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Best proximity point theorem in higher dimensions with an application

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Abstract

In this article, we introduce the notion of F_n-contractions $T : A^n \to B$ in standard metric spaces and explore the possibility of certain approximation results for these mappings. We prove the existence and uniqueness of n-tuple $(n \geq 2)$ best proximity points of F_n-contractions, not necessarily continuous, using the weak P-property in complete metric spaces. Additionally, suitable examples are presented to substantiate our main results. Moreover, we anticipate a fixed point result to prove the existence and uniqueness of the solution for a type of integral equations to elucidate our obtained theorems.

Keywords: F_n-contractions, best proximity points, P-property, weak P-property, n-tuple best proximity points.
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1 Introduction

The theory on the existence of a solution to the equation of the type $Tx = x$ has motivated mathematicians over the years due to its far-reaching applications to deal with many interesting problems that can be reformulated of ordinary differential equations, matrix equations and many others. Let A be a non-empty subset of a metric space (X, d) and $T : A \to X$ be a mapping. We know that the solutions of the equation $Tx = x$ are fixed points of T. Consequently, $T(A) \cap A \neq \emptyset$ is a necessary condition for the existence of a fixed point for the operator T. If this condition does not hold, then $d(x, Tx) > 0$ for any $x \in A$ and the mapping $T : A \to X$ does not have any fixed point. In fact, for given non-empty closed subsets A and B of a complete metric space (X, d), a contraction mapping $T : A \to B$, being non-self, does not necessarily has a fixed point. This gave birth to proximity theory. Let A, B be two non-empty subsets of X and let $T : A \to B$ be a non-self mapping. In this situation, it is quite natural to attempt to find an approximate solution x that is optimal in the sense that the distance $d(x, Tx)$ is minimum, i.e., $d(x, Tx) = d(A, B)$ is satisfied. We can say that it accomplishes the highest possible closeness between x and Tx. This x (if exists) is called a best proximity point for the mapping T. Best proximity point theory is regarded as the natural generalization of fixed point theory (when the mapping concerned is a non-self mapping). One of the most fascinating best approximation theorem is due to Ky Fan [6]. Over the years several authors [9, 10, 11] have extended

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this interesting theorem in different directions, sometimes by changing the restriction on mapping or by altering the structure of the space.

On the other hand, Bhaskar and Lakshmikantham [7] introduced the notion of mixed monotone mappings and proved some coupled fixed point theorems for mappings satisfying mixed monotone property in partially ordered metric spaces. The investigation of [7] was followed by similar investigations by several authors (for example see [1, 4, 12, 13, 14, 15, 16, 17, 18]). Of late, Ertürk et al. [5] introduced an idea of n-tuple fixed points for contractive type mappings in partially ordered metric spaces. Recently, Dey et al. [10] also proved a best proximity point theorem for general type F-contractions in a complete metric space. The concept of coupled best proximity points has been introduced by Sintunavarat et al. [17, 18] who investigated coupled best proximity theorem for cyclic contractions. For some more interesting findings in this direction, the readers are referred to [8, 20, 21].

The main objective of this article is to first introduce the notion of F_n-contractions ($n \geq 2$) and the notions of coupled, tripled, and inductively n-tuple best proximity points. Also we prove certain results for the existence and uniqueness of n-tuple best proximity points of these types of contraction mappings using weak P-property in a complete metric space. It is worthy to mention that, we deduce our claims without considering the continuity assumption of the concerning self-map. Moreover, we construct two examples to justify our results. To end with, as an application of the discussed results, we achieve a fixed point result of such type of contractions and execute it to solve a type of integral equations.

2 Preliminaries

In this section we recall some of the basic definitions and notations which are essential throughout the paper.

Definition 2.1. [7] Let A be a non-empty subset of a metric space X and $T : A \times A \to A$. A point $(x, x') \in A \times A$ is called a coupled fixed point of T if

$$x = T(x, x') \quad \text{and} \quad x' = T(x', x).$$

Definition 2.2. [5] Let A be a non-empty subset of a metric space X and $T : A^n \to A$. A point $(x^1, x^2, \cdots, x^n) \in A^n$ is called an n-tuple fixed point of T if

$$\begin{aligned} x^1 &= T(x^1, x^2, \cdots, x^n), \\ x^2 &= T(x^2, x^3, \cdots, x^1), \\ &\vdots \\ x^n &= T(x^n, x^1, \cdots, x^{n-1}). \end{aligned}$$

Definition 2.3. [19] Let (X, d) be a metric space and A and B be two non-empty subsets of X. Let $T : A \to B$ be any mapping. A point $x \in A$ is said to be a best proximity point of T if it satisfies the condition that

$$d(x, Tx) = d(A, B),$$

where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Note that if $B = A$, then the best proximity point reduces to usual fixed point of T.

Definition 2.4. [18] Let A and B be non-empty subsets of a metric space X and $T : A \times A \to B$. A point $(x, x') \in A \times A$ is called a coupled best proximity point of T if

$$d(x, T(x, x')) = d(x', T(x', x)) = d(A, B).$$

It is clear that if $A = B$, then coupled best proximity point reduces to coupled fixed point. Let A and B be two non-empty subsets of a metric space (X, d). We denote by A_0 and B_0 the following sets:

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\}$$

and

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In [9], Kirk et al. presented certain sufficient conditions for the sets A_0 and B_0 to be non-empty. Further, in [3], the authors introduced the notion of P-property which is given below.

Definition 2.5. [3] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

It is naturally true that for any non-empty subset A of (X, d), the pair (A, A) has the P-property.

Definition 2.6. [19] Let (A, B) be a pair of non-empty subsets of a metric space (X, d) with $A_0 \neq \emptyset$. Then the pair (A, B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1, y_1) = d(A, B) \text{ and } d(x_2, y_2) = d(A, B) \Rightarrow d(x_1, x_2) \leq d(y_1, y_2).$$

It is naturally true that if the pair (A, B) has the P-property, then it has the weak P-property. In [19], the authors gave an example to show that the converse is not true.

3 Main results

In this section we consider a new type of contraction mapping and we name this as F_n-contraction. Our main aim is to prove an n-tuple best proximity point theorem for F_n-contractions. Finally, we construct examples to support our results.

Definition 3.1. Let A and B be two non-empty closed subsets of a metric space X and $T : A^n \to B$ be a mapping from the product space A^n of A to B. Then T is said to be an F_n-contraction $(n \geq 2)$ if

$$d(T(x^1, x^2, \cdots, x^n), T(y^1, y^2, \cdots, y^n)) \leq \frac{k}{n}\{d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^n, y^n)\}$$

where $0 \leq k < 1$.

Example 3.2. Let us consider the metric space $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$ and let $A = [1, 3]$ and $B = [-1, 0]$. It is clear that $d(A, B) = 1$. Define $T : A \times A \times A \to B$ as

$$T(x, y, z) = \frac{-x - y - z}{9}.$$

Now

$$\begin{aligned} d(T(x, y, z), T(x', y', z')) &= \left|\frac{-x - y - z}{9} - \frac{-x' - y' - z'}{9}\right| \\ &\leq \frac{|x - x'| + |y - y'| + |z - z'|}{9} \\ &\leq \frac{k}{3}[d(x, x') + d(y, y') + d(z, z')]. \end{aligned}$$

This implies that T is an F_3-contraction with $k \in [\frac{1}{3}, 1)$.

Definition 3.3. Let A and B be two non-empty subsets of a metric space X and $T : A^n \to B$. A point $(x^1, x^2, \cdots, x^n) \in A^n$ is called an n-tuple best proximity point of T if

$$\begin{aligned} d(x^1, T(x^1, x^2, \cdots, x^n)) &= d(A, B), \\ d(x^2, T(x^2, x^3, \cdots, x^1)) &= d(A, B), \\ &\vdots \\ d(x^n, T(x^n, x^1, \cdots, x^{n-1})) &= d(A, B). \end{aligned}$$

Before proceeding to the main theorem, we prove the following lemma.

Lemma 3.4. Let (A,B) be a pair of non-empty closed subsets of a complete metric space (X,d) such that (A,B) satisfies the weak P-property. Then any n-tuple $(n \geq 2)$ best proximity point of an F_n-contraction $T : A^n \to B$ is of the form $\overbrace{(x, x, \cdots, x)}$ for some $x \in A$.

Proof . Let $(x^1, x^2, \cdots, x^n)$ be a best proximity point of T for some $x^i \in A,\ i = 1, 2, \cdots, n$. Then by the definition of best proximity points

$$d(x^1, T(x^1, x^2, \cdots, x^n)) = d(A,B),$$
$$d(x^2, T(x^2, x^3, \cdots, x^1)) = d(A,B),$$
$$\vdots$$

and

$$d(x^n, T(x^n, x^1, \cdots, x^{n-1})) = d(A,B).$$

Now by the weak P-property of (A,B) and since T is an F_n-contraction, we have

$$\begin{aligned} d(x^1, x^2) &\leq d(T(x^1, x^2, \cdots, x^n), T(x^2, x^3, \cdots, x^1)) \\ &\leq \frac{k}{n}\{d(x^1, x^2) + d(x^2, x^3) + \cdots + d(x^n, x^1)\}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d(x^i, x^{i+1}) &\leq d(T(x^i, x^{i+1}, \cdots, x^{i-1}), T(x^{i+1}, x^{i+2}, \cdots, x^i)) \\ &\leq \frac{k}{n}\{d(x^i, x^{i+1}) + d(x^{i+1}, x^{i+2}) + \cdots + d(x^{i-1}, x^i)\}, \end{aligned}$$

$$\vdots$$

$$\begin{aligned} d(x^n, x^1) &\leq d(T(x^n, x^1, \cdots, x^{n-1}), T(x^1, x^2, \cdots, x^n)) \\ &\leq \frac{k}{n}\{d(x^n, x^1) + d(x^1, x^2) + \cdots + d(x^{n-1}, x^n)\}. \end{aligned}$$

Adding all the above inequalities we get,

$$d(x^1, x^2) + d(x^2, x^3) + \cdots + d(x^n, x^1) \leq k\{d(x^1, x^2) + d(x^2, x^3) + \cdots + d(x^n, x^1)\},$$

Since $0 \leq k < 1$, $d(x^1, x^2) + d(x^2, x^3) + \cdots + d(x^n, x^1) = 0$. This shows that $x^1 = x^2 = \cdots = x^n$. Therefore best proximity is of the form $\overbrace{(x, x, \cdots, x)}$. $\square$

Now we establish the n-tuple $(n \geq 2)$ best proximity point theorem.

Theorem 3.5. Let (A,B) be a pair of non-empty closed subsets of a complete metric space (X,d) such that A_0 is non-empty and $T : A^n \to B$ be a mapping. Assume that the following conditions are satisfied:

(i) $T(A_0^n) \subseteq B_0$;
(ii) the pair (A,B) satisfies the weak P-property;
(iii) T is an F_n-contraction.

Then T has a unique n-tuple best proximity point in A^n.

Proof . Since A_0^n is non-empty, we can choose $x_0^1, x_0^2, \cdots, x_0^n \in A_0$. Clearly by condition (i),

$$T(x_0^1, x_0^2, \cdots, x_0^n) \in T(A_0^n) \subseteq B_0.$$

Then from the construction of the set A_0, we can find $x_1^1, x_1^2, \cdots, x_1^n \in A_0$ such that

$$d(x_1^1, T(x_0^1, x_0^2, \cdots, x_0^n)) = d(A,B),$$

$$d(x_1^2, T(x_0^2, x_0^3, \cdots, x_0^1)) = d(A, B),$$
$$\vdots$$
$$d(x_1^n, T(x_0^n, x_0^1, \cdots, x_0^{n-1})) = d(A, B).$$

Similarly, we can find $x_2^1, x_2^2, \cdots, x_2^n \in A_0$ such that

$$d(x_2^1, T(x_1^1, x_1^2, \cdots, x_1^n)) = d(A, B),$$
$$d(x_2^2, T(x_1^2, x_1^3, \cdots, x_1^1)) = d(A, B),$$
$$\vdots$$
$$d(x_2^n, T(x_1^n, x_1^1, \cdots, x_1^{n-1})) = d(A, B).$$

Recursively, we obtain sequences $(x_p^1), (x_p^2), \cdots, (x_p^n)$ in A_0 satisfying

$$d(x_{p+1}^1, T(x_p^1, x_p^2, \cdots, x_p^n)) = d(A, B),$$
$$d(x_{p+1}^2, T(x_p^2, x_p^3, \cdots, x_p^1)) = d(A, B),$$
$$\vdots$$
$$d(x_{p+1}^n, T(x_p^n, x_p^1, \cdots, x_p^{n-1})) = d(A, B).$$

Since (A, B) has the weak P-property, we have

$$\begin{aligned} d(x_{p+1}^1, x_p^1) &\le d(T(x_p^1, x_p^2, \cdots, x_p^n), T(x_{p-1}^1, x_{p-1}^2, \cdots, x_{p-1}^n)) \\ &\le \frac{k}{n}\{d(x_p^1, x_{p-1}^1) + d(x_p^2, x_{p-1}^2) + \cdots + d(x_p^n, x_{p-1}^n)\} \end{aligned}$$

$$\begin{aligned} d(x_{p+1}^2, x_p^2) &\le d(T(x_p^2, x_p^3, \cdots, x_p^1), T(x_{p-1}^2, x_{p-1}^3, \cdots, x_{p-1}^1)) \\ &\le \frac{k}{n}\{d(x_p^2, x_{p-1}^2) + d(x_p^3, x_{p-1}^3) + \cdots + d(x_p^1, x_{p-1}^1)\} \end{aligned}$$

for any $p \in \mathbb{N}$ and so on. Again

$$\begin{aligned} d(x_{p+1}^n, x_p^n) &\le d(T(x_p^n, x_p^1, \cdots, x_p^{n-1}), T(x_{p-1}^n, x_{p-1}^1, \cdots, x_{p-1}^{n-1})) \\ &\le \frac{k}{n}\{d(x_p^n, x_{p-1}^n) + d(x_p^1, x_{p-1}^1) + \cdots + d(x_p^{n-1}, x_{p-1}^{n-1})\}. \end{aligned}$$

for any $p \in \mathbb{N}$. Therefore

$$\begin{aligned} d(x_{p+1}^1, x_p^1) + d(x_{p+1}^2, x_p^2) + \cdots + d(x_{p+1}^n, x_p^n) &\le k\{d(x_p^1, x_{p-1}^1) + d(x_p^2, x_{p-1}^2) + \cdots + \\ &\quad d(x_p^n, x_{p-1}^n)\} \\ &\le k^2\{d(x_{p-1}^1, x_{p-2}^1) + d(x_{p-1}^2, x_{p-2}^2) + \cdots + \\ &\quad d(x_{p-1}^n, x_{p-2}^n)\} \\ &\quad \vdots \\ &\le k^p\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\}. \end{aligned}$$

Now for $m > p \geq n_1$,

$$\begin{aligned} d(x_m^1, x_p^1) + d(x_m^2, x_p^2) + \cdots + d(x_m^n, x_p^n) \leq & \{d(x_m^1, x_{m-1}^1) + d(x_{m-1}^1, x_{m-2}^1) + \cdots + d(x_{p+1}^1, x_p^1)\} \\ & + \{d(x_m^2, x_{m-1}^2) + d(x_{m-1}^2, x_{m-2}^2) + \cdots + d(x_{p+1}^2, x_p^2)\} \\ & + \cdots \\ & + \{d(x_m^n, x_{m-1}^n) + d(x_{m-1}^n, x_{m-2}^n) + \cdots + d(x_{p+1}^n, x_p^n)\} \\ = & \{d(x_m^1, x_{m-1}^1) + d(x_m^2, x_{m-1}^2) + \cdots + d(x_m^n, x_{m-1}^n)\} + \\ & \{d(x_{m-1}^1, x_{m-2}^1) + d(x_{m-1}^2, x_{m-2}^2) + \cdots + d(x_{m-1}^n, x_{m-2}^n)\} \\ & + \cdots + \{d(x_{p+1}^1, x_p^1) + d(x_{p+1}^2, x_p^2) + \cdots + d(x_{p+1}^n, x_p^n)\} \\ \leq & k^{m-1}\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \\ & + k^{m-2}\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \\ & + \cdots + k^p\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \end{aligned}$$

$$\begin{aligned} = & \quad (k^{m-1} + k^{m-2} + \cdots + k^p)\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \\ \leq & \quad (k^p + k^{p+1} + \cdots)\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \\ = & \quad \left(\frac{k^p}{1-k}\right)\{d(x_1^1, x_0^1) + d(x_1^2, x_0^2) + \cdots + d(x_1^n, x_0^n)\} \to 0 \text{ for } m, p \to \infty. \end{aligned}$$

From the above, we conclude that $(x_p^1), (x_p^2), \cdots, (x_p^n)$ are Cauchy sequences in A. A being a closed subset of a complete metric space X, so A is also complete. Hence $(x_p^1), (x_p^2), \cdots, (x_p^n)$ are convergent and let

$$\lim_{p\to\infty} x_p^1 = x^1, \ \lim_{p\to\infty} x_p^2 = x^2, \ \cdots, \ \lim_{p\to\infty} x_p^n = x^n.$$

Now we claim that $\lim_{p\to\infty} T(x_p^1, x_p^2, \cdots, x_p^n) = T(x^1, x^2, \cdots, x^n)$. If possible, let $\lim_{p\to\infty} T(x_p^1, x_p^2, \cdots, x_p^n) \neq T(x^1, x^2, \cdots, x^n)$. Then we have,

$$d(T(x_p^1, x_p^2, \cdots, x_p^n), T(x^1, x^2, \cdots, x^n)) \leq \frac{k}{n}\{d(x_p^1, x^1) + d(x_p^2, x^2) + \cdots + d(x_p^n, x^n)\}$$

as $p \to \infty$, this implies

$$d(T(x_p^1, x_p^2, \cdots, x_p^n), T(x^1, x^2, \cdots, x^n)) \leq 0$$

which is a contradiction. Hence our claim is established. Now as $p \to \infty$, we get,

$$d(x_{p+1}^1, T(x_p^1, x_p^2, \cdots, x_p^n)) = d(x^1, T(x^1, x^2, \cdots, x^n)) = d(A, B),$$

$$d(x_{p+1}^2, T(x_p^2, x_p^3, \cdots, x_p^1)) = d(x^2, T(x^2, x^3, \cdots, x^1)) = d(A, B),$$

$$\vdots$$

$$d(x_{p+1}^n, T(x_p^n, x_p^1, \cdots, x_p^{n-1})) = d(x^n, T(x^n, x^1, \cdots, x^{n-1})) = d(A, B).$$

Now, by Lemma 3.4, we can say $\overbrace{(x, x, \cdots, x)}^{n \ in \ total}$ is the best proximity point of T in A^n. Next we prove the uniqueness of the best proximity point of T. Let $\overbrace{(y, y, \cdots, y)}^{n \ in \ total}$ be another best proximity point of T. Therefore,

$$d(x, T\overbrace{(x, x, \cdots, x)}^{n \ in \ total}) = d(A, B) \text{ and } d(y, T\overbrace{(y, y, \cdots, y)}^{n \ in \ total}) = d(A, B).$$

Then using the weak P-property, we have,

$$\begin{aligned} d(x, y) & \leq d(T\overbrace{(x, x, \cdots, x)}^{n \ in \ total}, \overbrace{T(y, y, \cdots, y)}^{n \ in \ total}) \\ & \leq \frac{k}{n}\{\overbrace{d(x, y) + d(x, y) + \cdots + d(x, y)}^{n \ in \ total}\}. \end{aligned}$$

Therefore,

$$d(x,y) \leq k\{d(x,y)\}.$$

As $0 \leq k < 1$, we arrive at a contradiction. This completes the proof of the theorem. □

Remark 3.6. Recently, Zhang et al. [21] came by some interesting findings on the existence and uniqueness of the n-tuple best proximity points of a generalized contraction in the backdrop of partially ordered metric spaces via continuity assumption and weak P-monotone property. However, in our obtained Theorem 3.5, we attained our desired result without the continuity assumption of the contraction and considering the weak P-property. Also, we have proved our claim via less number of hypotheses.

Finally we illustrate our best proximity result by the following examples.

Example 3.7. Let $X = \mathbb{R}^2$ and $d((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$ be a metric on X. Consider the sets $A = \{(0,y) : |y| \geq 1, y \in \mathbb{R}\}$ and $B = \{(x,0) : x \in \mathbb{R}\}$ which are closed in X. Clearly $d(A,B) = 1$, where $A_0 = \{(0,1),(0,-1)\}$ and $B_0 = \{(0,0)\}$. As

$$d((0,1),(0,0)) = d((0,-1),(0,0)) = 1,$$

so the pair (A,B) does not satisfy weak P-property. We define $T : A \times A \to B$ by $T((0,x),(0,y)) = (\frac{|x|-|y|}{4}, 0)$. Now

$$\begin{aligned}
d(T((0,x),(0,x')),T((0,y),(0,y'))) &= d\left(\left(\frac{|x|-|x'|}{4},0\right),\left(\frac{|y|-|y'|}{4},0\right)\right) \\
&= \max\left\{\left|\frac{(|x|-|x'|)-(|y|-|y'|)}{4}\right|, |0-0|\right\} \\
&= \left|\frac{(|x|-|x'|)-(|y|-|y'|)}{4}\right| \\
&= \left|\frac{(|x|-|y|)-(|x'|-|y'|)}{4}\right| \\
&\leq \left|\frac{(|x-y|)-(|x'-y'|)}{4}\right| \\
&\leq \left|\frac{|x-y|}{4}-\frac{|x'-y'|}{4}\right| \\
&\leq \left|\frac{|x-y|}{4}\right| + \left|\frac{|x'-y'|}{4}\right| \\
&\leq \frac{k}{2}\{d((0,x),(0,y)) + d((0,x'),(0,y'))\},
\end{aligned}$$

where $k \in [\frac{1}{2}, 1)$. So T is an F_2-contraction. Note that even the pair (A,B) does not satisfy weak P-property. Here we see that $((0,1),(0,-1))$ is a best proximity point of T. But in that case $(0,1) \neq (0,-1)$ and there does not exist any other best proximity point.

Example 3.8. Let $X = \mathbb{R}^2$ and $d((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$ be a metric on X. Consider the sets $A = \{(x,0) : x \in \mathbb{R}\}$ and $B = \{(0,x) : |x| \geq 1, x \in \mathbb{R}\}$ which are closed in X. Clearly $d(A,B) = 1$, where $A_0 = \{(0,0)\}$ and $B_0 = \{(0,1),(0,-1)\}$. As

$$d((0,0),(0,-1)) = d((0,0),(0,1)) = 1,$$

so the pair (A,B) satisfies weak P-property but does not satisfy the P-property. We define $T : A \times A \to B$ by

$T((x,0),(y,0)) = \left(0, \frac{4+|x-y|}{4}\right)$. Now

$$\begin{aligned}
d(T((x,0),(x,0')),T((y,0),(y,0'))) &= d\left(\left(0,\frac{4+|x-x'|}{4}\right),\left(0,\frac{4+|y-y'|}{4}\right)\right)\\
&= \max\left\{|0-0|,\left|\frac{(|x-x'|)-(|y-y'|)}{4}\right|\right\}\\
&= \left|\frac{(|x-x'|)-(|y-y'|)}{4}\right|\\
&\leq \left|\frac{|(x-x')-(y-y')|}{4}\right|\\
&\leq \left|\frac{(|x-y|)-(|x'-y'|)}{4}\right|\\
&= \left|\frac{|x-y|}{4}-\frac{|x'-y'|}{4}\right|\\
&\leq \left|\frac{|x-y|}{4}\right|+\left|\frac{|x'-y'|}{4}\right|\\
&\leq \frac{k}{2}\{d((x,0),(y,0))+d((x',0),(y',0))\},
\end{aligned}$$

where $k \in [\frac{1}{2},1)$. So T is an F_2-contraction and note that all the conditions of Theorem 3.5 are satisfied. Therefore there exists a unique best proximity point of T in A^2. We see that $((0,0),(0,0))$ is a best proximity point of T and there does not exist any other best proximity point.

Example 3.9. Let $X = \mathbb{R}$ endowed with the usual metric $d(x,y) = |x-y|$. Let $A = [-3,-2] \cup [2,3]$ and $B = [-1,1]$ which are two closed subsets of $\mathbb{R}$. Define $T : A \times A \to B$ such that

$$T(x,x') = \begin{cases} \frac{|x-x'|-4}{4} & \text{if x, y are of same sign;}\\ \frac{||x|-|x'||-4}{4} & \text{if x, y are of opposite sign.}\end{cases}$$

Observe that $d(A,B) = 1$, where $A_0 = \{-2,2\}$ and $B_0 = \{-1,1\}$. Now $T(-2,-2) = T(2,2) = T(-2,2) = T(2,-2) = -1$, which imply

$$T(A_0 \times A_0) \subseteq B_0.$$

Also the pair (A,B) satisfies the weak P-property, since $d(-2,-1) = 1 = d(A,B) = d(1,2)$ implies $d(-2,1) = d(-1,2)$. Finally note that

$$d(T(x,x'),T(y,y')) \leq \frac{k}{2}\{d(x,y)+d(x',y')\},$$

where $k \in [\frac{1}{2},1)$. Therefore all the conditions of the Theorem 3.5 are satisfied. Hence there exists a best proximity point of T in A^2. Obviously $(-2,-2)$ is a best proximity point of T. Also, it is clear that there does not exist any other best proximity point of T.

4 Applications on fixed point

In this section, we state a fixed point result which is direct outcome of one of the results presented in the main section of this article. From Theorem 3.5, we derive the following theorem.

Theorem 4.1. Let (X,d) be a complete metric space and $T : \overbrace{X \times X \times \cdots \times X}^{n\ in\ total} \to X$ be a mapping such that the following condition holds:

$$d(T(x^1,x^2,\cdots,x^n),T(y^1,y^2,\cdots,y^n)) \leq \frac{k}{n}\{d(x^1,y^1)+d(x^2,y^2)+\cdots+d(x^n,y^n)\},$$

where $0 \leq k < 1$. Then T has a unique n-tuple fixed point.

Example 4.2. Let us consider a function $T:\mathbb{R}^2\to\mathbb{R}$ defined as $T(x,y)=\frac{x+y}{3}$ with the usual metric $d(x,y)=|x-y|$ on $\mathbb{R}$. Then

$$\begin{aligned} d(T(x_1,x_2),T(y_1,y_2)) &= d\Big(\frac{x_1+x_2}{3},\frac{y_1+y_2}{3}\Big)\\ &= \Big|\frac{x_1+x_2}{3}-\frac{y_1+y_2}{3}\Big|\\ &= \Big|\frac{x_1-y_1}{3}+\frac{x_2-y_2}{3}\Big|\\ &\le \frac{k}{2}\{d(x_1,y_1)+d(x_2,y_2)\}, \end{aligned}$$

where $\frac{2}{3}\le k<1$. Clearly $(0,0)$ is a coupled fixed point of T and it is unique.

Here we apply our fixed point result to study the existence and uniqueness of the solution to a type of integral equations. Let $a,b>0$ and $E=[0,a]\times[0,b]$. Moreover, we take $X=L^\infty(E)$. Suppose the mapping $d:X\times X\to[0,\infty)$ is defined by,

$$d(w_1,w_2)=\sup_{(x,y)\in E}|w_1(x,y)-w_2(x,y)|, \tag{4.1}$$

where $w_1,w_2\in X$ and $(x,y)\in E$. Then (X,d) is a complete metric space.

Theorem 4.3. Consider the integral equation

$$u(x,y)=h(x,y)+\iint_E\Big[K_1(x,y,\tau,s)+K_2(x,y,\tau,s)\Big]\Big(f(\tau,s,u(\tau,s))+g(\tau,s,v(\tau,s)\Big)d\tau ds, \tag{4.2}$$

where $a,b>0$ and $E=[0,a]\times[0,b]$. Suppose that

(i) $K_1,K_2:E\times E\to\mathbb{R}$ and $h\in L^\infty(E)$;

(ii) assume that there exists $\lambda_1,\lambda_2>0$ and $0\le k<1$ such that

$$\sup_{(x,y)\in E}\Big|\iint_E K_1(x,y,\tau,s)d\tau ds\Big|\le\frac{k}{2(\lambda_1+\lambda_2)}$$

and

$$\sup_{(x,y)\in E}\Big|\iint_E K_2(x,y,\tau,s)d\tau ds\Big|\le\frac{k}{2(\lambda_1+\lambda_2)};$$

(iii) $f,g:E\times\mathbb{R}\to\mathbb{R}$, such that

$$|f(t,s,u)-f(t,s,v)|\le\lambda_1|u-v|$$

and

$$|g(t,s,u)-g(t,s,v)|\le\lambda_2|u-v|,$$

for $(t,s)\in E$ and $u,v\in\mathbb{R}$.

Then there exists a unique solution of the integral equation (4.2) in $L^\infty(E)$.

Proof . Let $X=L^\infty(E)$. Then (X,d) is a complete metric space with metric defined in (4.1). Define $T:X\times X\to X$ by

$$\begin{aligned}(T(u,v))(x,y)=h(x,y) &+ \iint_E K_1(x,y,\tau,s)\Big(f(\tau,s,u(\tau,s))+g(\tau,s,v(\tau,s))\Big)d\tau ds\\ &+ \iint_E K_2(x,y,\tau,s)\Big(f(\tau,s,v(\tau,s))+g(\tau,s,u(\tau,s))\Big)d\tau ds\end{aligned}$$

for all $x, y \in E$ and $u, v \in X$. Now for $u_1, u_2, v_1, v_2 \in X$,

$$\begin{aligned}
d(T(u_1,v_1),T(u_2,v_2)) &= \sup_{(x,y)\in E} |T(u_1,v_1)(x,y) - T(u_2,v_2)(x,y)| \\
&= \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)\Big(f(\tau,s,u_1(\tau,s)) + g(\tau,s,v_1(\tau,s))\Big)d\tau ds \\
&+ \iint_E K_2(x,y,\tau,s)\Big(f(\tau,s,v_1(\tau,s)) + g(\tau,s,u_1(\tau,s))\Big)d\tau ds \\
&- \iint_E K_1(x,y,\tau,s)\Big(f(\tau,s,u_2(\tau,s)) + g(\tau,s,v_2(\tau,s))\Big)d\tau ds \\
&- \iint_E K_2(x,y,\tau,s)\Big(f(\tau,s,v_2(\tau,s)) + g(\tau,s,u_2(\tau,s))\Big)d\tau ds\Big| \\
&= \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)\Big[\Big(f(\tau,s,u_1(\tau,s)) - f(\tau,s,u_2(\tau,s))\Big) \\
&+ \Big(g(\tau,s,v_1(\tau,s)) - g(\tau,s,v_2(\tau,s))\Big)\Big]d\tau ds \\
&+ \iint_E K_2(x,y,\tau,s)\Big[\Big(f(\tau,s,v_1(\tau,s)) - f(\tau,s,v_2(\tau,s))\Big) \\
&+ \Big(g(\tau,s,u_1(\tau,s)) - g(\tau,s,u_2(\tau,s))\Big)\Big]d\tau ds\Big| \\
&\le \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)\Big[\Big(f(\tau,s,u_1(\tau,s)) - f(\tau,s,u_2(\tau,s))\Big) \\
&+ \Big(g(\tau,s,v_1(\tau,s)) - g(\tau,s,v_2(\tau,s))\Big)\Big]d\tau ds\Big| \\
&+ \sup_{(x,y)\in E} \Big| \iint_E K_2(x,y,\tau,s)\Big[\Big(f(\tau,s,v_1(\tau,s)) - f(\tau,s,v_2(\tau,s))\Big) \\
&+ \Big(g(\tau,s,u_1(\tau,s)) - g(\tau,s,u_2(\tau,s))\Big)\Big]d\tau ds\Big| \\
&\le \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)\Big[\lambda_1\Big(u_1(\tau,s) - u_2(\tau,s)\Big) + \\
&\quad \lambda_2\Big(v_1(\tau,s) - v_2(\tau,s)\Big)\Big]d\tau ds\Big| + \sup_{(x,y)\in E} \Big| \iint_E K_2(x,y,\tau,s) \\
&\quad \Big[\lambda_1\Big(v_1(\tau,s) - v_2(\tau,s)\Big) + \lambda_2\Big(u_1(\tau,s) - u_2(\tau,s)\Big)\Big]d\tau ds\Big| \\
&= \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)\Big[\lambda_1 d(u_1,u_2) + \lambda_2 d(v_1,v_2)\Big]d\tau ds\Big| \\
&+ \sup_{(x,y)\in E} \Big| \iint_E K_2(x,y,\tau,s)\Big[\lambda_1 d(v_1,v_2) + \lambda_2 d(u_1,u_2)\Big]d\tau ds\Big|
\end{aligned}$$

$$\begin{aligned}
&\le \sup_{(x,y)\in E} \Big| \iint_E K_1(x,y,\tau,s)d\tau ds\Big| \Big[\lambda_1 d(u_1,u_2) + \lambda_2 d(v_1,v_2)\Big] \\
&+ \sup_{(x,y)\in E} \Big| \iint_E K_2(x,y,\tau,s)d\tau ds\Big| \Big[\lambda_1 d(v_1,v_2) + \lambda_2 d(u_1,u_2)\Big] \\
&\le \frac{k}{2(\lambda_1+\lambda_2)}\Big[\lambda_1 d(u_1,u_2) + \lambda_2 d(v_1,v_2)\Big] + \frac{k}{2(\lambda_1+\lambda_2)}\Big[\lambda_1 d(v_1,v_2) + \lambda_2 d(u_1,u_2)\Big] \\
&= \frac{k}{2(\lambda_1+\lambda_2)}\Big[(\lambda_1+\lambda_2)d(u_1,u_2) + (\lambda_1+\lambda_2)d(v_1,v_2)\Big] \\
&= \frac{k}{2}[d(u_1,u_2) + d(v_1,v_2)].
\end{aligned}$$

Then by Theorem 4.1, $u(x,y)$ has a unique solution in $L^\infty(E)$. □

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