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Fixed point theory in digital topology

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Abstract

In this paper, we review some research works on exploring image processing in digital spaces using fixed point theorems. The basic concepts of digital images are mentioned. Moreover, we prove some theorems on digital metric spaces by replacing the conditions in the previously established theorem with a suitable condition.

Keywords: Fixed point theorems, Banach contraction principle, digital images, digital contraction, digital metric space
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1 Introduction

Fixed point theory is a blend of several areas of mathematics such as functional analysis, mathematical analysis and topology. Fixed point theory plays an important role in mathematics and other disciplines like engineering, game theory,image processing, computer graphics, digital images. Digital topology is the study of the topological properties of image arrays.In general, if a function f under certain conditions has atleast a point X such that $f(x) = x$, then such a point is termed as a fixed point. This theory was originated from Brower [4, 1, 2] fixed point theorem (1910) in R^n space. Later Banach [3] stated his fixed point theorem (1922) that in a complete metric space, a contraction must map a point to itself and that point is unique. Kannan [16, 17], then relaxed the continuity of the map considered in Banach Contraction Principle in his paper in 1968. Zamfirescu [23] and Rhoades [8], consequently developed more general contractions for a complete metric space. These contractions have been generalised to the other spaces also by various authors [13, 14, 22, 21]. The digital version of the topological concept was given by Boxer [5, 6, 7]. Digital topology was first studied by Rosenfield [19]. Kong [18], then introduced the digital fundamental group of a discrete object. Boxer [8] has given the digital versions of several notions from topology and [7] studied a variety of digital continuous functions. Ege and Karaca [20] defined a digital metric space and proved the famous Banach Contraction Principle for digital images. Ege and Karaca [10, 11] gave relative and reduced Lefschetz fixed point theorem for digital images.

2 Preliminaries

The following definitions will be used in the sequel.

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Definition 2.1. Let Z be the set of all integers and let n be a positive integer. Define the set Z^n as follows:

$$Z^n = \{(x_1, x_2, \dots, x_n)/x_i \in Z, 1 \leq i \leq n\}$$

Z^n is also called the set of all lattice points in the n dimensional Euclidean space.

Definition 2.2. Consider any two distinct points $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ in Z^n. Let m be a positive integer such that $1 \leq m \leq n$. We say that the two points p and q are k_m-adjacent in Z^n if there are at most m indices i such that $|p_i$ - $q_i| = 1$ and for all other indices j such that $|p_j$ - $q_j| \neq 1$, we have $p_j = q_j$.

Definition 2.3. A digital image is an ordered pair (X, k), where X is a finite subset of Z^n for some positive integer n and k is an adjacency relation for the members of X.

The following are the basic notions in digital images.

Definition 2.4. A k-neighbour of a point $p \in (X, k)$ is a point of X that is k-adjacent to p,where $k \in \{2, 4, 6, 8, 18, 26\}$ and $X \subset Z^n$, n = 1, 2, 3.

Definition 2.5. A digital interval is defined by $[a, b]_Z = \{z \in Z/a \leq z \leq b\}$, where $a, b \in Z$ and a < b.

Definition 2.6. A digital image (X,k) is k-connected if and only if different points $x, y \in X$, there is a set $\{x_0, x_1, x_2, ..., x_r\}$ points of digital image (X,k) such that $x = x_0$ and $y = x_r$ and x_i and x_{i+1} are k-neighbours, where $i = 0, 1, 2, ..., r - 1$

Definition 2.7. Let $(X, k_0) \subset Z^m$ and $(Y, k_1) \subset Z^n$ be digital images and T : X → Y be a function. If for every k_0-connected subset A of X, T(A) is k_1-connected subset of Y, then T is said to be (k_0,k_1)-continuous.

Definition 2.8. If in the above definition 1.8, T is (k_0, k_1)-continuous, bijective and T^{-1} is (k_1, k_0)-continuous, then T is called (k_0, k_1)-isomorphism. We denote it by X $\cong_{(k_0,k_1)}$ Y.

Definition 2.9. A point $x \in (X, d, k)$ is called a fixed point of the mapping $T : X \to X$ if $Tx = x$.
Let (X,k) be a digital image. we say that (X,k) has the fixed point property [22] if every (k, k)-continuous map $T : (X, k) \to (X.k)$ has a fixed point.

Definition 2.10. Let $(X, k) \subset Z^n$ be a digital image. Define a function $d : X \times X \to [0, \infty)$ by

$$d(p, q) = \left[\sum_1^n (p_i - q_i)^2\right]^{\frac{1}{2}}$$

Then we have the following properties satisfied by d for all $x, y, z \in X$

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

The digital image (X, k) together with the function d is called a digital metric space with k- adjacency. It is denoted by (X, d, k)

Definition 2.11. A sequence$\{x_n\}_1^\infty$of points of a digital metric space (X, d, k) is a Cauchy sequence if for all $\epsilon > 0$, there exists $N \in Z^+$ such that for all $m, n > N$, we have $d(x_m, x_n) < \epsilon$

Definition 2.12. A digital metric space (X, d, k) is said to be a complete metric space if every Cauchy sequence $\{x_n\}_1^\infty$ of points of (X, d, k) converge to a point L of (X, d, k).

Definition 2.13. Let (X, d, k) be a digital metric space. A function $T : (X, d, k) \to (X, d, k)$ is called right continuous if $\lim_{x \to a^+} Tx = Ta$ where $a \in X$.

Definition 2.14. Let (X, d, k) be any digital metric space and $T : (X, d, k) \to (X, d, k)$ be a digital self map.If there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$, $d(Tx, Ty) \leq \lambda d(x, y)$, then T is called a digital contraction map. Also the constant λ is a contractive factor.

Definition 2.15. Let (X, d, k) be a digital metric space. A self map $T : (X, d, k) \to (X, d, k)$ is called a strict digital contraction if for all $x, y \in X$, $x \neq y$, $d(Tx, Ty) < d(x, y)$.

Definition 2.16. Let (X, d, k) be a digital metric space. A self map $T : (X, d, k) \to (X, d, k)$ is called a weekly uniformly strict digital contraction if given $\epsilon > 0$, there exists $\delta > 0$ such that $\epsilon \leq d(x, y) < \epsilon + \delta$ implies $d(Tx, Ty) < \epsilon$ for all $x, y \in X$.

Many of the basic theorems in metric fixed point theory are extended to a digital metric space. We mention some of the important ones.
Brouwer's fixed point theorem in one dimension for digital images is as follows

Theorem 2.17. Every $(2, 2)$-continuous function $T : ([0, 1]_Z, d, 2) \to ([0, 1]_Z, d, 2)$ has a fixed point, where $d(x, y) = \mid x - y \mid$ for all $x, y \in [0, 1]_Z$.

Brouwer's fixed point theorem in two dimensions for digital images is as follows

Theorem 2.18. Let $X = \{(0, 0), (1, 0), (0, 1), (1, 1)\} \subset Z^2$ be a digital image with 4-adjacency. Then every (4, 4)-continuous function $T : (X, d, 4) \to (X, d, 4)$ has a fixed point, where $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in X$.

Ege and Karaca [11] formulated and proved the Banach contraction mapping principle for digital images in 2015 as follows:

Theorem 2.19. Let (X, d, k) be a complete digital metric space. Let $T : (X, d, k) \to (X, d, k)$ be a digital contraction map. Then T has unique fixed point, that is there exists a unique point $z \in X$ such that $Tz = z$.

Ege and Karaca [11] further generalized the above theorem as stated below. We observe that if the function $\psi(t)$ is taken as $\psi(t) = \lambda(t)$, where $\lambda \in [0, 1)$, we get the Banach contraction mapping principle as stated in theorem 2.19.

Theorem 2.20. Let (X, d, k) be a complete digital metric space and let $T : (X, d, k) \to (X, d, k)$ be a digital self map. Assume that there exists a right continuous real function $\psi : [0, v] \to [0, v]$, where v is sufficiently large real number such that $\psi(a) < a$ if $a > 0$ and let T satisfies $d\,(Tx_1, Tx_2) \leq \psi(d(x_1, x_2))$ for all $x_1, x_2 \in (X, d, k)$. Then T has a unique fixed point $z \in (X, d, k)$ and the sequence$\{T^n x\}_{n=1}^{\infty}$ converge to z for every $x \in X$.

Recently, Jyoti and Rani [15] presented an application of fixed point theory of digital metric space in image processing. They have proved that expansive mappings on complete digital metric space have a fixed point.

Theorem 2.21. Let $T : (X, d, k) \to (X, d, k)$ be a mapping on a complete digital metric space X. Let T be onto and satisfy $d(Tx, Ty) \geq \lambda d(x, y)$ for all $x, y \in X$ and $\lambda > 1$. Then T has a fixed point in X.

Remark: The mapping T in the above theorem can be replaced by a bijective mapping [15].

The condition on the mapping T in the above theorem 2.21 is replaced by another suitable condition and the following results are obtained.

Theorem 2.22. Let $(X, d, , k)$ be a complete digital self map which is continuous and onto on X. Let T satisfy the condition $d\,(Tx, Ty) \geq \lambda\mu$
where $\lambda > 1$, and

$$\mu = \mu(x, y) \in \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2} \right\}$$

then T has a fixed point.

Remark [15]: It has been proved that μ in the above theorem may be replaced by

$$\mu = \mu(x, y) \in \left\{ d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, d(x, Ty), d(y, Tx) \right\}$$

3 Main Results

In the following theorem the continuity condition on ψ in the theorem 2.20 is replaced by another suitable condition.

Theorem 3.1. Let (X, d, k) be a complete digital metric space and suppose that $T : (X, d, k) \to (X, d, k)$ satisfies $d(Tx, Ty) \le \psi(d(x, y))$ for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is monotone non-decreasing and satisfy $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$. Then T has a unique fixed point in (X, d, k).

Proof . Let x_0 be an arbitrary but fixed element in (X, d, k). Define a sequence of iterates $\{x_n\}_{n=1}^{\infty}$ in X by

$$x_1 = Tx_0, x_2 = Tx_1, x_3 = Tx_2, ..., x_n = Tx_{n-1}....$$

Note that,

$$\begin{aligned} 0 \le d(x_n + 1, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\le \psi(d(x_n, x_{n-1})) \\ &= \psi(d(Tx_{n-1}, Tx_{n-2})) \\ &\le \psi(\psi(d(x_{n-1}, x_{n-2}))) \\ &= \psi^2(d(x_{n-1}, x_{n-2})) \end{aligned}$$

Continuing in this way we get

$$0 \le d(x_{n+1}, x_n) \le \psi^n(d(x_1, x_0))$$

Thus

$$0 \le \lim_{n\to\infty} \sup d(x_{n+1}, x_n) \le \lim_{n\to\infty} \sup \psi^n(d(x_1, x_0)) = 0$$

Hence

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$$

We now show that the sequence $\{x_n\}_1^{\infty}$ is a Cauchy sequence. Also note that for any $\epsilon > 0$, $\psi(\epsilon) < \epsilon$. And since

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0,$$

so for $\epsilon > 0$, we can choose n such that $d(x_{n+1}, x_n) \le \epsilon - \psi(\epsilon)$. Now define the set $S = \{x \in X/d(x, x_n) < \epsilon\}$. Then for any $y \in S$, we have

$$\begin{aligned} d(Ty, x_n) &\le d(Ty, Tx_n) + d(Tx_n, x_n) && (3.1) \\ &\le \psi(d(y, x_n)) + d(x_{n+1}, x_n) && (3.2) \\ &\le \psi(\epsilon) + \epsilon - \psi(\epsilon) && (3.3) \\ &= \epsilon && (3.4) \end{aligned}$$

Thus $Ty \in S$. Hence $T(S) \subset S$. Therefore $d(x_m, x_n) \le \epsilon$ for all $m \ge n$. Hence the sequence $\{x_n\}_1^{\infty}$ is a Cauchy sequence in X. Since (X, d, k) is digital complete metric space, there is a limit z of $\{x_n\}_1^{\infty}$ in (X, d, k). Now we observe that the function T is (k, k)-continuous. If $a \in X$ and $\epsilon > 0$, then let $\delta = \epsilon$. Thus if $d(a, b) < \delta$, we have

$$\begin{aligned} d(Ta, Tb) &\le \psi(d(a, b)) \\ &< d(a, b) \\ &< \epsilon \end{aligned}$$

Thus T is (k, k)-continuous function. From the (k, k)-continuity of T we get

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_{n-1} = T\left[\lim_{n\to\infty} x_{n-1}\right] = Tz$$

Therefore, T has a fixed point z.
Uniqueness- Assume that $u, v \in X$ are fixed points of T. Then we have

$$d(u,v) = d(Tu,Tv) \leq \psi((d(u,v))) < d(u,v)$$

This imply $d(u,v) = 0$ and hence $u = v$.

□

Theorem 3.2. Let (X,d,k) be a complete digital metric space and $T : (X,d,k) \to (X,d,k)$ be a weakly uniformly strict digital contraction mapping. Then T has a unique fixed point z. Moreover, for any $x \in X$, $\lim_{n\to\infty} T_n x = z$.

Proof . We first observe that the weakly uniformly strict digital contraction imply the strict digital contraction. So let $x, y \in X$ be such that $x \neq y$. Then $d(x,y) > 0$. Let $\epsilon = d(x,y) > 0$.
Then by the condition of weakly uniformly strict digital contraction, there exists a $\delta > 0$ such that $\epsilon \leq d(x,y) < \epsilon + \delta$ implies $d(Tx,Ty) < \epsilon$ that is $d(Tx,Ty) < d(x,y)$.
We now prove that T is (k,k)-continuous. Let $a \in X$ and let $\epsilon > 0$. Let $\delta = \epsilon$. Then if $d(a,b) < \delta$, we have $d(Ta,Tb) < d(a,b) < \delta = \epsilon$. Thus given $\epsilon > 0$, there exists a $\delta > 0$ such that $d(a,b) < \delta$ implies $d(Ta,Tb) < \epsilon$. Hence the mapping T is (k,k)-continuous.
Next we show that if a fixed point of T exists then it is unique. Let $a, b \in X$ be fixed points of T. That is $Ta = a$ and $Tb = b$. Then we see by condition of strict digital contraction that, if $a \neq b$, then $d(Ta,Tb) = d(a,b) < d(a,b)$. Thus $d(a,b) = 0$ and hence $a = b$.
Next we proceed to show that the sequence $\{x_n\}_1^\infty = \{T^n x\}_{n=1}^\infty$ is a Cauchy sequence for every $x \in X$. Consider the sequence $\{u_n\}_{n=1}^\infty = d(x_n, x_{n+1})_{n=1}^\infty$. Since T satisfy the condition of strict digital contraction, we have x_n

$$\begin{aligned} d(x_n, x_{n+1}) &= d(T^n x, T^{n+1} x) \\ &= d(T(T^{n-1}x), T(T^n x)) \\ &< d(T^{n-1}x, T^n x) \\ &= d(x_{n-1}, x_n) \end{aligned}$$

Therefore $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$

Thus the sequence $\{u_n\}_{n=1}x\infty = \{d(x_n, x_{n+1})\}_{n=1}^\infty$ is decreasing sequence. It is also bounded below (by 0). Hence it is a convergent sequence. Let $\lim_{n\to\infty} u_n = L$. If $L > 0$ then letting $\epsilon = L > 0$, by the condition of weakly uniformly strict digital contraction, there exists a $\delta > 0$, such that $L \leq d(x_n, x_{n+1}) < L + \delta$ implies $d(x_{n+1}, x_{n+2} < L$. Then for all $m > n+1, n+2$, we have $d(x_m, x_{m+1}) < L$ (since the sequence $\{u_n\}_{n=1}^\infty$ is decreasing sequence). But then $\lim_{n\to\infty} u_n < L$ This is a contradiction. Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = L = 0$. Now we prove that the sequence $\{x_n\}_1^\infty = \{T^n x\}_{n=1}^\infty$ is a Cauchy sequence for all $x \in X$. This we show by contradiction method. So let us assume that $\{x_n\}_1^\infty = \{T^n x\}_{n=1}^\infty$ is not a Cauchy sequence for some $x \in X$. Then there exists $2\epsilon > 0$ such that

$$\lim_{n\to\infty} \sup d(x_{n+1}, x_n) > 2\epsilon$$

By hypothesis, there exists $\delta > 0$ such that $\epsilon \leq d(x,y) < \epsilon + \delta$ implies $d(Tx,Ty) < \epsilon$. This condition is true even if we replace δ by $\Delta = min(\delta, \epsilon)$. Since

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

,

we can find M such that $d(x_M, x_{M+1}) < \frac{\Delta}{3}$ Choose $m, n > M$ so that $d(x_m, x_n) > 2\epsilon$. For $m \leq j \leq n$, we have $|d(x_m, x_j) - d(x_m, x_{j+1})| \leq d(x_j, x_{j+1}) < \frac{\Delta}{3}$

This implies that there exists $m \leq j \leq n$ with $\epsilon + 2\frac{\Delta}{3} < d(x_m, x_j) < \epsilon + \Delta$ However, for all m and j, $d(x+m, x+j) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{j+1}) + d(x_{j+1}, x_j)$ Therefore,

$$d(x_m, x_j) \leq d(x_m, x_{m+1}) + \epsilon + d(x_j, x_{j+1}) < \frac{\Delta}{3} + \epsilon + \frac{\Delta}{3} = \epsilon + \frac{2\Delta}{3}.$$

This is a contradiction to the fact that

$$\epsilon + \frac{2\Delta}{3} < d(x_m, x_j) < \epsilon + \Delta$$

. Hence $\{x_n\}_1^\infty = \{T^n x\}_{n=1}^\infty$ must be Cauchy sequence for all $x \in X$. Since (X, d, k) is a complete digital metric space, there exists a point z_x such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x = z_x$ for all $x \in X$. Since T is (k, k)-continuous, we have $Tz_x = T\left(\lim_{n\to\infty} T^n x\right) = \lim_{n\to\infty} T^{n+1} x = z_x$. Thus z_x is a fixed point of T. As we have already observed that the fixed point is unique, we conclude that all the zx are same. Hence the theorem is proved.

□

4 Conclusion

In the first part, we give the required background about the digital images and digital topology. After that, we study the property of the completeness of digital metric spaces. In the next part, we state and prove the Banach fixed point theorem for digital images. Finally, some important theorems on digital metric space is proved.

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