m(s)-1})) &= min\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(R(q_{h(s)-1}, w_{h(s)-1}), q_{h(s)-1}),\\ &\quad d(R(q_{h(s)-1}, w_{h(s)-1}), q_{m(s)-1}),\\ &\quad d(R(q_{m(s)-1}, w_{m(s)-1}), q_{h(s)-1})\}\\ &= min\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(q_{h(s)}, q_{h(s)-1}), d(q_{h(s)}, q_{m(s)-1}), d(q_{m(s)}, q_{h(s)-1})\}\end{aligned}$$

Similarly,

$$\begin{aligned}\psi(d(w_{h(s)}, w_{m(s)})) &= \psi(d(R(w_{h(s)-1}, q_{h(s)-1}), d(R(w_{m(s)-1}, q_{m(s)-1})))\\ &\leq \psi(U((w_{h(s)-1}, q_{h(s)-1}), (w_{m(s)-1}, q_{m(s)-1})))\\ &\quad - \phi(U((w_{h(s)-1}, q_{h(s)-1}), (w_{m(s)-1}, q_{m(s)-1})))\\ &\quad + K\ (P((w_{h(s)-1}, q_{h(s)-1}), (w_{m(s)-1}, q_{m(s)-1}))),\end{aligned} \tag{2.28}$$

where

$$\begin{aligned}U((w_{h(s)-1}, q_{h(s)-1}), (w_{m(s)-1}, q_{m(s)-1}) &= max\{d(w_{h(s)-1}, w_{m(s)-1}), d(q_{h(s)-1}, q_{m(s)-1}),\\ &\quad d(R(w_{h(s)-1}, q_{h(s)-1}), w_{h(s)-1}),\\ &\quad d(R(w_{m(s)-1}, q_{m(s)-1}), w_{m(s)-1})\}\\ &= max\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(w_{h(s)}, w_{h(s)-1}), d(w_{m(s)}, w_{m(s)-1})\}.\end{aligned}$$

and

$$\begin{aligned}P((w_{h(s)-1}, q_{h(s)-1}), (w_{m(s)-1}, q_{m(s)-1}) &= min\{d(w_{h(s)-1}, w_{m(s)-1}), d(q_{h(s)-1}, q_{m(s)-1}),\\ &\quad d(R(w_{h(s)-1}, q_{h(s)-1}), w_{h(s)-1})\\ &\quad d(R(w_{h(s)-1}, q_{h(s)-1}), w_{m(s)-1}),\\ &\quad d(R(w_{m(s)-1}, q_{m(s)-1}), w_{h(s)-1})\}\\ &= min\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(w_{h(s)}, w_{h(s)-1}), d(w_{h(s)}, w_{m(s)-1}), d(w_{m(s)}, w_{h(s)-1})\}.\end{aligned}$$

From (2.27) and (2.28), we have

$$max\{\psi(d(q_{h(s)}, q_{m(s)}), d(w_{h(s)}, w_{m(s)}))\} \leq \psi(z_h) - \phi(z_h) + K\ (e_h),$$

where

$$\begin{aligned}z_h = max\{&d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1}),\\ &d(q_{m(s)}, q_{m(s)-1}), d(w_{m(s)}, w_{m(s)-1})\}.\end{aligned}$$

and

$$\begin{aligned}e_h &= min\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1}),\\ &\quad d(q_{h(s)}, q_{m(s)-1}), d(q_{m(s)}, q_{h(s)-1})\\ &\quad d(w_{h(s)}, w_{m(s)-1}), d(w_{m(s)}, w_{h(s)-1})\}\\ &\leq min\{d(q_{h(s)-1}, q_{m(s)-1}), d(w_{h(s)-1}, w_{m(s)-1}),\\ &\quad d(q_{h(s)}, q_{h(s)-1}), d(w_{h(s)}, w_{h(s)-1}),\\ &\quad d(q_{m(s)}, q_{m(s)-1}), d(w_{m(s)}, w_{m(s)-1})\}\end{aligned}$$

Finally letting $s \to \infty$ in last two inequalities and using (2.26), (2.17) and (2.10) and the condition of continuity of ϕ and ψ, we have

$$\psi(\varepsilon) \leq \psi(max(\varepsilon, 0, 0)) - \phi(max(\varepsilon, 0, 0)) + K\ (min((\varepsilon, 0, 0)) < \psi(\varepsilon)$$

and hence, $\phi(\varepsilon) = 0$. As ϕ is an altering distance function, therefore, $\varepsilon = 0$. This is a contradiction. Which proves our claim.

As W is a complete metric space, $\exists\ q, w \in W$ such that

$$\lim_{h\to\infty} q_h = q \text{ and } \lim_{h\to\infty} w_h = w.$$

Now we show that (q, w) is a coupled fixed point of R.

As, we have

$$q = \lim_{h\to\infty} q_{h+1} = \lim_{h\to\infty} R(q_h, w_h) = R(\lim_{h\to\infty} q_h, \lim_{h\to\infty} w_h) = R(q, w),$$
$$w = \lim_{h\to\infty} w_{h+1} = \lim_{h\to\infty} R(w_h, q_h) = R(\lim_{h\to\infty} w_h, \lim_{h\to\infty} q_h) = R(w, q).$$

Therefore, (q, w) is a coupled fixed point of R. □

Theorem 2.2. Presume all the hypothesis of Theorem 2.1 are gratified. Moreover, presuppose that W has the subsequent properties

(a) if a non-increasing sequence $\{w_h\} \to w$, then $w_h \geq w$, for each h,

(b) if a non-decreasing sequence$\{q_h\} \to q$, then $q_h \leq q$, for each h.

Then the conclusion of Theorem 2.1 also hold.

Proof . Following the proof of Theorem 2.1 we have to check only that (q, w) is a coupled fixed point of R.

Since $\{q_h\}$ is non-decreasing and $q_h \to q$ and $\{w_h\}$ is non-increasing and $w_h \to w$, by our assumption, $q_h \leq q$ and $w_h \geq w\ \forall\ h$.

Applying the contractive condition we have

$$\psi(d(R(q, w), R(q_h, w_h))) \leq \psi(U((q, w), (q_h, w_h))) - \phi(U((q, w), (q_h, w_h))) + K\ (P((q, w), (q_h, w_h)))$$

where

$$U((q, w), (q_h, w_h)) = max\{d(q, q_h), d(w, w_h), d(R(q, w), q), d(R(q_h, w_h), q_h)\}. \tag{2.29}$$

and

$$P((q, w), (q_h, w_h)) = max\{d(q, q_h), d(w, w_h), d(R(q, w), q), d(R(q, w), q_h), d(R(q_h, w_h), q)\} = 0.$$

and as ψ is nondecreasing, we obtain

$$d(R(q, w), R(q_h, w_h)) \leq U((q, w), (q_h, w_h)), \tag{2.30}$$

Letting $h \to \infty$ in (2.30) (and hence (2.29)), we obtain

$$d(q, R(q, w)) = 0,$$

and consequently $R(q, w) = q$.

On similar way, it can be showed that $w = R(w, q)$. □

for $(q, w), (u, v) \in W \times W$ there exists $(z, t) \in W \times W$ which is comparable to (q, w) and (p, v). (2.31)

Note that in $W \times W$ we consider the partial order relation given by

$$(q, w) \leq (p, v) \iff q \leq p \text{ and } w \geq v.$$

Theorem 2.3. In addition to the presumptions of Theorem 2.1 (resp. Theorem 2.2) condition (2.31) we acquire the uniqueness of the coupled fixed point of R.

Proof . Presuppose (q,w) and (q',w') are coupled fixed points of R, that is, $R(q,w)=q, R(w,q)=w, R(q',w')=q'$ and $R(w',q')=w'$. We shall show that $q=q', w=w'$.

Let (q,w) and (q',w') are not comparable. By presumption $\exists\ (z,t)\in W\times W$ comparable with (q,w) and (q',w'). Suppose that $(q,w)\geq(z,t)$.

We define sequences $\{z_h\},\{t_h\}$ as follows

$$t_0=t\ z_0=z,\ t_{h+1}=R(t_h,z_h)\ and z_{h+1}=R(z_h,t_h)\ \forall h.$$

As (z,t) is comparable with (q,w). We claim that $(q,w)\geq(z_h,t_h)$ for each $h\in N$.

We will use mathematical induction.

For $h=0$, as $(q,w)\geq(z,t)$, this means $z_0=z\leq q$ and $w\geq t=t_0$ and consequently, $(q,w)\geq(z_0,t_0)$.

Suppose that $(q,w)\geq(z_h,t_h)$; then using the property of mixed monotone R, we get

$$\begin{aligned} z_{h+1}&=R(z_h,t_h)\leq R(q,t_h)\leq R(q,w)=q,\\ t_{h+1}&=R(t_h,z_h)\geq R(w,z_h)\geq R(w,q)=w, \end{aligned}$$

and this proves our claim.

Now, since $z_h\leq q$ and $t_h\geq w$, using (2.1), we have

$$\begin{aligned} \psi(d(q,z_{h+1}))=\psi(d(R(q,w),R(z_h,t_h)))&\leq\psi(U((q,w),(z_h,t_h)))-\phi(U((q,w),(z_h,t_h))),\\ &+\ K\ (P((q,w),(z_h,t_h))) \end{aligned}\tag{2.32}$$

where

$$\begin{aligned} U((q,w),(z_h,t_h))&=max\{d(q,z_h),d(w,t_h),d(R(q,w),q),d(R(z_h,t_h),z_h)\}\\ &=max\{d(q,z_h),d(w,t_h)\}. \end{aligned}$$

and

$$P((q,w),(z_h,t_h))=max\{d(q,z_h),d(w,t_h),d(R(q,w),q),d(R(q,w),z_h),d(R(z_h,t_h),q)\}=0.$$

Therefore

$$\begin{aligned} \psi(d(q,z_{h+1}))&\leq\psi(max\{d(q,z_h),d(w,t_h)\})-\phi(max\{d(q,z_h),d(w,t_h)\})\\ &\leq\psi(max\{d(q,z_h),d(w,t_h)\}), \end{aligned}\tag{2.33}$$

and analogously

$$\psi(d(w,t_{h+1}))\leq\psi(max\{d(w,t_h),d(q,z_h)\}).\tag{2.34}$$

From (2.33) and (2.34) and using the fact that ψ is non-decreasing, we have

$$\begin{aligned} \psi(max\{d(q,z_{h+1}),d(w,t_{h+1})\})&=max\{\psi(d(q,z_{h+1}),\psi(d(w,t_{h+1}))\}\\ &\leq\psi(max\{d(q,z_h),d(w,t_h)\})-\phi(max\{d(q,z_h),d(w,t_h)\}). \end{aligned}\tag{2.35}$$

This imples that

$$max\{d(q,z_{h+1}),d(w,t_{h+1})\}\leq max\{d(q,z_h),d(w,t_h)\},$$

and consequently the sequence $max\{d(q,z_{h+1}),d(w,t_{h+1})\}$ is non-increasing and nonnegative and so,

$$\lim_{h\to\infty}max\{d(q,z_{h+1}),d(w,t_{h+1})\}=x,\tag{2.36}$$

for certain $x\geq0$. Using (2.36) and letting $h\to\infty$ in (2.35), we have

$$\psi(x)\leq\psi(x)-\phi(x)\leq\psi(x),$$

and consequently $\psi(x) = 0$ and thus $x = 0$.
Finally, as

$$\lim_{h\to\infty} max\{d(q, z_{h+1}), d(w, t_{h+1})\} = 0. \tag{2.37}$$

This implies

$$\lim_{h\to\infty} d(q, z_{h+1}) = \lim_{h\to\infty} d(w, t_{h+1}) = 0. \tag{2.38}$$

Similarly

$$\lim_{h\to\infty} d(q', z_{h+1}) = \lim_{h\to\infty} d(w', t_{h+1}) = 0. \tag{2.39}$$

From (2.38) and (2.39), we have $q = q', w = w'$. The proof is complete.
□

Theorem 2.4. In addition to the hypotheses of Theorem 2.1(resp. Theorem 2.2), suppose that q_0 and w_0 in W are comparable, then $q = w$.

Proof . Suppose that $q_0 \le w_0$. We claim that

$$q_h \le w_h, \forall h \in N. \tag{2.40}$$

From the mixed monotone property of R, we have

$$q_1 = R(q_0, w_0) \le R(w_0, w_0) \le R(w_0, q_0) = w_1.$$

Assume that $q_h \le w_h$, for some h. Now,

$$q_{h+1} = R(q_h, w_h) \le R(w_h, w_h) \le R(w_h, q_h) = w_{h+1}.$$

Hence, this proves our claim.

Now, using (2.40) and the contractive condition, we get

$$\begin{aligned} \psi(d(q_{h+1}, w_{h+1})) = \psi(d(w_{h+1}, q_{h+1})) &= \psi(R(w_h, q_h), R(q_h, w_h)) \\ &\le \psi(U((w_h, q_h), (q_h, w_h))) - \phi(U((w_h, q_h), (q_h, w_h)) \\ &\quad + K\ (P((w_h, q_h), (q_h, w_h))) \end{aligned} \tag{2.41}$$

where

$$\begin{aligned} U(R(w_h, q_h), R(q_h, w_h)) &= max\{d(w_h, q_h), d(q_h, w_h), d(R(w_h, q_h), w_h), d(R(q_h, w_h), q_h)\} \\ &= max\{d(w_h, q_h), d(w_{h+1}, w_h), d(q_{h+1}, q_h)\}. \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} P(R(w_h, q_h), R(q_h, w_h)) &= max\{d(w_h, q_h), d(q_h, w_h), d(R(w_h, q_h), w_h), d(R(w_h, q_h), q_h), d(R(q_h, w_h), w_h)\} \\ &= max\{d(w_h, q_h), d(q_h, w_h), d(w_{h+1}, w_h), d(w_{h+1}, q_h), d(q_{h+1}, w_h)\}. \end{aligned} \tag{2.43}$$

Thus, $\lim_{h\to\infty} d(q_h, w_h) = r$ for certain $r \ge 0$.

Taking $h \to \infty$ in (2.41)(and hence (2.42) and (2.43)), and using continuity of ψ and ϕ, we obtain

$$\psi(r) \le \psi(r) - \phi(r),$$

and we get $r = 0$.

As $q_h \to q$ and $w_h \to w$ and $\lim_{h\to\infty} d(q_h, w_h) = 0$. We have $0 = \lim_{h\to\infty} d(q_h, w_h) = d(\lim_{h\to\infty} q_h, \lim_{h\to\infty} w_h) = d(q, w)$ and thus $q = w$.

This finishes the proof. □

Example 2.5. Presume $W = \mathbb{R}$ with usual metric and order. Define $R : W \times W \to W$ as $R(q, w) = \frac{1}{7}(q - 5w)$ for all $q, w \in W$.

Presume $\psi, \phi : [0, \infty) \to [0, \infty)$ is defined as $\psi(t) = t$ and $\phi(t) = \frac{3}{5}(t)$. Apparently, ψ, ϕ are altering distance functions.

Now, let $q \leq p$ and $w \geq v$. So, we obtain

$$\begin{aligned}
\psi(d(R(q,w), R(p,v))) = d(R(q,w), R(p,v)) &= |\frac{1}{7}(q - 5w) - \frac{1}{7}(p - 5v)| \\
&= \frac{1}{7}|(q - p) - 5(w - v)| \\
&= \frac{1}{7}[d(q,p) + 5d(w,v)] \\
&\leq \frac{2}{5} max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\} \\
&= max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\} \\
&\quad - \frac{3}{5} max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\} \\
&\leq \psi(max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\}) \\
&\quad - \phi(max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\}) \\
&\quad + K\ (min\{d(q,p), d(w,v), d(R(q,w), p), d(R(p,v), q)\})
\end{aligned}$$

Therefore, all of the presumptions of Theorem 2.1 are contented. Also, $(0, 0)$ is the coupled fixed point of R.

Theorem 2.6. Presuppose $(W, \leq, d)$ be a complete metric space. Presume a continuous mapping $R : W \times W \to W$ possesses the mixed monotone property on W

$$\psi(d(R(q,w), R(p,v))) \leq \psi(d(U((q,w), (p,v))) - \phi(d(U((q,w), (p,v))) \tag{2.44}$$

where

$$U((q,w), (p,v)) = max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\}$$

$\forall\, p, q, v, w \in W$ for $q \geq p$ and $w \leq v$, here ψ and ϕ are altering distance functions. Presuppose either R is continuous or W has the following properties

(a) if a non-decreasing sequence $\{q_h\} \to q$, then $q_h \leq q$, for each h,

(b) if a non-increasing sequence $\{w_h\} \to w$, then $w_h \geq w$, for each h.

Presuppose that there exists $q_0, w_0 \in W$ such that $q_0 \leq R(q_0, w_0)$ and $w_0 \geq R(w_0, q_0)$, then R has a coupled fixed point.

Corollary 2.7. Presume $(W, \leq, d)$ a complete metric space. Let $R : W \times W \to W$ is a continuous mapping on W having the mixed monotone property such that $\exists\, k \in [0, 1)$ satisfying

$$\psi(d(R(q,w), R(p,v))) \leq k \max\{d(q,p), d(w,v), d(R(q,w), q), d(R(p,v), p)\}$$

$\forall\, p, q, w, v \in W$ with $q \geq p$ and $w \leq v$. Suppose either R is continuous or W has the following properties

(a) if a non-decreasing sequence $\{q_h\} \to q$, then $q_h \leq q$, for each h,

(b) if a non-increasing sequence $\{w_h\} \to w$, then $w_h \geq w$, for each h.

If there exists $q_0, w_0 \in W$ such that $q_0 \leq R(q_0, w_0)$ and $w_0 \geq R(w_0, q_0)$, then R has a coupled fixed point.

Corollary 2.8. Let R satisfy the contractive condition of Theorems 2.6 and 2.2 except that assumption (2.44) is changed by the subsequent assumption. There exists a non-negative Lebesgue- integrable function μ on $\mathbb{R}_+$ such that $\int_0^{\varepsilon} \mu(t)dt > 0$, for every $\varepsilon > 0$ and that then, R has a coupled fixed point.

$$\int_0^{\psi(d(R(q,w),R(p,v)))} \leq \int_0^{\psi(U((q,w),(p,v)))} \mu(t)dt - \int_0^{\phi(U((q,w),(p,v)))} \mu(t)dt, \tag{2.45}$$

where

$$U((q,w),(p,v)) = max\{d(q,p), d(w,v), d(R(q,w),q), d(R(p,v),p)\}.$$

Proof . Consider the mapping $\psi : [0,\infty) \to [0,\infty)$ defined by

$$\Gamma = \int_0^q \mu(t)dt.$$

is an altering distance function.

Then (2.45) becomes

$$\Gamma(\psi(d(R(q,w),R(p,v)))) \leq \Gamma(\psi(U((q,w),(p,v)))) - \Gamma(\phi(U((q,w),(p,v)))),$$

where

$$U((q,w),(p,v)) = max\{d(q,p), d(w,v), d(R(q,w),q), d(R(p,v),p)\}.$$

Taking $\psi_1 = \Gamma o \psi$, $\phi_1 = \Gamma o \phi$ and applying Theorem 2.1 and Theorem 2.2, we obtain the result. □

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