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Strong bi-ideals in bi-near subtraction semigroup

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Abstract

We study certain things connecting strong bi-ideals of bi-near subtraction semigroup in a sub commutative S-bi-near subtraction semigroup. A bi-ideals B of X is strong bi-ideals if $X_1B_1^2 \subseteq B_1$ (or $X_2B_2^2 \subseteq B_2$) denotes $XB^2 \subseteq B$.

Keywords: S-bi-near subtraction semigroup, sub commutative, bi-ideals.
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1 Introduction

This paper belongs to a bi-near subtraction semigroup, simply we can say zero-symmetric right bi-near subtraction semigroup. We can refer Pilz [4] for basic definition. Zelinka [7] conversed a problem offered by schein regarding the structure of multiplication in subtraction semigroup. And also he solved the complications of subtraction algebras. Jun et al. [3] presented the view of ideas in subtraction algebras and recommended characterization of ideas. In Jun and Kim recognized the ideal generated by a set and conferred many related results. Inspired by the study of strong bi-ideals in "A study on regularities in near rings" by S. Jayalakshmi and T. Tamizh Chelvam [1] and also inspired by the study of bi-ideals in "Bi-ideals of near subtraction semigroup" by S. Maharasi, V. Mahalakshmi, S. Jayalakshmi. With this in mind, primarily we study certain properties connecting strong bi-ideals of a bi-near subtraction semigroup in a sub commutative S-bi-near subtraction semigroup. More it shown the concept of strong bi-ideals and bi-ideals are equal in sub commutative S-bi-near subtraction semigroup. We characterized strong bi-ideals by a bi-sub algebra B of X. In case of zero-symmetric and sub commutative S-bi-near subtraction semigroup, we recognized essential and necessary condition for strong bi-ideals to be bi-ideals in bi-near subtraction semigroup X.

2 Preliminaries

Definition 2.1. Let X is nonempty set, then the operation "-" is subtraction algebra, if the condition holds.

- $u(v - u) = u$
- $u - (u - v) = v - (v - u)$
- $(u - v) - w = (u - w) - v$ for every $u, v, w \in X$.

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Definition 2.2. Let X is nonempty set, then the operation "-"and "$\cdot$"is near subtraction semigroup (right), if the condition holds.

- $(X, -)$ is subtraction algebra.
- $(X, \cdot)$ is semigroup.
- $(x - y)z = xz - yz$ for every $x, y, z \in X$.

Definition 2.3. Let X is nonempty set, then the operation "-"and "$\cdot$"is subtraction semigroup, if the condition holds.

- $(X, -)$ is subtraction algebra.
- $(X, \cdot)$ is semigroup.
- $a(b - c) = ab - bc$ for every $a, b, c \in X$.

Definition 2.4. Let $(X, -, \cdot)$ be nonempty set. We call X to be bi-near subtraction semigroup, if $X = X_1 \cup X_2$

- $(X_1, -, \cdot)$ is near subtraction group (right).
- $(X_2, -, \cdot)$ is subtraction semigroup.

Definition 2.5. Let S be nonempty subset. Then a near subtraction semigroup is sub algebra of X, if $u - v \in S$, for every $u, v \in S$.

Definition 2.6. Let S be nonempty subset. Then S be bi-sub algebra of $(X, -)$ if

- $S = S_1 \cup S_2$.
- $(S_1, -)$ is near sub algebra of $(X_1, -)$.
- $(S_2, -)$ is sub algebra of $(X_2, -)$.

Definition 2.7. Let each element of X is bi-regular. Then for all $a \in X_1 \cup X_2$, there exists $b \in X_1 \cup X_2$, such that $a = aba$.

Definition 2.8. Let X is strongly bi-regular if every element of X is strongly regular. For each $a \in X_1 \cup X_2$, there exists $b \in X_1 \cup X_2$, such that $a = ba^2$.

Definition 2.9. Let X is sub commutative, if $Xa = aX$ for all $a \in X_1 \cup X_2$.

Definition 2.10. Let every element of X is left bi-potent, i.e., $Xa = aX$ for all $a \in X_1 \cup X_2$.

Definition 2.11. A S-bi-near subtraction semigroup of X, if $a \in Xa$ for all $a \in X_1 \cup X_2$.

Definition 2.12. A bi-sub algebra B of $(X, -)$ is bi-ideals of X. If B_1 s sub algebra of $(X_1, -)$ is bi-ideal of X_1 or if B_2 is sub algebra of $(X_2, -)$ is bi-ideal of X_2. (i.e.,) $B_1X_1B_1 \subseteq B_1$(or $B_2X_2B_2 \subseteq B_2$) which implies $BXB \subseteq B$.

Result 2.13. Let X be bi-near subtraction semi group then Xx is bi-ideals of X for every $x \in X_1 \cup X_2$.

Definition 2.14. A sub algebra Q of $(X, -)$ is quasi-ideal of X if $QX \cap XQ \cap X^*Q \subset Q$. In case of zero-symmetric, a bi-sub algebra Q of $(X, -)$ is s quasi-ideal of X if and only if $QX \cap XQ \subseteq Q$.

Result 2.15. Every quasi-ideal Q of bi-near subtraction semigroup X is also a bi-ideal B of bi-near subtraction semigroup X.

Proof . By the definition of quasi-ideal $QX \cap XQ \subseteq Q$. It is enough to prove Q is bi-ideal of X. (i.e.,) to prove $QXQ \subseteq Q$. Let $x \in QXQ$ then $x = qx_1q \in QX$ and $x = qx_1q \in XQ$. Thus $x \in QX \cap XQ \subseteq Q$. Hence $x \in QXQ$ implies $x \in Q$ which gives $QXQ \subseteq Q$. Therefore Q is bi-ideal of X. □

Lemma 2.16. A $\bar{S}$-b-near subtraction semigroup X with property (α) is bi-regular if and only if $B = BXB$ for every bi-ideals B of X.

Definition 2.17. If X is $S_k(S'_k)$ bi-near subtraction semigroup then X is said to be $\bar{S}$-bi-near subtraction semigroup if $a \in Xa^k$ $(a \in a^kX)$ for all $a \in X_1 \cup X_2$ for any positive integer k.

Remark 2.18. Every $S_k(S'_k)$-bi-near subtraction semigroup is $S(S')$-bi-near subtraction semigroup. i.e., every $\bar{S}$-bi-near subtraction semigroup is bi-near subtraction semigroup.

Proof . Let X be $\bar{S}$-bi-near subtraction semigroup for $k = 2$. Then for all $x \in X_1 \cup X_2$ and $x \in Xx^k \subseteq Xx$. Hence X is $\bar{S}$-bi-near subtraction semigroup. □

Definition 2.19. If X is $P_k(P'_k)$-bi-near subtraction semigroup there exist positive integer k such that $x^kX = xXx(Xx^k = xXx)$, for all $x \in X_1 \cup X_2$.

Theorem 2.20. The following subdivisions are equivalent.

(i) A bi-near subtraction semigroup of X is GNF.
(ii) A bi-near subtraction semigroup of X is bi-regular and each idempotent is central.
(iii) A bi-near subtraction semigroup of X is bi-regular and sub commutative.

Theorem 2.21. Let X be sub commutative $\bar{S}$-bi-near subtraction semigroup. The following conditions are equivalent.

(i) $B = X^mXB^n$ for each generalized (m,n) bi-ideals B of X.
(ii) X is regular.
(iii) X is strongly bi-regular.
(iv) X is left bi-potent.
(v) $aXa = Xa = Xa^2$, for all $a \in X_1 \cup X_2$.
(vi) $B = BXB$, for each bi-ideals B of X.

3 Strong Bi-ideals

In this section, first we study certain properties of Strong bi-ideals in a left permutable S-bi-near subtraction semigroup. A bi-near subtraction semigroup of X is said to be left permutable if $xyz = yxz$ for all $x, y, z \in X_1 \cup X_2$.

Definition 3.1. A bi-ideals B of $(X, -)$ is strong bi-ideals of a bi-near subtraction semigroup, if B_1 is sub algebra of $(X_1, -)$ is strong bi-ideal of X_1 or if B_2 is sub algebra of $(X_2, -)$ is strong bi-ideal of X_2. (i.e.,) $X_1B_1^2 \subseteq B_1$(or $X_2B_2^2 \subseteq B_2$) implies $XB^2 \subseteq B$.

Example 3.2. Let X be non-empty set. Where X_1 is proper subset in X. Then $X_1 = \{0, x, y, z\}$ be near subtraction semigroup (right) took from Klein's four group scheme (0, 7, 0, 9) defined as follows.

Table 1: Strong bi-ideals in near subtraction semigroup

−	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	y	x	0

·	0	x	y	z
0	0	0	0	0
x	0	x	0	z
y	0	0	0	0
z	0	x	0	z

Consider $B_1 = \{0, z\}$ be non empty subset. To prove B_1 is sub algebra of X_1. For every $0, z \in B_1 \Rightarrow 0 - z = 0 \in B_1$ & $z - 0 = z \in B_1$. Therefore, $(B_1, -)$ is sub algebra of $(X_1, -)$. Now to prove B_1 is a strong bi-ideal of X_1. Then $X_1B_1^2 \subseteq B_1 \Rightarrow X_1\{0, z\}^2 \subseteq \{0, z\} \Rightarrow \{0, x, y, z\}\{0, z\} \subseteq \{0, z\} \Rightarrow \{0, z\} \subseteq \{0, z\}$.

Or X_2 is proper subset in X. Then $X_2 = \{0, a, b, c\}$ be subtraction semigroup defined on Klein's four group scheme (0, 7, 13, 9) (P. 408, Pilz [29]) defined as follows.

Table 2: Strong bi-ideals in subtraction semigroup

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	0	0	0
c	0	a	b	c

Consider $B_2 = \{0, a\}$ be nonempty subset. To prove B_2 is sub algebra of X_2. For every $0, a \in B_2 \Rightarrow 0 - a = 0 \in B_2$ & $a - 0 = a \in B_2$. Therefore, $(B_2, -)$ is sub algebra of $(X_2, -)$. Now to prove B_2 is a strong bi-ideal of X_2. Then $X_2B_2^2 \subseteq B_2 \Rightarrow X_2\{0, a\}^2 \subseteq \{0, a\} \Rightarrow \{0, a, b, c\}\{0, a\} \subseteq \{0, a\} \Rightarrow \{0, a\} \subseteq \{0, a\}$. Therefore X is a strong bi-ideals of a bi-near subtraction semigroup.

Remark 3.3. Every strong bi-ideals of X is also a bi-ideals of X.

Example 3.4. Let X is non-empty set. Where X_1 is proper subset in X. Then $X_1 = \{0, x, y, z\}$ be near subtraction semigroup (right) took from Klein's four group scheme (0, 7, 0, 9) defined as follows

Table 3: Strong bi-ideals as well as bi-ideals in near subtraction semigroup

$-$	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	y	x	0

$\cdot$	0	x	y	z
0	0	0	0	0
x	0	x	0	z
y	0	0	0	0
z	0	x	0	z

Consider $B_1 = \{0, z\}$ be non empty subset. To prove B_1 is sub algebra of X_1. For every $0, z \in B_1 \Rightarrow 0 - z = 0 \in B_1$ & $z - 0 = z \in B_1$. Therefore, $(B_1, -)$ is sub algebra of $(X_1, -)$. Now to prove B_1 is a strong bi-ideal of X_1. Then $X_1B_1^2 \subseteq B_1 \Rightarrow X_1\{0, z\}^2 \subseteq \{0, z\} \Rightarrow \{0, x, y, z\}\{0, z\} \subseteq \{0, z\} \Rightarrow \{0, z\} \subseteq \{0, z\}$. Also to prove B_1 is a bi-ideal of X_1. Then $B_1X_1B_1 \subseteq B_1 \Rightarrow \{0, z\}X_1\{0, z\} \Rightarrow \{0, z\}\{0, x, y, z\}\{0, z\} \Rightarrow \{0, z\} \subseteq \{0, z\}$.

Or X_2 is proper subset in X. Then $X_2 = \{0, a, b, c\}$ be subtraction semigroup defined on Klein's four group scheme (0, 7, 11, 1) defined as follows:

Table 4: Strong bi-ideals as well as bi-ideals in subtraction semigroup

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Consider $B_2 = \{0, a\}$ be nonempty subset. To prove B_2 is sub algebra of X_2. For every $0, a \in B_2 \Rightarrow 0 - a = 0 \in B_2$ & $a - 0 = a \in B_2$. Therefore, $(B_2, -)$ is sub algebra of $(X_2, -)$. Now to prove B_2 is a strong bi-ideal of X_2. Then $X_2B_2^2 \subseteq B_2 \Rightarrow X_2\{0, a\}^2 \subseteq \{0, a\} \Rightarrow \{0, a, b, c\}\{0, a\} \subseteq \{0, a\} \Rightarrow \{0, a\} \subseteq \{0, a\}$. Also to prove B_2 is a bi-ideal of X_2. Then $B_2X_2B_2 \subseteq B_2 \Rightarrow \{0, a\}X_2\{0, a\} \Rightarrow \{0, a\}\{0, a, b, c\}\{0, a\} \Rightarrow \{0, a\} \subseteq \{0, a\}$.

Clearly every strong bi-ideals of bi-near subtraction semigroup X is also a strong bi-ideals of X.

Remark 3.5. Every bi-ideals of bi-near subtraction semigroup X is different from strong bi-ideals of bi-near subtraction semigroup X.

Example 3.6. Let X is nonempty set. Where X_1 is proper subset in X. Then $X_1 = \{0, x, y, z\}$ be near subtraction semi group(right) defined on Klein's four group scheme (0, 1, 1, 1) defined as follows:

Table 5: Bi-ideals is different from strong bi-ideals in near subtraction semigroup

−	0	x	y	z
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
z	z	y	x	0

·	0	x	y	z
0	0	0	0	0
x	0	x	x	x
y	0	y	y	y
z	0	z	z	z

Consider $B_1 = \{0, y\}$ be nonempty subset. To prove B_1 is sub algebra of X_1. For every $0.y \in B_1 \Rightarrow 0 - y = 0 \in B_1 \& y - 0 = y \in B_1$. Therefore, $(B_1, -)$ is sub algebra of $(X_1, -)$. Now, to prove B_1 is a bi-ideal of X_1. Then $B_1X_1B_1 \subseteq B_1 \Rightarrow \{0, y\}X_1\{0, y\} \subseteq \{0, y\} \Rightarrow \{0, y\}\{0, x, y, z\}\{0, y\} \subseteq \{0, y\} \Rightarrow \{0, y\} \subseteq \{0, y\}$. Also to prove B_1 is different from strong bi-ideal of X_1. Then $X_1B_1^2 \not\subseteq B_1 \Rightarrow X_1\{0, y\}^2 \not\subseteq \{0, y\} \Rightarrow \{0, x, y, z\}\{0, y\} \not\subseteq \{0, y\} \Rightarrow \{0, x, y, z\} \not\subset \{0, y\}$.

Or, X_2 is proper subset in X. Then $X_2 = \{0, a, b, c\}$ be subtraction semi group defined on Klein's four group scheme (0, 0, 1, 1) defined as follows:

Table 6: Bi-ideals is different from Strong bi-ideals in subtraction semigroup

−	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

·	0	a	b	c
0	0	0	0	0
a	0	0	a	a
b	0	0	b	b
c	0	0	c	c

Consider $B_2 = \{0, b\}$ be nonempty subset. To prove B_2 is sub algebra of X_2. For every $0.b \in B_2 \Rightarrow 0 - b = 0 \in B_2 \& b - 0 = b \in B_2$. Therefore, $(B_2, -)$ is sub algebra of $(X_2, -)$. Now, to prove B_2 is a bi-ideal of X_2. Then $B_2X_2B_2 \subseteq B_2 \Rightarrow \{0, b\}X_2\{0, b\} \subseteq \{0, b\} \Rightarrow \{0, b\}\{0, a, b, c\}\{0, b\} \subseteq \{0, b\} \Rightarrow \{0, b\} \subseteq \{0, b\}$. Also to prove B_2 is different from strong bi-ideal of X_2. Then $X_2B_2^2 \not\subseteq B_2 \Rightarrow X_2\{0, b\}^2 \not\subseteq \{0, b\} \Rightarrow \{0, a, b, c\}\{0, b\} \not\subseteq \{0, b\} \Rightarrow \{0, a, b, c\} \not\subset \{0, b\}$. One can verify that $\{0, b\}$ is a bi-ideals but not a strong bi-ideals of bi-near subtraction semigroup X.

Result 3.7. Let X be a bi-near subtraction semigroup, then Xx is strong bi-ideals of X for every $x \in X_1 \cup X_2$.

Proof . Let X be bi-near subtraction semigroup. By Result 2.13, then Xx is strong bi-ideal of X. Let us consider $X(Xx)^2 = X(Xx)(Xx) = XxXx \subseteq Xx$. The above gives Xx is strong bi-ideals of X. □

Proposition 3.8. A set of all strong bi-ideals of X form a moore system on X.

Proof . Let $\{B_i\}_{i\in I}$ be set of all strong bi-ideals of X. To prove $B = \bigcap_{i\in I} B_i$ is strong bi-ideals of X. Now to prove $XB^2 \subseteq B$. Let $\{B_i\}_{i\in I}$ be strong bi-ideals of X if $XB_i^2 \subseteq B_i$ implies $X_1B_i^2 \subseteq B_i$ (or $X_2B_i^2 \subseteq B_i$). Clearly $B \subseteq B_i$. We can say as $B^2 \subseteq B_i^2$. Then $X_1B_1^2 \subseteq X_1B_i^2 \subseteq B_i \subseteq B$(or $X_2B_2^2 \subseteq X_2B_i^2 \subseteq B_i \subseteq B$) implies $X_1B_1^2 \subseteq B_i \subseteq B$ (or $X_2B_2^2 \subseteq B_i \subseteq B$) which implies $XB_i^2 \subseteq B_i \subseteq B \Rightarrow XB^2 \subseteq B$. Hence $B = \bigcap_{i\in I} B_i$ is a strong bi-ideals of X. □

Proposition 3.9. Let B be strong bi-ideals of X and S is sub bi-near subtraction semigroup of X, then $B \cap S$ is strong b-ideals of S.

Proof . Let B is strong bi-ideals of X implies $XB^2 \subseteq B$ and S is sub bi-near subtraction semigroup of X implies $SS \subseteq S$. To prove $C = B \cap S$ is strong bi-ideals of S. Now to prove $SC^2 \subseteq C$. Now $SC^2 = SCC = S(B \cap S)(B \cap S) = S(BB \cap SS) \subseteq SB^2 \cap SS^2 \subseteq SB^2 \cap S$. (Since S is sub bi-near)$\in XB^2 \cap S \subseteq B \cap S = C$. (i.e.,) $SC^2 \subseteq C$. Hence $B \cap S$ is strong bi-ideals of S. □

Proposition 3.10. Let B be a Strong bi-ideals of X. If elements of B are strongly regular, then B is a quasi-ideal of X.

Proof . Let B be a Strong bi-ideals of X if $X_1B_1^2 \subseteq B_1$ (or $X_2B_2^2 \subseteq B_2$) implies $XB^2 \subseteq B$. To prove B is a quasi-ideal of X. (i.e) To prove $B_1X_1 \cap X_1B_1 \subseteq B_1$ (or $B_2X_2 \cap X_2B_2 \subseteq B_2$). If $x \in B_1X_1 \cap X_1B_1$, then $x = b_1x_1 = m_1b_2$, for some $b_1, b_2 \in B$ and $n_1, m_1 \in X_1$. Since the elements of B are strongly regular, $b_1 = c_1b_1^2$ and $b_2 = c_2b_2^2$, for some $c_1, c_2 \in B$. Hence $x = b_1x_1 = (c_1b_1^2)x_1 = (c_1(b_1.b_1))x_1 = (c_1b_1)(b_1x_1) = (c_1b_1)(m_1b_2) = (c_1b_1)(m_1c_2b_2^2) = (c_1b_1m_1c_2)b_2^2 \in X_1B_1^2 \subseteq B_1$ (i.e) $B_1X_1 \cap X_1B_1 \subseteq B_1$. Similarly $x \in B_2X_2 \cap X_2B_2$, we can prove that $B_2X_2 \cap X_2B_2 \subseteq B_2$. Hence B is a quasi-ideal of X. □

Proposition 3.11. Let X be sub commutative. If B is a Strong bi-ideals of X then xB and Bx' are strong bi-ideals of bi-near subtraction semigroup X, where $x, x' \in X$ and x is distributive.

Proof . To prove xB is a Strong bi-ideals of X. First to prove xB is a bi sub algebra of X. (i.e.,) to prove $X_1(xB_1)^2$(or $X_2(xB_2)^2$) $\subseteq X(xB) \subseteq xB$. Let $n \in X(xB)^2$ then $n = x_1xbxb'$ where $x \in X_1 \cup X_2$ and $b, b' \in B$. Now $n = x_1xbxb' = xx_1bxb'$ (Since X is sub commutative) $xx_1xbb' \in x(XB^2) \subseteq xB$. (i.e) $X(xB)^2 \subseteq xB$. Hence xB is a Strong bi-ideals of X.

Similarly we can prove that Bx' is Strong bi-ideals of X. First to prove Bx' is a bi sub algebra of X. Now, to prove $X(Bx')^2 \subseteq Bx'$. Let $n \in X(Bx')^2$ then $n = x_1bx'b'x'$ where $x' \in X_1' \cup X_2'$ and $b,' \in B$. Now $n = x_1bb'x'x' \in X(B^2x') = X(Bx')^2 \subseteq Bx'$. (i.e) $X(Bx')^2 \subseteq Bx'$. Hence Bx' is a Strong bi-ideals of X. □

Proposition 3.12. Let X be strongly bi-regular, then $B = XB^2$, for each strong bi-ideals B of X.

Proof . Let B be strong bi-ideals of X then $XB^2 \subseteq B$ i.e., $X_1B_1^2 \subseteq B_1$ (or $X_2B_2^2 \subseteq B_2$). It is enough prove $B \subseteq X_1B_1^2$. Let $b \in B$. Since X is strongly bi-regular, there exist $x \in X_1 \cup X_2$ such that $b = xb^2 \in X_1B_1^2$(or $X_2B_2^2$), (i.e., $B \subseteq X_1B_1^2$ (or $B \subseteq X_2B_2^2$) implies $B \subseteq XB^2$. Hence $B = XB^2$, for each strong b-ideals B of X. □

Proposition 3.13. Let X be sub commutative S-bi-near subtraction semigroup of X. Then X is strongly bi-regular, if and only if $B = XB^2$, for each strong bi-ideals B of X.

Proof . Let X be sub commutative S-bi-near subtraction semigroup of X. Suppose X is strongly bi-regular, then by the Proposition 3.12, $B = XB^2$.

Conversely assume that $B = XB^2$. To prove X is strongly bi-regular. (i.e) to prove $x \in Xx^2$ for all $x \in X_1 \cup X_2$. By the Result 3.7, Xx is a Strong bi-ideals of X. Since X is S-bi-near subtraction semigroup, $x \in Xx = X(Xx)^2 \subseteq XxXx$. (i.e) for $x_1, x_2 \in X_1 \cup X_2$, $x = x_1xx_2x = x_1x_2xx \in Xx^2$. Thus $x \in Xx$ implies $x \in Xx^2$ which gives that X is strongly bi-regular. □

Proposition 3.14. Let X be sub commutative S-bi-near subtraction semigroup of X. Then X is left bi-potent if and only if $B = XB^2$, for each strong bi-ideals B of X.

Proof . Every left bi-potent is strongly bi-regular and by Proposition 3.10. □

Proposition 3.15. Let X be sub commutative. Then B is bi-ideal of X if and only if B is strong bi-ideals of X.

Proof . If part, every bi-ideals of X with sub commutative is strong bi-ideals of X.

Conversely assume B is bi-ideals of X, then $BXB \subseteq B$. (i.e.,) $B_1X_1B_1 \subseteq B_1$ (or $B_2X_2B_2 \subseteq B_2$). Now to prove B is strong bi-ideals of X. (i.e.,) to prove $XB^2 \subseteq B$. Let $x \in XB^2$, then $x = x_1b^2$ where $x \in X_1 \cup X_2$ and $b \in B$. Now $x = x_1b^2 = x_1bb = bx_1b$ (Since X is sub commutative). Therefore $x = bx_1b \in B_1X_1B_1 \subseteq B_1$(or $B_2X_2B_2 \subseteq B_2$) implies $x \in B$. Therefore $XB^2 \subseteq B$. Hence B is strong bi-ideals of X. □

Proposition 3.16. Let X be sub commutative S-bi-near subtraction semigroup and B be bi-ideals of X. Then $B = BXB$ if and only if $B = XB^2$, for each strong bi-ideals B of X.

Proof . Assume that $B = BXB$, for each bi-ideals B of X. By the definition, if B is bi-ideals of X implies $BXB \subseteq B$ and X is sub commutative. From Proposition 3.15, B is a strong bi-ideals of X implies $XB^2 \subseteq B$. (i.e) $X_1B_1^2 \subseteq B_1$(or $X_2B_2^2 \subseteq B_2$). Let $x \in B = BXB$ then $x = bx_1b$ for some $b \in B$ then $b \in X_1 \cup X_2$ and $x_1 \in X_1 \cup X_2$. Since X is left permutable, $x = x_1bb = x_1b^2 \in XB^2$. (i.e) $B \subseteq XB^2$ and so $B = XB^2$, for each strong bi-ideals B of X.

Conversely let $B = XB^2$. To prove $B = BXB$. From the definition of bi-ideals we get $BXB \subseteq B$. It is enough to prove $B \subseteq BXB$. Let $x \in B = XB^2$ then $x = x_1b^2$ where $x_1 \in X_1 \cup X_2$. Now $x = x_1b^2 = x_1bb = bx_1b \in BXB$. That is $B \subseteq BXB$ and hence $B = BXB$. □

Theorem 3.17. Let X be sub commutative S-bi-near subtraction semigroup of X. Then the following conditions are equivalent.

(i) $B = B^m X B^n$, for each generalized (m, n) bi-ideals B of X, for any positive integers m, n.
(ii) $XB^2 = B$, for each strong bi-ideals B of X.
(iii) $B = BXB$, for each bi-ideals B of X.
(iv) $Q = QXQ$, for each quasi-ideal Q of X.

Proof . $(i) \Rightarrow (ii)$ Consider $B = B^m X B^n$. By the definition, if B be strong bi-ideals of X implies $XB^2 \subseteq B$. (i.e) $X_1 B_1^2 \subseteq B_1$(or $X_2 B_2^2 \subseteq B_2$). It is enough to prove $B \subseteq XB^2$. Since every strong bi-ideals is a bi-ideals implies B is a bi-ideals of X. Since every bi-ideals of X is a generalized (m, n) bi-ideals of X, we get $B = B_1^m X_1 B_1^n$(or $B_2^m X_2 B_2^n) \subseteq B_1^2$ or (or $X_2(B_2^{n-2} B_2^2)) \subseteq X_1 B_1^2$(or $X_2 B_2^2) \subseteq XB^2$. Hence $XB^2 = B$, for each strong bi-ideals B of X.

$(ii) \Rightarrow (iii)$ Consider $XB^2 = B$. By the Proposition 3.16, $B = BXB$, for each bi-ideals B of X.

$(iii) \Rightarrow (iv)$ Suppose $B = BXB$. By Result 2.15, every quasi-ideal Q of X is a bi-ideals of X, we acquire $Q = QXQ$.

$(iv) \Rightarrow (i)$ Consider $Q = QXQ$. Let B be generalized (m, n) bi-ideals B of X. Since X is S-bi-near subtraction semigroup. Let $a \in X_1 \cup X_2$, then Xa is quasi-ideal of bi-near subtraction semigroup X. Now $a \in Xa = (Xa)X(Xa)$, $a = x_1 a x_1 x_1 a$ for some $x_1 \in X_1 \cup X_2$. (Since X is sub commutative) $= a x_1^3 a \in aXa$ (i.e) $Xa = aXa$ for all $a \in X_1 \cup X_2$. In a sub commutative bi-near subtraction semigroup $a \in Xa^2$ implies $a \in xa = xaa = axa \in aXa$ then $xa \subseteq aXa \subseteq Xa$ for all $a \in X_1 \cup X_2$ and hence $aXa = Xa = Xa^2$ for all $a \in X_1 \cup X_2$. Now by the Theorem 2.21, hence $B = B^m X B^n$, for all positive integers m and n. □

4 Conclusion

We concluded that every strong bi-ideals of X is also a bi-ideals of X. And we have seen the properties of strong bi-ideals in bi-near subtraction semigroup.

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