Int. J. Nonlinear Anal. Appl. 13 (2022) 2, 1585–1589
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2021.24402.2735

Hilbert matrix operator on Zygmund spaces

Ayyoub Manavi[a], Mostafa Hassanlou[b,*]

[a] *Department of Mathematics, Sarab Branch, Islamic Azad University, Sarab, Iran*

[b] *Engineering Faculty of Khoy, Urmia University of Technology, Urmia, Iran*

(Communicated by Ali Jabbari)

Abstract

Let $\mathcal{H}_\mu = (\mu_{n+k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$ induces the operator $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n,k}a_k\right)z^n$ on the space of all analytic functions $f(z) = \sum_{n=0}^{\infty}a_n z^n$ in the unit disk $\mathbb{D}$, where μ is a positive Borel measure on the interval $[0,1)$. In this paper, we characterize the boundedness and compactness of the operator $\mathcal{H}_\mu$ on Zygmund type spaces.

Keywords: Boundedness, Compactness, Hilbert matrix operator, Operator, Zygmund space
2020 MSC: 47B38, 30H30

1 Introduction

Let $\mathbb{D}$ be unit ball (disk) $\{z \in \mathbb{C} : |z| < 1\}$, $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$ and $C(\overline{\mathbb{D}})$ be the space of all continuous functions on $\overline{\mathbb{D}}$. For two numbers U and V by $U \preceq V$, we mean that there exists a positive constant $c \in \mathbb{R}$ such that $U \leq cV$.

The class of all $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ such that

$$\|f\| = \sup \frac{\left|f\left(e^{i(\theta+h)}\right) + f\left(e^{i(\theta-h)}\right) - 2f\left(e^{i(\theta)}\right)\right|}{h} < \infty,$$

where the supremum is taken over all $e^{i(\theta)} \in \partial\mathbb{D}$ and $h > 0$, is denoted by $\mathcal{Z}$. The Closed Graph Theorem together [3, Theorem 5.3] implies that $f \in \mathcal{Z}$ if and only if $\sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f''(z)| < \infty$. Moreover, $\mathcal{Z}$ by the following norm is a Banach space

$$\|f\|_{\mathcal{Z}} = |f(0)| + |f'(0)| + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f''(z)|,$$

for all $f \in \mathcal{Z}$. The Banach space $(\mathcal{Z}, \|\cdot\|_{\mathcal{Z}})$ is called the Zygmund space.

Let $I \subset \partial\mathbb{D}$ be a subarc. The Carleson box based on I is defined as follows:

$$S(I) = \left\{z \in \mathbb{D} : 1 - |I| \leq |z| < 1 \quad \text{and} \quad \frac{z}{|z|} \in I\right\},$$

where $|I| = \frac{1}{2\pi}\int_I |d\xi|$ is the normalized length of the arc I.

*Corresponding author
Email addresses: ayobmanaviii@gmail.com (Ayyoub Manavi), m.hassanlou@uut.ac.ir (Mostafa Hassanlou)

Received: August 2021 *Accepted:* October 2021

Let μ be a positive Borel measure. Following [2], for any $0 < s < \infty$, μ is called an s-Carleson measure on $\mathbb{D}$ if

$$\|\mu\| = \sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^s} < \infty, \tag{1.1}$$

where the supremum taken over all subarcs I of $\partial\mathbb{D}$. For $s = 1$, μ is called the Carleson measure. Following [8], (1.1), means that there exists $0 < C < \infty$ such that

$$|\mu|(1-t, 1) \leq Ct. \tag{1.2}$$

Let μ be a finite positive Borel measure on $[0, 1)$ and let $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ be the Hankel matrix with entries

$$\mu_{n,k} = \int_{[0,1)} t^{n+k} \mathrm{d}\mu(t).$$

The matrix $\mathcal{H}_\mu$ induces an operator, denoted also by $\mathcal{H}_\mu$, on $H(\mathbb{D})$ by multiplication of the matrix with the sequence of the Taylor coefficient of the function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$,

$$\{a_n\}_{n=0}^{\infty} \longrightarrow \left\{ \sum_{k=0}^{\infty} \mu_{n,k} a_n \right\}_{n=0}^{\infty}.$$

More precisely, for any $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$,

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k \right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. An integral representation of $\mathcal{H}_\mu$ has obtained in [4] as follows:

$$I_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz} \mathrm{d}\mu(t),$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. Whenever, μ is a the Lebesgue measure on the interval $[0, 1)$, then $\mathcal{H}_\mu$ is the classical Hilbert matrix $\mathcal{H} = \left\{ \frac{1}{n+k+1} \right\}_{n,k\geq 0}$.

Some properties of the operator $\mathcal{H}_\mu$ on various Banach spaces such as Hardy, Bergman, Bloch, weighted Bloch, Dirichlet and BMOA spaces are studied in [1, 5, 6, 4, 7].

In the next section, we give a necessary and sufficient condition for boundedness and compactness of the operator $\mathcal{H}_\mu$ on the Zygmund spaces.

2 The operator $H_\mu : \mathcal{Z} \longrightarrow \mathcal{Z}$

We commence with the following technical lemmas that have essential roles in the proof of our main result in this paper.

Lemma 2.1. [9] Let X and Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose that

(i) The point evaluation functions on X are continuous.

(ii) The closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.

(iii) $T : X \longrightarrow Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.

Then, T is a compact operator if and only if given a bounded sequence $\{f_n\}_{n=1}^{\infty}$ in X such that $f_n \longrightarrow 0$ uniformly on compact sets, then the sequence $\{Tf_n\}_{n=1}^{\infty}$ converges to zero in the norm of Y.

Lemma 2.2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative numbers that converges 0. Then $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{Z}$ if and only if $\sup_n n(n-1)a_n < \infty$.

Proof . As $a_n \downarrow 0$, there exists $C > 0$ such that

$$\frac{n(n-1)a_n}{2} \leq \frac{1}{n}\sum_{k=2}^{n} k(k-1)a_k \leq C \qquad (n \in \mathbb{N}). \tag{2.1}$$

Moreover, for any $j \geq 2$, we have $1/e \leq \left(1-j^{-1}\right)^{j-2} \leq 1$. This implies that there exists $C' > 0$ such that $j^{-1}C'\left(1-j^{-1}\right)^{j-2} \geq j^{-1}$. Hence,

$$j^{-1}\left(1-j^{-1}\right)^{j-2} \succeq j^{-1}. \tag{2.2}$$

If $f \in \mathcal{Z}$, then by employing (2.2),

$$\begin{aligned}
\|f\|_{\mathcal{Z}} =& a_0 + a_1 + \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f''(z)| \\
\geq& \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)|f''(z)| \\
=& \sup_{z\in\mathbb{D}}\left(1-|z|^2\right)\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \\
\geq& \sup_{z=1-j^{-1}}\left(1-|z|^2\right)\sum_{n=2}^{j} n(n-1)a_n z^{n-2} \\
\geq& j^{-1}\left(1-j^{-1}\right)^{j-2}\sum_{n=2}^{j} n(n-1)a_n \\
\succeq& j^{-1}\sum_{n=2}^{j} n(n-1)a_n.
\end{aligned}$$

Then by (2.1), we have $\sup_{j\in\mathbb{N}}\sum_{n=2}^{j} n(n-1)a_n < \infty$. This implies that $\sup_{n\in\mathbb{N}} n(n-1)a_n < \infty$.

Conversely, assume that $\sup_{n\in\mathbb{N}} n(n-1)a_n < \infty$. Then by (2.1), we have $\sup_{j\in\mathbb{N}}\sum_{n=2}^{j} n(n-1)a_n < \infty$. Thus, for any $k \in \mathbb{N}$,

$$\sum_{n=2^k}^{2^{k+1}-1} a_n \preceq 1. \tag{2.3}$$

Then by the above inequality, for any $z \in \mathbb{D}$, we have

$$\begin{aligned}
|f''(z)| =& \left|\sum_{n=2}^{\infty} n(n-1)a_n z^{n-2}\right| \\
=& \left|\sum_{k=2}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n(n-1)a_n z^{n-2}\right| \\
\leq& \sum_{k=2}^{\infty}\sum_{n=2^k}^{2^{k+1}-1} n(n-1)a_n |z|^{n-2} \\
\leq& \sum_{k=2}^{\infty} 2^{k+1}\left(2^{k+1}-1\right)\sum_{n=2^k}^{2^{k+1}-1} a_n |z|^{n-2} \\
\leq& \sum_{k=2}^{\infty} 2^{k+1}\left(2^{k+1}-1\right)|z|^{2^k-1}\sum_{n=2^k}^{2^{k+1}-1} a_n \\
\preceq& \sum_{k=2}^{\infty} 2^{2k+2}|z|^{2^k-1}
\end{aligned}$$

$$\begin{aligned}&\leq \sum_{k=2}^{\infty} 2^k |z|^{2^k-1}\\ &\preceq (1-|z|)^{-1}.\end{aligned}$$

This means that $f \in \mathcal{Z}$. $\square$

Theorem 2.3. Let μ be a positive measure on $[0,1)$. Then the following statements are equivalent:

(i) The operator $H_\mu : \mathcal{Z} \longrightarrow \mathcal{Z}$ is bounded.

(ii) μ is a Carleson measure.

(iii) The operator $H_\mu : \mathcal{Z} \longrightarrow \mathcal{Z}$ is compact.

Proof . (i)→(ii) For any $0 < \alpha < 1$, one can choose $n \in \mathbb{N}$ such that $1 - \frac{1}{n} \leq \alpha < 1 - \frac{1}{n+1}$. Then

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = e^{-1} \leq \lim_{n\to\infty} \alpha^n \quad \text{and} \quad n \leq \frac{1}{1-\alpha} < n+1. \tag{2.4}$$

Let $f(z) = 1$. Clearly, $f \in \mathcal{Z}$ and we have

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k\right) z^n = \sum_{n=0}^{\infty} \mu_{n,0} z^n \in \mathcal{Z}.$$

As $\{\mu_{n,0}\}_{n=1}^{\infty}$ is a nonnegative and decreasing. Then by (2.4) and Lemma 2.2, we have

$$\begin{aligned}\frac{\mu([\alpha,1))}{e(1-\alpha)^2} &\leq n(n-1)\alpha^n \int_\alpha^1 \mathrm{d}\mu(t)\\ &\leq n(n-1) \int_0^1 t^n \mathrm{d}\mu(t)\\ &= n(n-1)\mu_{n,0} < \infty.\end{aligned}$$

Thus, μ is a Carleson measure.

(ii)→(i) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{Z}$. Since μ is a Carleson measure, we have $\sup_{n\in\mathbb{N}} \mu_n(n+1) < \infty$. Thus, there exists $0 < C < \infty$, such that

$$\sum_{k=0}^{\infty} \mu_{n,k} a_k \leq C \sum_{k=0}^{\infty} \frac{a_k}{n+k+1} \leq C\|f\|_{\mathcal{Z}}. \tag{2.5}$$

Hence, $\mathcal{H}_\mu(f)(z) \in H(\mathbb{D})$. Then by a similar argumentation in the proof of [4, Proposition 1.1], we have

$$\sum_{k=0}^{\infty} \mu_{n,k} a_k = \int_{[0,1)} t^n f(t)\mathrm{d}\mu(t), \quad n \in \mathbb{N}. \tag{2.6}$$

This implies that

$$\mathcal{H}_\mu(f)(z) = \sum_{n=1}^{\infty} \left(\int_{[0,1)} t^n f(t)\mathrm{d}\mu(t)\right) z^n = \int_{[0,1)} \frac{f(t)}{1-tz}\mathrm{d}\mu(t), \quad z \in \mathbb{D}. \tag{2.7}$$

For any $f \in \mathcal{Z}$, we have

$$\int_{[0,1)} f(t)\mathrm{d}\mu(t) \leq \|f\|_{\mathcal{Z}} \int_{[0,1)} \mathrm{d}\mu(t) < \infty. \tag{2.8}$$

Then for any $g \in H(\mathbb{D})$ with $\|g\|_\infty = \sup_{z\in\mathbb{D}} |g(z)| < \infty$ and $0 < r < 1$, (2.8) implies that

$$\int_0^{2\pi} \int_{[0,1)} \left| \frac{f(t) g\left(e^{i\theta}\right)}{1 - rte^{i\theta}} \right| \mathrm{d}\mu(t)\mathrm{d}\theta < \infty. \tag{2.9}$$

Then the above equation can be represented as follows:

$$\int_0^{2\pi} I_\mu(f)\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\mathrm{d}\theta=\int_{[0,1)} f(t)\overline{g(rt)}\mathrm{d}\mu(t), \tag{2.10}$$

for all $f\in\mathcal{Z}$ and $g\in H(\mathbb{D})$. Then by (2.10), we have

$$\begin{aligned}\left|\int_0^{2\pi} I_\mu(f)\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\mathrm{d}\theta\right| &\left|\int_{[0,1)} f(t)\overline{g(rt)}\mathrm{d}\mu(t)\right|\\ &\le\|f\|_{\mathcal{Z}}\int_{[0,1)}|g(rt)|\mathrm{d}\mu(t)\\ &\le\|\mu\|\ \|f\|_{\mathcal{Z}}\ \|g\|_\infty.\end{aligned}$$

This shows that $\mathcal{H}_\mu$ is bounded.

(ii)$\to$(iii) Let μ be a Carleson measure. Then by $\mathcal{H}_\mu(f)=I_\mu(f)$, for all $f\in\mathcal{Z}$, and by (i), $\mathcal{H}_\mu$ is bounded. Let $\{f_n\}_{n=1}^\infty$ be a bounded sequence in the unit ball of $\mathcal{Z}$ that converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. This implies that $\sup_{z\in\mathbb{D}}|f_n(z)|\to 0$ as $n\to\infty$. By employing (2.10), for any $g\in H(\mathbb{D})$ with $\|g\|_\infty=\sup_{z\in\mathbb{D}}|g(z)|<\infty$ and $0<r<1$, we have

$$\begin{aligned}\left|\int_0^{2\pi} I_\mu(f_n)\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\mathrm{d}\theta\right| &=\left|\int_{[0,1)} f_n(t)\overline{g(rt)}\mathrm{d}\mu(t)\right|\\ &\le\sup_{t\in(0,1)}|f_n(t)|\int_{[0,1)}|g(rt)|\mathrm{d}\mu(t)\\ &\le\sup_{t\in(0,1)}|f_n(t)|\|\mu\|\ \|g\|_\infty.\end{aligned}$$

This implies that $\lim_{n\to\infty}\int_0^{2\pi} I_\mu(f_n)\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\mathrm{d}\theta=0$. Thus,

$$\lim_{n\to\infty}\mathcal{H}_\mu(f_n)=\lim_{n\to\infty}I_\mu(f)=0.$$

Hence, H_μ is compact.

(iii)$\to$(i) is trivial. □

References

[1] G. Bao and H. Wulan, *Hankel matrices acting on Dirichlet spaces*, J. Math. Anal. Appl. **409** (2014), 228-–235.

[2] P. Duren, *Extension of a theorem of Carleson*, Bull. Amer. Math. Soc. **75** (1969), 143-–146.

[3] P. Duren, *Theory of H^p spaces*, Academic Press, New York, 1973.

[4] P. Galanopoulos and J. Peláez, *A Hankel matrix acting on Hardy and Bergman spaces*, Studia Math. **200** (2010), 201–220.

[5] D. Girela and N. Merchán, *A Hankel matrix acting on spaces of analytic functions*, Integr. Equ. Oper. Theory **89** (2017), 581–594.

[6] D. Girela and N. Merchán, *Hankel Matrix acting on the Hardy space H^1 and on Dirichlet spaces*, Rev. Mat. Complut. **32** (2019), 799–822.

[7] S. Li and J. Zhou, *Essential norm of generalized Hilbert matrix from Bloch type spaces to BMOA and Bloch space*, AIMS Math. **6** (2021), no. 4, 3305–3318.

[8] V. Peller, *Hankel operators and their applications*, Springer Monogr. Math., Springer, New York, 2003.

[9] M. Tjani, *Compact composition operators on some Moebius invariant Banach spaces*, Thesis (PhD.), Michigan State University, 1996.