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A convergence theorem for a common solution of f-fixed point, variational inequality and generalized mixed equilibrium problems in Banach spaces

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Abstract

The purpose of this paper is to construct an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces.

Keywords: Generalized mixed equilibrium problem, Variational inequality problem, f-pseudocontractive mapping, monotone mapping, reflexive Banach spaces
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1 Introduction

Let E be a reflexive real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. Let $F : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real valued function, and $B : C \to E^*$ be a nonlinear mapping. The *Generalized Mixed Equilibrium Problem (GMEP)* (Ceng and Yao [8]) is to find $x \in C$ such that

$$H(x, y) := F(x, y) + \varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $GMEP(F, \varphi, B)$. In particular, if $\varphi \equiv 0$, the problem (1.1) reduces to the *Generalized Equilibrium problem (GEP)* (Mouda and Thera [13]) which is to find $x \in C$ such that

$$\overline{H}(x, y) := F(x, y) + \langle Bx, y - x\rangle \geq 0, \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $GEP(F, B)$.
If in (1.1), we consider $F \equiv 0$, then problem (1.1) reduces to finding $x \in C$ such that

$$\varphi(y) - \varphi(x) + \langle Bx, y - x\rangle \geq 0, \forall y \in C, \tag{1.3}$$

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which is called the *Mixed Variational Inequality of Browder type (MVI)* [7]. The set of solutions to (1.3) is denoted by $MVI(C, B, \varphi)$.
If $F \equiv 0$ and $\varphi(y) \equiv 0$ for all $y \in C$, problem (1.1) reduces to finding $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \forall y \in C, \tag{1.4}$$

which is the classical *Variational Inequality Problem (VIP)*. The set of solutions to (1.4) is denoted by $VI(C, B)$.
If in (1.2), $B \equiv 0$, then problem (1.2) reduces to the *Equilibrium problem (EP)* (Blum and Oettli [3]) which is to find $x \in C$ such that

$$F(x, y) \geq 0, \forall y \in C. \tag{1.5}$$

The set of solutions to (1.5) is denoted by $EP(F)$.
We say that a bi-function F satisfies "**Condition A**" if the following four properties hold:

(A1) $F(x, x) = 0, \forall x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;
(A3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \forall x, y, z \in C$;
(A4) for each $x \in C, y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Some of the applications of the equilibrium problem are given below.
Optimization: Let $\phi : C \to \mathbb{R}$ be a convex and lower semi-continuous function. The minimization problem is to find $x^* \in C$ such that

$$\phi(x^*) \leq \phi(y), \forall y \in C. \tag{1.6}$$

Setting $F(x, y) := \phi(y) - \phi(x)$, problem (1.6) coincides with (1.5).
Saddle Point Problem: Let $\varphi : C_1 \times C_2 \to \mathbb{R}$. Then $x^* = (x_1^*, x_2^*)$ is called a saddle point of the function φ if and only if for $x^* = (x_1^*, x_2^*)$,

$$\varphi(x_1^*, y_2) \leq \varphi(y_1, x_2^*), \forall (y_1, y_2) \in C_1 \times C_2. \tag{1.7}$$

If $C := C_1 \times C_2$, and $F : C \times C \to \mathbb{R}$ is defined by

$$F((x_1, x_2), (y_1, y_2)) := \varphi(y_1, x_2) - \varphi(x_1, y_2),$$

then $x^* = (x_1^*, x_2^*)$ is a solution of (1.5) if and only if $x^* = (x_1^*, x_2^*)$ satisfies (1.7).
Nash Equilibrium in Non-cooperative Games: Let I be a finite set of players and let C_i be a strategy set of the i^{th} player, for each $i \in I$. Let $f_i : C := \prod_{i \in I} C_i \to \mathbb{R}$ be a loss function of the i^{th} player depending on the strategies of all players, for all $i \in I$. For $x = (x_i)_{i \in I} \in C$, we find $x_{-i} = (x_j)_{j \in I | j \neq i}$. The point $x^* = (x^*)_{i \in I} \in C$ is called Nash Equilibrium if for $i \in I$, the following holds:

$$f_i(x^*) \leq f_i(x_{-i}^*, y_i), \forall y_i \in C_i, \tag{1.8}$$

(that is, no player can reduce his loss by varying his strategy alone). If $F : C \times C \to \mathbb{R}$ is given by

$$F(x, y) := \sum_{i \in I} (f_i(x_{-i}, y_i) - f_i(x)),$$

then $x^* \in C$ is a Nash equilibrium if and only if x^* satisfies (1.5).

Let $f : E \to (-\infty, +\infty]$ be a proper, lower semi-continuous and convex function. We denote the domain of f by $domf = \{x \in E : f(x) < \infty\}$. The subdifferential of f at x is the convex set given by

$$\partial f(x) = \{x^* \in E^* : f(y) - f(x) \geq \langle y - x, x^* \rangle, \forall y \in E\}.$$

The Fenchel conjugate of f is a function $f^* : E^* \to (-\infty, +\infty]$, defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

A function $f : E \to (-\infty, +\infty]$ is called *strongly coercive* if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = \infty.$$

For any $x \in int(domf)$ and any $y \in E$, we denote by $f^0(x,y)$ the right-hand derivative of f at x in the direction of y, that is,

$$f^0(x,y) = \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}.$$

The function f is called *Gâteaux differentiable* at x if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any $y \in E$. In this case, the gradient of f at x, $\nabla f(x)$, coincides with $f^0(x,y)$ for all $y \in E$. It is called *Gâteaux differentiable* if it is Gâteaux differentiable at every point $x \in int(domf)$. We note that if the subdifferential of f is single-valued, then $\partial f = \nabla f$. The function $f : E \to \mathbb{R}$ is called *uniformly convex* if there exists a continuous increasing function $g : [0,+\infty) \to \mathbb{R}$, $g(0) = 0$, such that

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - t(1-t)g(\|x-y\|), \tag{1.9}$$

for all $x, y \in domf$. The function g is called a *modulus of convexity* of f. If f is a uniformly convex and Gâteaux differentiable function in $domf$ with modulus of convexity g, then $\langle x-y, \nabla f(x) - \nabla f(y)\rangle \ge 2g(\|x-y\|), \forall x, y \in domf$, or equivalently, $f(y) \ge f(x) + \langle y-x, \nabla f(x)\rangle + g(\|x-y\|), \forall x, y \in domf$. The functional f is called *strongly convex* if f is uniformly convex with the modulus of convexity $g(t) = ct^2$, $c > 0$. If a function f is strongly convex with constant $\mu > 0$ and Gâteaux differentiable in $(domf)$, then $\langle x-y, \nabla f(x) - \nabla f(y)\rangle \ge \mu\|x-y\|^2, \forall x, y \in domf$, or equivalently, $f(y) \ge f(x) + \langle y-x, \nabla f(x)\rangle + \frac{\mu}{2}\|x-y\|^2, \forall x, y \in domf$. If E is a smooth and strictly convex Banach space, the function $f(x) = \|x\|^2, \forall x \in E$ is strongly convex with constant $\mu \in (0,1]$ (see, Phelps [15]).

A mapping $A : D(A) \subset E \to E^*$, is said to be *monotone* if for each $x, y \in D(A)$, the following inequality holds:

$$\langle x-y, Ax-Ay\rangle \ge 0. \tag{1.10}$$

A mapping $A : D(A) \subset E \to E^*$, is said to be γ-*inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \ge \gamma\|Ax-Ay\|^2. \tag{1.11}$$

If A is γ-inverse strongly monotone, then it is Lipschitz continuous with constant $\frac{1}{\gamma}$, that is, $\|Ax-Ay\| \le \frac{1}{\gamma}\|x-y\|, \forall x, y \in D(A)$, and hence uniformly continuous.

Closely related to the class of monotone mappings is the class type of f-pseudocontractive mappings.

A mapping $T : E \to E^*$, is said to be *f-pseudocontractive* mapping (see, Zegeye and Wega [25]) if for each $x, y \in E$, we have

$$\langle x-y, T(x)-T(y)\rangle \le \langle x-y, \nabla f(x) - \nabla f(y)\rangle. \tag{1.12}$$

A mapping T is said to be γ-strictly f-pseudocontractive if for all $x, y \in C$, there exists $\gamma > 0$ such that

$$\langle x-y, T(x)-T(y)\rangle \le \langle x-y, \nabla f(x) - \nabla f(y)\rangle - \gamma\|(\nabla f(x) - \nabla f(y)) - (Tx-Ty)\|^2. \tag{1.13}$$

The *f-fixed point problem* with respect to T is to find a point $p \in C$ such that $Tp = \nabla f(p)$. The set of f-fixed points of T is denoted by $F_f(T)$, that is, $F_f(T) = \{p \in C : Tp = \nabla f(p)\}$. A mapping T is said to be semi-pseudocontractive if $\langle x-y, T(x)-T(y)\rangle \le \langle x-y, J(x)-J(y)\rangle$, $\forall x, y \in E$. We remark that if E is smooth and strictly convex and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*}, and the notion of f-pseudocontractive mapping reduces to the notion of semi-pseudocontractive mapping and f-fixed point of T reduces to semi-fixed point of T. If, in addition, $E = H$, a real Hilbert space, then f-pseudocontractive mapping becomes pseudocontractive mapping. The mapping T is f-pseudocontractive if and only if $A = \nabla f - T$ is monotone and T is strictly f-pseudocontractive if and only if $A = \nabla f - T$ is γ-inverse strongly monotone. In this case, the zero of A corresponds to f-fixed point of T. In fact, if T and ∇f are continuous on E then A is maximal monotone and the set of zeros of A and hence the set of f-fixed points of an f-pseudocontractive mapping T is closed and convex (see, Zegeye and Wega [25]).

The above formulation of fixed point problem was treated as equilibrium problem as follows.
Fixed Point Problem: Let $T : E \to E$ be a given mapping. If $F(x,y) = \langle x - T(x), y - x\rangle$, $\forall x, y \in E$, then p is a solution of (1.5) if and only if it is a fixed point of T.

A method for solving the fixed point problem of pseudocontractive mapping with the use of the resolvent mapping was introduced by Zegeye [24] in Hilbert spaces. Let f be a self contraction on C, and let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and

$$x_{n+1} = \alpha_n f(x) + (1-\alpha_n)K^{T_1}K^{T_2}x_n, \tag{1.14}$$

where $\{\alpha_n\} \subset [0,1]$ with $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $K_{r_n}^{T_1}$ and $K_{r_n}^{T_2}$ with $\{r_n\} \subset (0,\infty)$, $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ where $K_{r_n}^{T_i}x = \{z \in C : \langle y-z, T_i z\rangle - \frac{1}{r_n}\langle y-z, (1+r_n)z - x\rangle \le 0, \forall y \in C\}$, where T_i's, $i = 1, 2$, are continuous pseudocontractive mappings. He proved that if $\mathcal{F} = \bigcap_{i=1}^{2} Fix(T_i) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to $z = \Pi_{\mathcal{F}} f(z)$.

Recently, several authors have proposed algorithms for approximating a common solution of a variational inequality, an equilibrium problem, and semi-fixed points of a continuous semi-pseudocontractive mapping in the framework of Hilbert spaces and Banach spaces (see, [9, 11]).

In 2019, Shahzad and Zegeye [21] proved the following convergence theorem for a common solution of fixed point, equilibrium and variational inequality problems in Hilbert spaces.

Theorem 1.1. Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a Lipschitz monotone mapping with Lipschitz constant $L > 0$, $F : C \times C \to \mathbb{R}$ be a bi-functional satisfying **Condition A**, and $T : C \to H$ be a continuous pseudocontractive mapping with $\mathcal{F} := F(T) \bigcap VI(A, C) \bigcap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} u, x_0 \in C, \\ z_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)(\beta y_n + (1 - \beta) u_n), \end{cases} \tag{1.15}$$

where P_C is the metric projection from H onto C, $y_n = K_{r_n}^T T_{r_n}^F x_n$ with $T_{r_n}^F$ and $K_{r_n}^S$ as the resolvent mappings for F and T, respectively, $\{r_n\} \subset [a, \infty)$, for some $a > 0$, $u_n = P_C(x_n - \lambda A z_n)$, $\lambda \in [a, b] \subset (0, \frac{1}{L})$ and $\{\alpha_n\} \subset (0, c] \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a point $P_{\mathcal{F}} u$.

In 2019, Khonchaliew et al. [10] studied two shrinking extragradient algorithms for finding a common solution set of equilibrium problems for a finite family of pseudomonotone bifunctions and set of fixed points of quasinonexpansive mappings in real Hilbert spaces.

In 2020, Nnakwe and Okeke [14] constructed a new Halpern-type iterative algorithm and proved the following result in uniformly smooth and uniformly convex real Banach spaces. Let $B_i : C \to E^*$, $i = 1, 2$ be a continuous and monotone mappings, $F_i : C \times C \to \mathbb{R}$, $i = 1, 2$ be a bi-functionals satisfying **Condition A**, and $T_i : C \to E^*$, $i = 1, 2$ be a continuous semi-pseudocontractive mappings with $\mathcal{F} := \bigcap_{i=1}^{2} (F_s(T_i) \bigcap GEP(F_i, B_i)) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_1 \in C, \\ z_n = T_{r_n}^{\overline{H}_1} T_{r_n}^{\overline{H}_2} x_n, \\ x_{n+1} = J^{-1}(\alpha_n J x_1 + (1 - \alpha_n) J K_{r_n}^{T_1} K_{r_n}^{T_2} z_n]), \forall n \geq 1, \end{cases} \tag{1.16}$$

where $T_{r_n}^{\overline{H}_i}$ and $K_{r_n}^{T_i}$ are the resolvent mappings for $\overline{H}_i$ and T_i, $i = 1, 2$, respectively, and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_1$.

In 2021, Bello and Nnakwe [2] studied a new Halpern-type subgradient extragradient iterative algorithm and proved strong convergence in a uniformly smooth and 2-uniformly convex real Banach space. Let $A : C \to E^*$ be a Lipschitz monotone mapping with Lipschitz constant $L > 0$, $F : C \times C \to \mathbb{R}$ be a bi-functional satisfying **Condition A**, and $T : C \to E^*$ be a continuous semi-pseudocontractive mapping with $\mathcal{F} := F_s(T) \bigcap VI(C, A) \bigcap EP(F) \neq \emptyset$. Let the sequence $\{x_n\}$ be generated by

$$\begin{cases} x_0 \in C, \\ z_n = \Pi_C J^{-1}(J x_n - \lambda A x_n), \\ T_n = \{w \in E : \langle w - z_n, J x_n - \lambda A x_n - J z_n \rangle \leq 0\}, \\ x_{n+1} = J^{-1}(\alpha_n J x_0 + (1 - \alpha_n)[\beta J v_n + (1 - \beta) J w_n]), \end{cases} \tag{1.17}$$

where $v_n = T_{r_n}^F K_{r_n}^T x_n$ with $T_{r_n}^F$ and $K_{r_n}^S$ are the resolvent mappings of F and T, respectively, $\{r_n\} \subset [a, \infty)$, for some $a > 0$, $w_n = \Pi_{T_n} J^{-1}(J x_n - \lambda A z_n)$, $\lambda \in (0, 1)$ with $\lambda < \frac{c}{L}$ and $\{\alpha_n\} \subset (0, 1)$ with $\lim_{n\to\infty} \alpha_n = 0$ and $\sum \alpha_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a point $\Pi_{\mathcal{F}} x_0$.

Motivated and inspired by the above results, it is our purpose in this paper to propose an algorithm for approximating a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for Lipschitz monotone mappings in real reflexive Banach spaces.

2 Preliminaries

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The function $D_f : domf \times int(domf) \to [0, +\infty)$, defined by

$$D_f(y,x) = f(y) - f(x) - \langle y - x, \nabla f(x)\rangle, \forall x, y \in E. \tag{2.1}$$

is called the *Bregman distance* with respect to f (see, Bregman [5]).
The Bregman distance has the following two important properties (see, Reich and Sabach [16]), called the *three-point identity*: for any $x \in domf$ and $y, z \in int(domf)$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle x - y, \nabla f(z) - \nabla f(y)\rangle, \tag{2.2}$$

and the *four-point identity*: for any $y, w \in domf$ and $x, z \in int(domf)$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle y - w, \nabla f(z) - \nabla f(x)\rangle. \tag{2.3}$$

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable convex function. The function $\nu_f : int(domf) \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\nu_f(x,t) = \inf_{y \in int(domf)} \{D_f(y,x) : \|x - y\| = t\}$$

is called the *Modulus of total convexity* of f at $x \in int(domf)$ and f is called *totally convex* if

$$\nu_f(x,t) > 0, \text{ for all } (x,t) \in int(domf) \times \mathbb{R}^+.$$

We remark that f is totally convex on bounded subsets of E if and only if f is uniformly convex on bounded subsets of E (see, Butnariu and Resmerita [6], Theorem 2.10, Page 9).
The Bregman projection of $x \in int(domf)$ onto the nonempty, closed and convex set $C \subset domf$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}.$$

If E is a smooth and strictly convex Banach space and $f(x) = \frac{1}{2}\|x\|^2$ for all $x \in E$, then we have that $\nabla f = J$, where J is the normalized duality mapping from E into 2^{E^*} and the Bregman distance with respect to f, D_f, reduces to the Lyapunov functional $\phi : E \times E \to [0, +\infty)$ defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \forall x, y \in E. \tag{2.4}$$

The function f is called Legendre if it satisfies the following two properties:

(L1) the interior of the domain of f, $int(domf)$, is nonempty, f is Gâteaux differentiable and $dom(\nabla f) = int(domf)$;

(L2) the interior of the domain of f^*, $int(domf^*)$, is nonempty, f^* is Gâteaux differentiable and $dom(\nabla f^*) = int(domf^*)$;

Since E is reflexive, $(\partial f)^{-1} = \partial f^*$. This, with (L1) and (L2), imply the following equalities:

$$\nabla f = (\nabla f^*)^{-1}, R(\nabla f) = dom(\nabla f^*) = int(domf^*),$$

and

$$R(\nabla f^*) = dom(\nabla f) = int(domf),$$

where $R(\nabla f)$ denotes the range of ∇f.

If a function $f : E \to (-\infty, +\infty]$ is a Legendre function and E is a reflexive Banach space, then $\nabla f^* = (\nabla f)^{-1}$(see, Bonnans and Shapiro [4]).

One of the important and interesting Legendre function in a smooth and strictly convex Banach space is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$ with its conjugate function $f^*(x) = \frac{1}{q}\|x\|^q$ $(1 < q < \infty)$ (see, for example, Bauschke et al. [1]), where $\frac{1}{p} + \frac{1}{q} = 1$. In this case, the gradient of f, ∇f, coincides with the generalized duality mapping, J_p, of E; that is, $\nabla f = J_p$, where $J_p : E \to 2^{E^*}$ is defined by

$$J_p(x) = \{y^* \in E^* : \langle x, y^*\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\}, \forall x \in E.$$

If $p = 2$, we write $J_2 = J$, called the *normalized duality mapping* and if $E = H$, a real Hilbert space, then $J = I$, where I is the identity mapping on H.
Let $f : E \to \mathbb{R}$ be a Legendre function. We make use of the function $V_f : E \times E^* \to \mathbb{R}$ defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \text{ for all } x \in E \text{ and } x^* \in E^*.$$

We note that V_f is a nonnegative function which satisfies (see, Senakka and Cholamjiak [20])

$$V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \text{ for all } x \in E \text{ and } x^* \in E^*, \tag{2.5}$$

and

$$V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*), \text{ for all } x \in E \text{ and } x^*, y^* \in E^*. \tag{2.6}$$

Lemma 2.1. (Phelps [15]) If $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is a proper, weak* lower semi-continuous and convex function and for any $x \in E$, $\{y_k\}_{k=1}^N \subseteq E$ and $\{c_k\}_{k=1}^N \subseteq (0, 1)$ with $\sum_{k=1}^N c_k = 1$ the following holds:

$$D_f\left(x, \nabla f^*\left(\sum_{k=1}^N c_k \nabla f(y_k)\right)\right) \leq \sum_{k=1}^N c_k D_f(x, y_k). \tag{2.7}$$

Lemma 2.2. (Reich and Sabach [17]) If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is norm-to-norm uniformly continuous on bounded subsets of E and hence both f and ∇f are bounded on bounded subsets of E.

Lemma 2.3. (Bunariu and Resmerita [6]) Let $f : E \to \mathbb{R}$ be a totally convex and Gâteaux differentiable function, and $x \in E$. Let C be a nonempty, closed and convex subset of E. The Bregman projection P_C^f from E onto C has the following properties:

(i) $z = P_C^f(x)$ if and only if $\langle y - z, \nabla f(x) - \nabla f(z) \rangle \leq 0$, $\forall y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x)$, $\forall y \in C$.

Lemma 2.4. (Reich and Sabach [18]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x \in E$ and the sequence $\{D_f(x_n, x)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.

Lemma 2.5. (Reich and Sabach [18]) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then, the following assertions are equivalent:

(i) $\lim_{n\to\infty} D_f(x_n, y_n) = 0$;

(ii) $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6. (Wega and Zegeye [23]) Let f be a strongly convex function with constant $\mu > 0$. Then, for all $y \in dom f$ and $x \in int(dom f)$,

$$D_f(y, x) \geq \frac{\mu}{2}\|x - y\|^2,$$

where $D_f(y, x)$ is a Bregman distance with respect to f.

Lemma 2.7 (Darvish [9]). Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let C be a closed and convex subset of a real reflexive Banach space E. Assume that $B : C \to E^*$ is a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ is a lower semi-continuous and convex function and let $F : C \times C \to \mathbb{R}$ be a bi-function satisfying **Condition A**. For $r > 0$ and $x \in E$, define a mapping $T_H^{f,r} : E \to C$ as follows:

$$T_H^{f,r} x = \{z \in C : H(z, y) + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x) \rangle \geq 0, \forall y \in C\}, \tag{2.8}$$

where $H(z, y) := F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz \rangle$. Then, $T_H^{f,r}(x) \neq \emptyset$, and the following hold:

(1) $T_H^{f,r}$ is single-valued;
(2) $F(T_H^{f,r}) = GMEP(F, \varphi, B)$;
(3) $GMEP(F, \varphi, B)$ is closed and convex;
(4) $T_H^{f,r}$ is quasi-Bregman nonexpansive;
(5) $D_f(p, T_H^{f,r}x) + D_f(T_H^{f,r}x, x) \leq D_f(p, x), \forall p \in F(T_H^{f,r})$.

Lemma 2.8. Let $f : E \to (-\infty, +\infty]$ be a coercive and Gâteaux differentiable function. Let E^* be the dual space of a real reflexive Banach space E and C be a closed and convex subset E. Let $T : C \to E^*$ be a continuous f-pseudocontractive mapping. For $r > 0$ and $x \in E$, define a mapping $K_T^{f,r} : E \to C$ as follows:

$$K_T^{f,r}x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r}\langle y - z, (1 + r)\nabla f(z) - \nabla f(x)\rangle \leq 0, \forall y \in C\}. \tag{2.9}$$

Then, $K_T^{f,r}(x) \neq \emptyset$, and the following hold:

(1) $K_T^{f,r}$ is single-valued;
(2) $F(K_T^{f,r}) = F_f(T)$
(3) $F_f(T)$ is closed and convex;
(4) $K_T^{f,r}$ is quasi-Bregman nonexpansive;
(5) $D_f(p, K_T^{f,r}x) + D_f(K_T^{f,r}x, x) \leq D_f(p, x), \forall p \in F(K_T^{f,r})$.

Proof. Let $B := \nabla f - T$. Then, B is monotone and continuous. Putting $F \equiv 0$ and $\varphi \equiv 0$ in Lemma 2.7. Then, there exists $z \in C$ such that

$$\langle y - z, B(z)\rangle + \frac{1}{r}\langle y - z, \nabla f(z) - \nabla f(x)\rangle \geq 0, \forall y \in C.$$

Equivalently,

$$\langle y - z, T(z)\rangle - \frac{1}{r}\langle y - z, (1 + r)\nabla f(z) - \nabla f(x)\rangle \leq 0, \forall y \in C.$$

Furthermore, applying Lemma 2.7, we get the results (1)-(5) of Lemma 2.8. This completes the proof.

Lemma 2.9. (Xu [22]) Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n b_n, n \geq n_0,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\} \subset \mathbb{R}$ satisfying the following conditions: $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\limsup_{n\to\infty} b_n \leq 0$, or $\sum_{n=1}^{\infty} |\alpha_n b_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.10. (Maingé [12]) Suppose $\{s_n\}$ is a sequence of real numbers such that there exists a subsequence $\{s_i\}$ of $\{n\}$ such that $s_{n_i} < s_{n_i+1}$ for all $i \in \mathbb{N}$. Let the sequence of $\{m_k\}$ be defined by $m_k = \max\{j \leq k : s_j < s_{j+1}\}$. Then, $\{m_k\}$ is a nondecreasing sequence satisfying $m_k \to \infty$ as $k \to \infty$ and the following properties hold:

$$s_{m_k} \leq s_{m_k+1} \text{ and } s_k \leq s_{m_k+1},$$

for all $k \geq N_0$, for some $N_0 > 0$.

Lemma 2.11. (Rockafellar [19]) Let C be a nonempty, closed and convex subset of a real Banach space E and let A be a monotone and hemicontinuous mapping from C into E^* with $C = D(A)$. Let $B : E \to 2^{E^*}$ be a mapping defined as follows:

$$Bv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

where $N_C(v) := \{w \in E^* : \langle v - u, w\rangle \geq 0, \forall u \in C\}$ is called the normal cone to C at $v \in C$. Then B is maximal monotone and $B^{-1}(0) = VI(A, C)$.

3 Main Results

The following assumptions will be used in the sequel.
Assumption 3.1.

(B1) Let C be a nonempty, closed and convex subset of a reflexive real Banach space E with its dual E^*;

(B2) Let $T_i : E \to E^*$, $i = 1, 2, \cdots, N$ be continuous f-pseudocontractive mappings;

(B3) Let $B_t : C \to E^*$, $t = 1, 2, \cdots, M$ be continuous monotone mappings;

(B4) Let $F_t : C \times C \to \mathbb{R}$, $t = 1, 2, \cdots, M$ be bi-functionals satisfying **Condition A**;

(B5) Let $\varphi_t : C \to \mathbb{R}$, $t = 1, 2, \cdots, M$ be real valued functions;

(B6) Let $A_j : C \to E^*$ be Lipschitz monotone mappings with Lipschitz constants L_j, for $j = 0, 1, 2, \ldots, K$.

(B7) Let the common set of solutions, denoted by $\mathcal{F}$, be nonempty, that is

$$\mathcal{F} := \left[\bigcap_{i=1}^{N} F_f(T_i)\right] \cap \left[\bigcap_{j=0}^{K} VI(C, A_j)\right] \cap \left[\bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)\right] \neq \emptyset.$$

(C1) Let f be a strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu > 0$ on bounded subsets of E.

Let $\{x_n\}$ be the sequence generated by the iterative scheme:

$$\begin{cases} u, x_0 \in C, \\ z_n = P_C^f \nabla f^*(\nabla f(x_n) - \lambda_n A_n x_n), \\ d_n = P_C^f \nabla f^*(\nabla f(x_n - \lambda_n A_n z_n), \\ u_n = T_{H_M}^{f,r_n} \circ T_{H_{N-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)), \end{cases} \tag{3.1}$$

where $A_n = A_{n \mod (K+1)}$ and ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\beta_n, \gamma_n \in [c, 1)$ for some $c > 0$, and $d_n = P_C^f \nabla f^*(\nabla f(x_n - \lambda_n A_n z_n)$, $0 < a \leq \lambda_n \leq b < \frac{\mu}{L}$, for $L = \max_{0\leq i\leq K} L_i$.

Lemma 3.1. Assume that Conditions $(B1) - (B7)$, and $(C1)$ hold. Then, the sequence $\{x_n\}$ generated by (3.1) is bounded.

Proof. Let $a_0 = b_0 = I$, where I is the identity mapping on E, $a_i = K_{T_i}^{f,r_n} \circ K_{T_{i-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n}$ for $i = 1, 2, \ldots, N$, and , $b_t = T_{H_t}^{f,r_n} \circ T_{H_{t-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n}$ for $t = 1, 2, \ldots, M$. Let $p \in \mathcal{F}$. Then, by Lemma 2.7 and 2.8, we get

$$\begin{aligned} D_f(p, u_n) &\leq D_f(p, b_{M-1}(x_n)) - D_f(u_n, b_{M-1}(x_n)) \\ &\leq D_f(p, b_{M-2}(x_n)) - D_f(b_{M-1}(x_n), b_{M-2}(x_n)) - D_f(u_n, b_{M-1}(x_n)), \end{aligned}$$

and, by induction we obtain

$$D_f(p, u_n) \leq D_f(p, x_n) - \sum_{t=0}^{M-1} D_f(b_{t+1}(x_n), b_t(x_n)). \tag{3.2}$$

Similarly,

$$D_f(p, v_n) \leq D_f(p, u_n) - \sum_{t=0}^{N-1} D_f(a_{t+1}(u_n), a_t(u_n)). \tag{3.3}$$

Thus, from (3.2), (3.3) and Lemma 2.6, we obtain

$$\begin{aligned} D_f(p,v_n) &\leq D_f(p,x_n) - \sum_{t=0}^{M-1} D_f(b_{t+1}(x_n), b_t(x_n)) - \sum_{i=0}^{N-1} D_f(a_{i+1}(u_n), a_i(u_n)) \\ &\leq D_f(p,x_n) - \frac{\mu}{2}\left(\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right) \qquad (3.4) \\ &\leq D_f(p,x_n). \qquad (3.5) \end{aligned}$$

Let $w_n = \nabla f^*(\nabla f(x_n) - \lambda_n A_n z_n)$. By Lemma 2.3 and the fact that $\lambda_n \leq \frac{\mu}{L}$, we get

$$\begin{aligned} D_f(p,d_n) &= D_f(p, P_C^f w_n) \leq D_f(p,w_n) - D_f(d_n,w_n) \qquad (3.6) \\ &= f(p) - f(w_n) - \langle p - w_n, \nabla f(w_n)\rangle - [f(d_n) - f(w_n) - \langle d_n - w_n, \nabla f(w_n)\rangle] \\ &= f(p) - \langle p - d_n, \nabla f(w_n)\rangle - f(d_n) \\ &= f(p) - \langle p - d_n, \nabla f(x_n) - \lambda_n A_n z_n\rangle - f(d_n) \\ &= f(p) - \langle p - d_n, \nabla f(x_n)\rangle + \langle p - d_n, \lambda_n A_n z_n\rangle - f(d_n) \\ &= f(p) - \langle p - x_n, \nabla f(x_n)\rangle - f(x_n) - [f(d_n) - \langle d_n - x_n, \nabla f(x_n)\rangle - f(x_n)] \\ &\quad + \langle p - d_n, \lambda_n A_n z_n\rangle \\ &= D_f(p,x_n) - D_f(d_n,x_n) + \langle p - d_n, \lambda_n A_n z_n\rangle \\ &= D_f(p,x_n) - D_f(d_n,x_n) + \langle p - z_n, \lambda_n A_n z_n\rangle + \langle z_n - d_n, \lambda_n A_n z_n\rangle \\ &= D_f(p,x_n) - D_f(d_n,x_n) + \lambda_n \langle p - z_n, A_n z_n - A_n p\rangle \\ &\quad + \lambda_n \langle p - z_n, A_n p\rangle + \langle z_n - d_n, \lambda_n A_n z_n\rangle \\ &\leq D_f(p,x_n) - D_f(d_n,x_n) + \langle z_n - d_n, \lambda_n A_n z_n\rangle. \end{aligned}$$

Now, from (2.2), we obtain

$$D_f(d_n,x_n) = D_f(d_n,z_n) + D_f(z_n,x_n) + \langle d_n - z_n, \nabla f(z_n) - \nabla f(x_n)\rangle. \qquad (3.7)$$

Thus, from (3.6), (3.7) and Lemma 2.6, we get

$$\begin{aligned} D_f(p,d_n) &\leq D_f(p,x_n) - D_f(d_n,z_n) - D_f(z_n,x_n) + \langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n)\rangle \\ &\leq D_f(p,x_n) - \frac{\mu}{2}\left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right] \qquad (3.8) \\ &\quad + \langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n)\rangle. \end{aligned}$$

Using the fact that A_i is Lipschitz monotone for $i = 0,1,2,\ldots,K$ and Lemma 2.3, we have that

$$\begin{aligned} \langle z_n - d_n, \lambda_n A_n z_n + \nabla f(z_n) - \nabla f(x_n)\rangle &= \langle d_n - z_n, \lambda_n A_n x_n - \lambda_n A_n z_n\rangle \qquad (3.9) \\ &\quad + \langle d_n - z_n, \nabla f(x_n) - \lambda_n A_n x_n - \nabla f(z_n)\rangle \\ &\leq \lambda_n \langle d_n - z_n, A_n x_n - A_n z_n\rangle \\ &\leq \lambda_n \|d_n - z_n\| \|A_n x_n - A_n z_n\| \\ &\leq L\lambda_n \|d_n - z_n\| \|x_n - z_n\| \\ &\leq \frac{1}{2} L\lambda_n \left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right]. \end{aligned}$$

Thus, from (3.8), (3.9) and the fact that $\lambda_n \leq \frac{\mu}{L}$, we get

$$\begin{aligned} D_f(p,d_n) &\leq D_f(p,x_n) - \frac{1}{2}(\mu - L\lambda_n)\left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right] \qquad (3.10) \\ &\leq D_f(p,x_n). \qquad (3.11) \end{aligned}$$

By (3.4), (3.10), $\lambda_n \leq \frac{\mu}{L}$ and Lemma 2.1, we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n))) \\ &\leq \alpha_n D_f(p,u) + \theta_n D_f(p,x_n) + \beta_n D_f(p,d_n) + \gamma_n D_f(p,v_n) \\ &\leq \alpha_n D_f(p,u) + (1-\alpha_n) D_f(p,x_n) \qquad (3.12) \\ &\quad -\frac{1}{2}\beta_n(\mu - L\lambda_n)\left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right] \\ &\quad -\gamma_n \frac{\mu}{2}\left[\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right] \\ &\leq \alpha_n D_f(p,u) + (1-\alpha_n) D_f(p,x_n) \\ &\leq \max\{D_f(p,u), D_f(p,x_n)\}. \qquad (3.13) \end{aligned}$$

Therefore, by induction, we get

$$D_f(p, x_n) \leq \max\{D_f(p,u), D_f(p,x_0)\}, \text{ for all } n \geq 0. \qquad (3.14)$$

This implies that $\{D_f(p, x_n)\}$ is bounded. Therefore, by Lemma 2.4 we have, $\{x_n\}$ is bounded and also the sequences $\{z_n\}$, $\{d_n\}$, $\{u_n\}$ and $\{v_n\}$ are bounded.

Theorem 3.2. Assume that Conditions $(B1)-(B7)$ and $(C1)$ hold. Then, the sequence $\{x_n\}$ generated by (3.1) converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

Proof. Let $p = P^f_{\mathcal{F}} u$. From (2.5), (2.6), (3.4), (3.10) and Lemma 2.1, we obtain

$$\begin{aligned} D_f(p, x_{n+1}) &= D_f(p, \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n))) \\ &= V_f(p, \alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)) \\ &\leq V_f(p, \alpha_n \nabla f(p) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)) \\ &\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p)) - \nabla f(u)\rangle \\ &= D_f(p, \nabla f^*(\alpha_n \nabla f(p) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n))) \\ &\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(u)\rangle \\ &\leq \alpha_n D_f(p,p) + \theta_n D_f(p,x_n) + \beta_n D_f(p,d_n) + \gamma_n D_f(p,v_n) \\ &\quad -\alpha_n \langle x_{n+1} - p, \nabla f(p) - \nabla f(u)\rangle \\ &= (1-\alpha_n) D_f(p,x_n) - \frac{1}{2}\beta_n(\mu - L\lambda_n)\left[\|d_n - z_n\|^2 + \|x_n - z_n\|^2\right] \qquad (3.15) \\ &\quad -\gamma_n \frac{\mu}{2}\left[\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right] \\ &\quad +\alpha_n \langle x_{n+1} - p, \nabla f(u) - \nabla f(p)\rangle \\ &\leq (1-\alpha_n) D_f(p,x_n) + \alpha_n \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle \\ &\quad +\alpha_n \langle x_{n+1} - x_n, \nabla f(u) - \nabla f(p)\rangle \\ &\leq (1-\alpha_n) D_f(p,x_n) + \alpha_n \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle \qquad (3.16) \\ &\quad +\alpha_n \|x_{n+1} - x_n\| \|\nabla f(u) - \nabla f(p)\|. \end{aligned}$$

Now, we divide the rest of the proof into two parts as follows.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{D_f(p, x_n)\}$ is decreasing for all $n \geq n_0$. It then follows that $\{D_f(p, x_n)\}$ is convergent and hence $D_f(p, x_n) - D_f(p, x_{n+1}) \to 0$ as $n \to \infty$. Thus, from (3.15) and the conditions on α_n, β_n, γ_n, and λ_n, we get

$$\lim_{n\to\infty} \|d_n - z_n\|^2 + \|x_n - z_n\|^2 = 0, \qquad (3.17)$$

and

$$\lim_{n\to\infty} \left[\sum_{t=0}^{M-1} \|b_{t+1}(x_n) - b_t(x_n)\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_n) - a_i(u_n)\|^2\right] = 0, \qquad (3.18)$$

which imply

$$\lim_{n\to\infty}\|d_n - z_n\| = \lim_{n\to\infty}\|x_n - z_n\| = 0, \text{ and hence, } \lim_{n\to\infty}\|x_n - d_n\| = 0, \tag{3.19}$$

$$\lim_{n\to\infty}\|b_{t+1}(x_n) - b_t(x_n)\| = 0, \qquad 0 \le t \le M-1, \text{ and hence, } \lim_{n\to\infty}\|u_n - x_n\| = 0, \tag{3.20}$$

and

$$\lim_{n\to\infty}\|a_{i+1}(u_n) - a_i(u_n)\| = 0, \qquad 0 \le i \le N-1, \text{ and hence, } \lim_{n\to\infty}\|v_n - u_n\| = 0. \tag{3.21}$$

Now,

$$\begin{aligned} \|\nabla f(x_{n+1}) - \nabla f(x_n)\| &= \|\left(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \beta_n \nabla f(d_n) + \gamma_n \nabla f(v_n)\right) - \nabla f(x_n)\| \\ &\le \alpha_n \|\nabla f(u) - \nabla f(x_n)\| + \beta_n \|\nabla f(d_n) - \nabla f(x_n)\| \\ &\quad + \gamma_n \|\nabla f(v_n) - \nabla f(x_n)\|, \end{aligned} \tag{3.22}$$

and from (3.19), (3.20), (3.21), the fact that $\alpha_n \to 0$ as $n \to \infty$ and uniform continuity of ∇f, we get $\|\nabla f(x_{n+1}) - \nabla f(x_n)\| \to 0$ as $n \to \infty$. Moreover, the uniform continuity of ∇f^* implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.23}$$

Now, for $j = 0, 1, \dots, K$, we have

$$\|d_{n+j} - x_n\| \le \|d_{n+j} - x_{n+j}\| + \sum_{l=n}^{n+j-1} \|x_{l+1} - x_l\|. \tag{3.24}$$

Then, from (3.19), (3.23) and (3.24), we obtain that

$$\lim_{n\to\infty}\|d_{n+j} - x_n\| = 0, \text{ for } j = 0, 1, \dots, K. \tag{3.25}$$

Since $\{x_n\}$ is bounded in E, there exists $q \in E$ and a subsequence $\{x_{n_s}\}$ of $\{x_n\}$ such that $x_{n_s} \rightharpoonup q$ and

$$\limsup_{n\to\infty} \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle = \lim_{s\to\infty} \langle x_{n_s} - p, \nabla f(u) - \nabla f(p)\rangle. \tag{3.26}$$

Then, from (3.20), (3.21) and (3.25), we have that $b_t(x_{n_s}) \rightharpoonup q$, $a_i(u_{n_s}) \rightharpoonup q$, $d_{n_s+j} \rightharpoonup q$ for $t \in \{1, 2, \dots, M\}$, $i \in \{1, 2, \dots, N\}$ and $j \in \{1, 2, \dots, K\}$. Now, we show that $q \in \mathcal{F}$.

Step 1. First we show that $q \in \bigcap_{j=0}^{K} VI(C, A_j)$.

Let

$$B_j v = \begin{cases} A_j v + N_C v, & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C, \end{cases}$$

where N_C is the normal cone to C at $v \in C$ given by $N_C = \{w \in E^* : \langle v - x, w\rangle \ge 0, \forall x \in C\}$. Then, by Lemma 2.11, B_j is maximal monotone and $B_j^{-1}(0) = VI(C, A_j)$. Let $w \in B_j v$. Then, we have $w \in A_j v + N_C v$ and hence $w - A_j v \in N_C v$. Thus, we obtain that

$$\langle v - x, w - A_j v\rangle \ge 0, \forall x \in C. \tag{3.27}$$

Let $\{n_s + j\}, s \ge 1$ be such that $A_{n_s+j} = A_j$ for all $s \in \mathbb{N}$ where $j = 0, 1, 2, \dots, K$. Then, since $d_{n_s+j} = P_C^f \nabla f^*(\nabla f(x_{n_s+j}) - \lambda_{n_s+j} A_j z_{n_s+j})$, and $v \in C$, we have

$$\langle v - d_{n_s+j}, \nabla f(d_{n_s+j}) - (\nabla f(x_{n_s+j}) - \lambda_{n_s+j} A_j z_{n_s+j})\rangle \ge 0,$$

and so

$$\left\langle v - d_{n_s+j}, \frac{\nabla f(d_{n_s+j}) - \nabla f(x_{n_s+j})}{\lambda_{n_s+j}} + A_j z_{n_s+j} \right\rangle \ge 0. \tag{3.28}$$

From (3.27), (3.28) and A_j is monotone mapping, we get that

$$\begin{aligned}
\langle v-d_{n_s+j},w\rangle &\geq \langle v-d_{n_s+j},A_jv\rangle\\
&\geq \langle v-d_{n_s+j},A_jv\rangle-\left\langle v-d_{n_s+j},\frac{\nabla f(d_{n_s+j})-\nabla f(x_{n_s+j})}{\lambda_{n_s+j}}+A_jz_{n_s+j}\right\rangle\\
&= \langle v-d_{n_s+j},A_jv-A_jd_{n_s+j}\rangle+\langle v-d_{n_s+j},A_jd_{n_s+j}-A_jz_{n_s+j}\rangle\\
&\quad -\left\langle v-d_{n_s+j},\frac{\nabla f(d_{n_s+j})-\nabla f(x_{n_s+j})}{\lambda_{n_s+j}}\right\rangle\\
&\geq \langle v-d_{n_s+j},A_jd_{n_s+j}-A_jz_{n_s+j}\rangle-\left\langle v-d_{n_s+j},\frac{\nabla f(d_{n_s+j})-\nabla f(x_{n_s+j})}{\lambda_{n_s+j}}\right\rangle\\
&\geq \langle v-d_{n_s+j},A_jd_{n_s+j}-A_jz_{n_s+j}\rangle-\|v-d_{n_s+j}\|\frac{\|\nabla f(d_{n_s+j})-\nabla f(x_{n_s+j})\|}{\lambda_{n_s+j}}\\
&\geq \langle v-d_{n_s+j},A_jd_{n_s+j}-A_jz_{n_s+j}\rangle-R\frac{\|\nabla f(d_{n_s+j})-\nabla f(x_{n_s+j})\|}{\lambda_{n_s+j}}, \qquad (3.29)
\end{aligned}$$

where $R=\max\limits_{0\leq j\leq K}\sup\limits_{s\geq 0}\|v-d_{n_s+j}\|$. Taking limits on both sides of the inequality (3.29) as $s\to\infty$ and using the fact that $\lambda_n\geq a>0$, for all $n\geq 0$, ∇f is uniformly continuous, and (3.19), we get that $\langle v-q,w\rangle\geq 0$ as $s\to\infty$ for each j. Therefore, the maximality of B_j gives that $q\in B_j^{-1}(0)=VI(C,A_j)$ for each j. Therefore, $q\in\bigcap\limits_{j=0}^{K}VI(C,A_j)$.

Step 2. We show that $q\in\bigcap\limits_{j=1}^{N}F_f(T_j)$. Let $a_i(u_{n_s})=K_{T_i}^{f,r_{n_s}}a_{i-1}(u_{n_s})$. By Lemma 2.8 (2), we get that

$$\langle y-a_i(u_{n_s}),T_ia_i(u_{n_s})\rangle-\frac{1}{r_{n_s}}\langle y-a_i(u_{n_s}),(1+r_{n_s})\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\rangle\leq 0,\forall y\in C.$$

Since C is convex, $y_\lambda=\lambda y+(1-\lambda)q\in C$, where $\lambda\in[0,1]$ and $y\in C$. Thus,

$$\begin{aligned}
\langle a_i(u_{n_s})-y_\lambda,T_iy_\lambda\rangle &\geq \langle a_i(u_{n_s})-y_\lambda,T_iy_\lambda\rangle+\langle y_\lambda-a_i(u_{n_s}),T_ia_i(u_{n_s})\rangle\\
&\quad -\frac{1}{r_{n_s}}\langle y_\lambda-a_i(u_{n_s}),(1+r_{n_s})\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\rangle\\
&= \langle a_i(u_{n_s})-y_\lambda,T_iy_\lambda-T_ia_i(u_{n_s})\rangle\\
&\quad -\frac{1}{r_{n_s}}\langle y_\lambda-a_i(u_{n_s}),(1+r_{n_s})\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\rangle\\
&\geq \langle a_i(u_{n_s})-y_\lambda,\nabla f(y_\lambda)-\nabla f(a_i(u_{n_s}))\rangle\\
&\quad -\frac{1}{r_{n_s}}\langle y_\lambda-a_i(u_{n_s}),(1+r_{n_s})\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\rangle\\
&= \langle a_i(u_{n_s})-y_\lambda,\nabla f(y_\lambda)\rangle\\
&\quad -\frac{1}{r_{n_s}}\langle y_\lambda-a_i(u_{n_s}),\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\rangle\\
&\geq \langle a_i(u_{n_s})-y_\lambda,\nabla f(y_\lambda)\rangle\\
&\quad -\|y_\lambda-a_i(u_{n_s})\|\frac{\|\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\|}{r_{n_s}}\\
&\geq \langle a_i(u_{n_s})-y_\lambda,\nabla f(y_\lambda)\rangle \qquad (3.30)\\
&\quad -W\frac{\|\nabla f(a_i(u_{n_s}))-\nabla f(a_{i-1}(u_{n_s}))\|}{r_{n_s}},
\end{aligned}$$

where $W=\max\limits_{1\leq i\leq N}\sup\limits_{s\geq 0}\|y_\lambda-a_i(u_{n_s})\|$. From the facts that $a_i(u_{n_s})\rightharpoonup q$, ∇f is uniformly continuous, (3.21), $r_n\geq c_1$, for all $n\geq 0$ and taking the limits on both sides of the inequality (3.30) as $s\to\infty$, we obtain that

$$\langle q-y_\lambda,T_iy_\lambda\rangle\geq\langle q-y_\lambda,\nabla f(y_\lambda)\rangle. \qquad (3.31)$$

Thus, from inequality (3.31), we obtain

$$\langle q-y, T_i(q+\lambda(y-q))\rangle \geq \langle q-y, \nabla f(q+\lambda(y-q))\rangle, \forall y \in E. \tag{3.32}$$

Using the fact that T_i is continuous and ∇f is uniformly continuous on bounded subset of E and letting $\lambda \downarrow 0$, we have from inequality (3.32) that

$$\langle q-y, T_i q\rangle \geq \langle q-y, \nabla f(q)\rangle, \forall y \in C \Leftrightarrow 0 \geq \langle q-y, \nabla f(q) - T_i q\rangle, \forall y \in E. \tag{3.33}$$

Now, set $y = \nabla f^*(T_i q)$. Since E is reflexive and ∇f^* is monotone, we get that

$$\langle q - \nabla f^*(T_i q), \nabla f(q) - T_i q\rangle = 0, \tag{3.34}$$

which implies that $T_i q = \nabla f(q)$. Hence $q \in F_f(T_i)$, for each $i = 1, 2, \ldots, N$ and $q \in \bigcap_{i=1}^{N} F_f(T_i)$.

Step 3. We show that $q \in \bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)$.

Set $b_t(x_{n_s}) = T_{H_t}^{f,r_{n_s}} b_{t-1}(x_{n_s})$. Then,

$$H_t(b_t(x_{n_s}), y) + \frac{1}{r_{n_s}} \langle y - b_t(x_{n_s}), \nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s}))\rangle \geq 0, \forall y \in C.$$

Thus, by Condition (A2), we have

$$\begin{aligned} H_t(y, b_t(u_{n_s})) \leq -H_t(b_t(x_{n_s}), y) &\leq \frac{1}{r_{n_s}} \langle y - b_t(x_{n_s}), \nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s}))\rangle \\ &\leq \|y - b_t(x_{n_s})\| \frac{\|\nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s}))\|}{r_{n_s}} \\ &\leq P \frac{\|\nabla f(b_t(x_{n_s})) - \nabla f(b_{t-1}(x_{n_s})\|}{r_{n_s}}, \end{aligned} \tag{3.35}$$

where $P = \max_{1\leq t\leq M} \sup_{s\geq 0} \|y - b_t(x_{n_s})\|$. From the facts that $b_t(x_{n_s}) \rightharpoonup q$, **Condition A** (A4), $r_n \geq c_1$, for all $n \geq 0$ and taking limits on both sides of the inequality (3.35) as $s \to \infty$, we obtain that

$$H_t(y, q) \leq 0, \forall y \in C. \tag{3.36}$$

Set $y_\lambda = \lambda y + (1-\lambda)q, \lambda \in (0,1]$ and $y \in C$. Consequently, we get $y_\lambda \in C$. From (3.36) and **Condition A** (A1), we obtain

$$\begin{aligned} 0 &= H_t(y_\lambda, y_\lambda) \leq \lambda H_t(y_\lambda, y) + (1-\lambda) H_t(y_\lambda, q) \\ &\leq H_t(q + \lambda(q-y), y). \end{aligned} \tag{3.37}$$

If $\lambda \downarrow 0$, using **Condition A** (A3), we have

$$H_t(q, y) \geq 0, \forall y \in C.$$

Hence, $q \in GMEP(F_t, \varphi_t, B_t)$, for each $t = 1, 2, \ldots\ldots, M$. Therefore, $q \in \bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_t)$.

Finally, we show that $\{x_n\}$ converge strongly to the point p.

From (3.26) and Lemma 2.3, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} \langle x_n - p, \nabla f(u) - \nabla f(p)\rangle &= \lim_{s\to\infty} \langle x_{n_s} - p, \nabla f(u) - \nabla f(p)\rangle \\ &= \langle q - p, \nabla f(u) - \nabla f(p)\rangle \leq 0. \end{aligned} \tag{3.38}$$

Thus, using (3.16), (3.23), (3.38) and Lemma 2.9, we conclude that

$$\lim_{n\to\infty} D_f(p, x_n) = 0.$$

Hence, Lemma 2.5 implies that $x_n \to p$ as $n \to \infty$.

Case 2. Suppose that there exists $\{n_s\}$ of $\{n\}$ such that $D_f(p, x_{n_s}) < D_f(p, x_{n_s+1})$, for all $s \geq 0$. It follows from Lemma 2.10 that there exists a nondecreasing sequence$\{k_s\} \subset \mathbb{N}$ such that $k_s \to \infty$ as $s \to \infty$ and

$$\max\{D_f(p, x_{k_s}), D_f(p, x_s)\} < D_f(p, x_{k_s+1}), \tag{3.39}$$

for all $s \geq 0$. Thus, from (3.15) and the conditions on α_n, β_n, γ_n, and λ_n, we get

$$\lim_{n\to\infty} \|d_{k_s} - z_{k_s}\|^2 + \|x_{k_s} - z_{k_s}\|^2 = 0, \tag{3.40}$$

and

$$\lim_{s\to\infty} \left[\sum_{t=0}^{M-1} \|b_{t+1}(x_{k_s}) - b_t(x_{k_s})\|^2 + \sum_{i=0}^{N-1} \|a_{i+1}(u_{k_s}) - a_{i-1}(u_{k_s})\|^2\right] = 0. \tag{3.41}$$

Then

$$\lim_{s\to\infty} \|d_{k_s} - z_{k_s}\| = \lim_{s\to\infty} \|x_{k_s} - z_{k_s}\| = 0 \text{ and hence } \lim_{s\to\infty} \|x_{k_s} - d_{k_s}\| = 0, \tag{3.42}$$

$$\lim_{s\to\infty} \|b_{t+1}(x_{k_s}) - b_t(x_{k_s})\| = 0, \quad 0 \leq t \leq M-1, \lim_{s\to\infty} \|u_{k_s} - x_{k_s}\| = 0, \tag{3.43}$$

and

$$\lim_{s\to\infty} \|a_i(u_{k_s}) - a_{i-1}(u_{k_s})\| = 0, \quad 0 \leq i \leq N-1, \lim_{s\to\infty} \|v_{k_s} - u_{k_s}\| = 0. \tag{3.44}$$

Moreover, following the methods used in ***Case 1***, we get

$$\limsup_{s\to\infty} \langle x_{k_s} - p, \nabla f(u) - \nabla f(p)\rangle \leq 0. \tag{3.45}$$

Therefore, from (3.16), (3.23), (3.45) and Lemma 2.9, we obtain that

$$\lim_{s\to\infty} D_f(p, x_{k_s}) = 0. \tag{3.46}$$

This together with (3.16) imply that

$$\lim_{s\to\infty} D_f(p, x_{k_s+1}) = 0. \tag{3.47}$$

Thus, from (3.39), and (3.47) we have that

$$\lim_{s\to\infty} D_f(p, x_s) = 0.$$

This together with Lemma 2.5 imply that $x_s \to p$ as $s \to \infty$. Therefore, from ***Case 1*** and ***Case 2***, we can conclude that $\{x_n\}$ converges strongly to the point p in $\mathcal{F}$. The proof is complete.

We note that the method of proof of Theorem 3.2 provides the following theorem for approximating a common solution of f-fixed point, variational inequality and generalized mixed equilibrium problems in real Banach spaces.

Theorem 3.3. Assume that Conditions $(B1)-(B7)$ and $(C1)$ are satisfied with $N = K = M = 1$. Then, the sequence $\{x_n\}$ generated by (3.1) with $N = K = M = 1$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

If, in Theorem 3.2, we assume that $A_j \equiv 0$, for $j = 0, 1, 2, \ldots, K$, then Theorem 3.2 provides the following corollary.

Corollary 3.4. Assume that Conditions $(B1)-(B5)$, and $(C1)$ hold.
Let $\mathcal{F} := \left[\bigcap_{i=1}^N F_f(T_i)\right] \cap \left[\bigcap_{t=1}^M GMEP(F_t, \varphi_t, B_t)\right] \neq \emptyset$. Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$ by

$$\begin{cases} u_n = T_{H_M}^{f,r_n} \circ T_{H_{M-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \gamma_n \nabla f(v_n)), \end{cases} \tag{3.48}$$

where ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^\infty \alpha_n = \infty$ and $\gamma_n \in [c, 1)$ for some $c > 0$. Then, the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

If, in Corollary 3.4, we assume that $F_i \equiv 0$, for $i = 1, 2, \dots, K$, then Corollary 3.2 provides the following corollary for approximating the common solution of a finite family of mixed variational inequality of Browder type problems for continuous monotone mappings and f-fixed point problems for continuous f-pseudocontractive mapping in a reflexive real Banach space.

Corollary 3.5. Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$ by

$$\begin{cases} u_n = T_{H_M}^{f,r_n} \circ T_{H_{M-1}}^{f,r_n} \circ \cdots \circ T_{H_2}^{f,r_n} \circ T_{H_1}^{f,r_n} x_n, \\ v_n = K_{T_N}^{f,r_n} \circ K_{T_{N-1}}^{f,r_n} \circ \cdots \circ K_{T_2}^{f,r_n} \circ K_{T_1}^{f,r_n} u_n, \\ x_{n+1} = \nabla f^*(\alpha_n \nabla f(u) + \theta_n \nabla f(x_n) + \gamma_n \nabla f(v_n)), \end{cases} \tag{3.49}$$

where ∇f is the gradient of f on E; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \gamma_n \in (0,1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\gamma_n \in [c, 1)$ for some $c > 0$. If the Conditions $(B1) - (B3)$, $(B5)$ and $(C1)$ are satisfied and $\mathcal{F} := \left[\bigcap_{i=1}^{N} F_f(T_i)\right] \cap \left[\bigcap_{t=1}^{M} VI(B_t, \varphi_t, C)\right] \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the Bregman distance.

If we assume that E is smooth and strictly convex, then $f(x) = \frac{1}{2}\|x\|^2$ is strongly coercive, bounded and uniformly Fréchet differentiable Legendre function which is strongly convex with constant $\mu = 1$ and conjugate $f^*(x^*) = \frac{1}{2}\|x^*\|^2$. In this case, we have $\nabla f = J$, $\nabla f^* = J^{-1}$ and for $r > 0$ and $x \in E$, we have

$$T_H^r x = \{z \in C : H(z, y) + \frac{1}{r} \langle y - z, J(z) - J(x)\rangle \geq 0, \forall y \in C\}, \tag{3.50}$$

where $H(z, y) := F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$, and

$$K_T^r x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r} \langle y - z, (1 + r)J(z) - J(x)\rangle \leq 0, \forall y \in C\}. \tag{3.51}$$

In this case, Theorem 3.2 reduces to the following corollary:

Corollary 3.6. Let C be nonempty, closed and convex subset of a smooth and strictly convex reflexive real Banach space E with its dual E^*. Assume that Conditions $(B1) - (B7)$ hold. Let $\{x_n\}$ be a sequence generated from arbitrary $u_0, x_0 \in C$ by

$$\begin{cases} z_n = \Pi_C J^{-1}(J(x_n) - \lambda_n A_n x_n) \\ d_n = \Pi_C J^{-1}(J(x_n - \lambda_n A_n z_n), \\ u_n = T_{H_M}^{r_n} \circ T_{H_{N-1}}^{r_n} \circ \cdots \circ T_{H_2}^{r_n} \circ T_{H_1}^{r_n} x_n, \\ v_n = K_{T_N}^{r_n} \circ K_{T_{N-1}}^{r_n} \circ \cdots \circ K_{T_2}^{r_n} \circ K_{T_1}^{r_n} u_n, \\ x_{n+1} = J^{-1}(\alpha_n J(u) + \theta_n J(x_n) + \beta_n J(d_n) + \gamma_n J(v_n)), \end{cases} \tag{3.52}$$

where $A_n = A_{n \mod (K+1)}$, and Π_C is the generalized metric projection from E onto C; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$, $\lim\limits_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\beta_n, \gamma_n \in [c, 1)$ for some $c > 0$, and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$, for $L = \max\limits_{0 \leq i \leq K} L_i$. Then, the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the generalized metric projection.

If, in Corollary 3.6, we assume that $E = H$, a real Hilbert space, and $f(x) = \frac{1}{2}\|x\|^2$, then we have $\nabla f = J = I$ and $\nabla f^* = J^{-1} = I$, were I is identity mapping on H. Moreover, f-pseudocontractive mapping reduces to pseudocontractive mapping. In this case, for $r > 0$ and $x \in E$, we have

$$T_H^r x = \{z \in C : H(z, y) + \frac{1}{r} \langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \tag{3.53}$$

where $H(z, y) := F(z, y) + \varphi(y) - \varphi(z) + \langle y - z, Bz\rangle$, and

$$K_T^r x = \{z \in C : \langle y - z, T(z)\rangle - \frac{1}{r} \langle y - z, (1 + r)z - x\rangle \leq 0, \forall y \in C\}. \tag{3.54}$$

Thus, we have the following corollary.

Corollary 3.7. Let C be a nonempty, closed and convex subset of a real Hilbert space H and let $T_i : H \to H$, $i = 1, 2, \cdots, N$ be continuous pseudocontractive mappings. Let $\{x_n\}$ be a sequence generated from an arbitrary $u, x_0 \in C$ by

$$\begin{cases} z_n = P_C(x_n - \lambda_n A_n x_n) \\ d_n = P_C(x_n - \lambda_n A_n z_n), \\ u_n = T^{r_n}_{H_M} \circ T^{r_n}_{H_{M-1}} \circ \cdots \circ T^{r_n}_{H_2} \circ T^{r_n}_{H_1} x_n, \\ v_n = K^{r_n}_{T_N} \circ K^{r_n}_{T_{N-1}} \circ \cdots \circ K^{r_n}_{T_2} \circ K^{r_n}_{T_1} u_n, \\ x_{n+1} = \alpha_n u + \theta_n x_n + \beta_n d_n + \gamma_n v_n, \end{cases} \tag{3.55}$$

where $A_n = A_{n \mod (K+1)}$, and P_C is metric projection of H onto C; $\{r_n\} \subset [c_1, \infty)$ for some $c_1 > 0$, $\alpha_n, \theta_n, \beta_n, \gamma_n \in (0, 1)$, $\forall n \geq 0$ such that $\alpha_n + \theta_n + \beta_n + \gamma_n = 1$, $\lim_{n\to\infty} \alpha_n = 0$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\beta_n, \gamma_n \in [c, 1)$ for some $c > 0$, and $0 < a \leq \lambda_n \leq b < \frac{1}{L}$, for $L = \max_{0\leq i\leq K} L_i$. If Conditions $(B3) - (B7)$ are satisfied, then the sequence $\{x_n\}$ converges strongly to p in $\mathcal{F}$ which is nearest to u with respect to the metric projection.

4 Numerical Example

In this section, we present an example to illustrate the main result of our paper.

Example 4.1. Let $E = L_2^{\mathbb{R}}([0, 1])$ with norm $\|x\|_{L_2^{\mathbb{R}}} = (\int_0^1 |x(s)|^2 ds)^{\frac{1}{2}}$, for $x \in E$ and $C = \{x \in E : \|x\|_{L_2^{\mathbb{R}}} \leq 1\}$. Define $f : E \to \mathbb{R}$ by $f(x) = \frac{1}{2}\|x\|^2_{L_2^{\mathbb{R}}}$, then $\nabla f = J = I$ and $\nabla f^* = J = I$, where I is identity mapping on E. Let A_j, T_i, $B_t : C \to E$ be defined by $A_j(x)(s) = (1 + j)\nabla f(x)(s)$, $j = 0, 1, \ldots, K$; $T_i(x)(s) = -s^i \nabla f(x)(s)$, $i = 1, \ldots, N$ and $B_t(x)(s) = \frac{t+1}{2t+1}\nabla f(x)(s)$, $t = 1, \ldots, M$, for all $x(s) \in C, s \in [0, 1]$, respectively. Let $F_t : C \times C \to \mathbb{R}$ be defined by $F_t(x, y) = \frac{t}{2t+1} \langle y - x, \nabla f(x)\rangle, \forall x, y \in C$. Then A_j, for $j = 0, 1, \ldots, K$ are Lipschitz monotone mappings with $\bigcap_{j=0}^{K} VI(C, A_j) = \{0\}$; T_i, for $i = 1, \ldots, N$ are continuous f-pseudocontractive with $\bigcap_{i=1}^{N} F_f(T_i) = \{0\}$; B_t, for $t = 1, \ldots, M$ are continuous monotone mappings, and F_t, for $t = 1, \ldots, M$ are bi-function satisfying **Condition A**. Thus, a common solution set of the generalized equilibrium problems is $\bigcap_{t=1}^{M} GMEP(F_t, \varphi_t, B_k) = \{0\}$, where $\varphi_t \equiv$ constant. Now, for implementation, we choose $K = 0$, $N = M = 1$, $r_n = 1$, $\theta_n = \beta_n = \gamma_n = \frac{1}{3}(1 - \alpha_n)$, $\lambda_n = 0.00001 + \frac{1}{100n}$, for $n \geq 0$ and we compute the $(n + 1)^{th}$ iteration as follows:

$$\begin{cases} z_n(s) = \min\{1, \frac{1}{\|w_n\|_{L_2^{\mathbb{R}}}}\} w_n(s), \\ d_n(s) = \min\{1, \frac{1}{\|h_n\|_{L_2^{\mathbb{R}}}}\} h_n(s), \\ u_n(s) = \frac{1}{r_n+1} x_n(s), \\ v_n(s) = \frac{1}{1+r_n(1+s)} u_n, \\ x_{n+1}(s) = \alpha_n u(s) + \theta_n x_n(s) + \beta_n d_n(s) + \gamma_n v_n(s), \end{cases} \tag{4.1}$$

where $w_n(s) = x_n(s) - \lambda_n(1 + s)x_n(s)$ and $h_n(s) = x_n(s) - \lambda_n(1 + s)z_n(s)$.

Now, taking different initial points, $x_0(s) = 2s$, $x_0(s) = 2s^5$, $x_0(s) = 2s^{10}$ and fixed $u_0(s) = 2s^2$ in C and $\alpha_n = \frac{1}{10000n+10}$, the numerical experiment result provides that the sequence $\{\|x_n - p\|\}$ approaches zero as $n \to \infty$ (see, Figure 1 below), where $p = 0$. In this case, we observe that the sequence $\{x_n\}$ converges faster when the power of s gets large.

Next, we obtain the same numerical tests of algorithm 4.1 by taking initial points $u_0(s) = 2s^2$, $x_0(s) = 2s^{10}$ and different control parameters, $\alpha_n = \frac{1}{100n+10}$, $\alpha_n = \frac{1}{(100)^2 n+10}$, $\alpha_n = \frac{1}{(100)^3 n+10}$. In this case, we observe that the rate of convergence looks the same through out (see, Figure 2).

5 Conclusion

In this paper, we constructed a new algorithm to approximate a common element of the set of solutions of a finite family of generalized mixed equilibrium problems, the set of f-fixed points of a finite family of f-pseudocontractive mappings and the set of solutions of a finite family of variational inequality problems for a finite family of Lipschitz monotone mappings in reflexive real Banach spaces. We proved a strong convergence theorem for the developed algorithm in reflexive real Banach spaces. In addition, a numerical example is given to illustrate the implementability

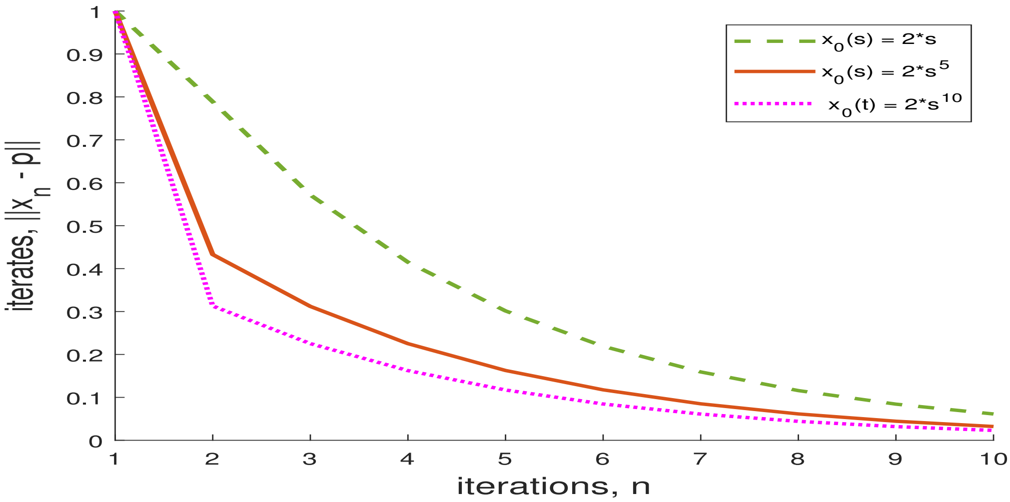

Figure 1: Figure 1: Convergence of the sequence $\{||x_n - p||\}$ as n gets large.

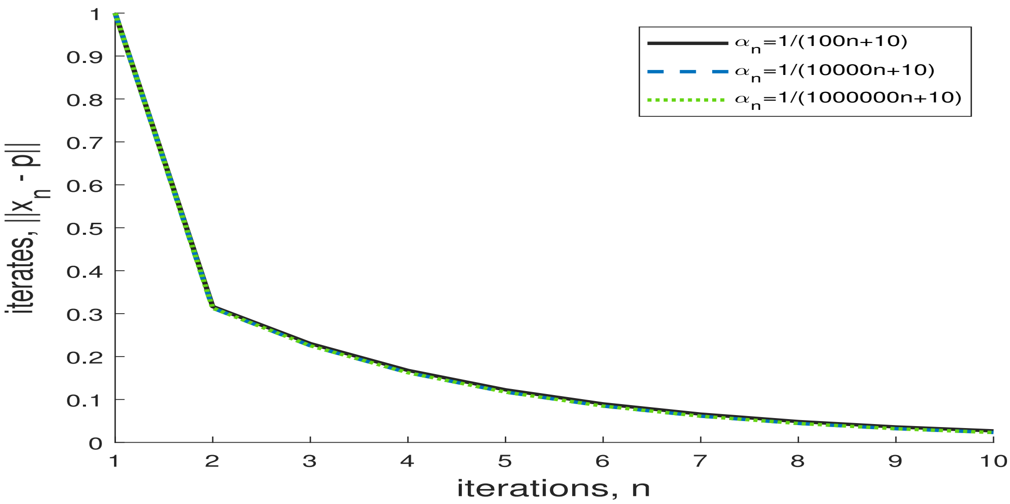

Figure 2: Figure 2: Convergence of the sequence $\{||x_n - p||\}$ as n gets large.

of our algorithm. Specifically, the result of our method improve the result obtained by Shahzad and Zegeye [21] from a *Hilbert spaces* to a *reflexive Banach spaces*, from continuous pseudocontractive to *continuous f-pseudocontractive* and from *equilibrium problem* to *generalized mixed equilibrium problem.* In addition, Theorem 3.2 extends Theorem 3.1 of Bello and Nnakwe [2] from *2-uniformly convex and uniformly smooth spaces* to *reflexive Banach spaces*, from *continuous semi-pseudocontractive* to *continuous f-pseudocontractive* and from *equilibrium problem* to *generalized mixed equilibrium problem.*

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