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Banach fixed point theorem on incomplete orthogonal S-metric spaces

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Abstract

In this paper, we are interested in obtaining fixed point theorem for mappings in S-metric space by weakening the completeness of S-metric space using relations. As a consequence, an application to existence and uniqueness of solution of integral equation is given.

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1 Introduction

The concept of standard metric spaces is a fundamental tool in topology, functional analysis and nonlinear analysis. This structure has attracted a considerable attention from mathematicians because of the development of the fixed point theory in standard metric spaces.

In recent years, several generalizations of standard metric spaces have appeared [6, 7, 8, 9, 11]. Sedghi et al. [10] have introduced the concept of S-metric spaces and gave some of their properties. Then a common fixed point theorem for a self-mapping on complete S-metric spaces have given.

Sedghi et al. [10] considered the concept of S-metric spaces as follows:

Definition 1.1. [10] Let X be a nonempty set. A S-metric on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following condition, for each $x, y, z, a \in X$,

1. $S(x, y, z) \geq 0$,
2. $S(x, y, z) = 0$ if and only if $x = y = z$,
3. $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S-metric space.

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Definition 1.2. [10] A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$.

Definition 1.3. [10] A sequence $\{x_n\}$ in X is called Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

Definition 1.4. [10] The S-metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Lemma 1.5. [10] In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.

They also proved the following fixed point theorem in S-metric spaces [10].

Theorem 1.6. [10] Let (X, d) be a complete S-metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point $x^* \in X$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} f^n(x) = x^*$ with

$$S(f^n(x_0), f^n(x_0), x^*) \leq \frac{2L^n}{1-L} S(x, x, f(x)).$$

Eshaghi and et. al [2] introduced the notion of orthogonal sets as follows (also see [1, 3, 4, 5, 12, 13, 14, 15]):

Definition 1.7. [2] Let $X \neq \phi$ and $\perp \subseteq X \times X$ be a binary relation. If $\perp$ satisfies the following condition

$$\exists x_0;\ ((\forall y; y \perp x_0) \quad or \quad (\forall y; x_0 \perp y)),$$

it is called an orthogonal set (briefly O-set). We denote this O-set by $(X, \perp)$.

Definition 1.8. [2] Let $(X, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called orthogonal sequence (briefly O-sequence) if

$$((\forall n; x_n \perp x_{n+1}) \quad or \quad (\forall n; x_{n+1} \perp x_n)).$$

Definition 1.9. [2] Let $(X, d, \perp)$ be an orthogonal metric space ($(X, \perp)$ is an O-set and (X, d) is a metric space). The space X is orthogonally complete (briefly O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete and the converse is not true (see [2]). For instance, let $X = [0, 1)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is easy to see that (x, d) is incomplete metric space. If we consider $\perp = \leq$, then one can show that X is O-complete metric space.

Definition 1.10. [2] Let $(X, d, \perp)$ be an orthogonal metric space and $0 < k < 1$.

1. A mapping $f : X \to X$ is said to be orthogonal contractive ($\perp-$contractive) mapping with Lipchitz constant k if
$$d(fx, fy) \leq kd(x, y) \qquad\qquad if x \perp y.$$
2. A mapping $f : X \to X$ is called orthogonal preserving ($\perp-$preserving) mapping if $x \perp y$ then $f(x) \perp f(y)$.
3. A mapping $f : X \to X$ is orthogonal continuous ($\perp-$continuous) mapping in $a \in X$ if for each O-sequence $\{a_n\}_{n\in\mathbb{N}}$ in X if $a_n \to a$ then $f(a_n) \to f(a)$. Also f is $\perp-$continuous on X if f is $\perp-$continuous in each $a \in X$.

They also proved the following theorem which can be considered as a real extension of Banach fixed point theorem [1, 2, 3, 4, 5, 12, 13, 14, 15].

Theorem 1.11. [2] Let $(X, d, \perp)$ be an O-complete metric space (not necessarily complete metric space). Let $f : X \to X$ be $\perp-$continuous, $\perp-$contraction (with Lipschitz constant k) and $\perp-$preserving, then f has a unique fixed point x^* in X. Also, f is a Picard operator, that is, $\lim f^n(x) = x^*$ for all $x \in X$.

One of the most important conditions in Banach contraction principle is the completeness of the space. Also, in many generalizations of this theorem in different spaces such as S-metric spaces and fuzzy metric spaces the completeness of spaces is one of the most important condition and here, there is a question that how we can weaken the completeness condition of the space.

Let us consider the following integral equation

$$x(t) = \int_0^T K(t, s, x(s))ds + g(t), \quad t \in I = [0, T], \tag{1.1}$$

where $T > 0$. Inspired and motivated by the above results, in this paper, we are interested in weakening the completeness condition of S-metric space by considering a relation on S-metric space and by using this relation. As an application, we find the existence and uniqueness of solution of integral equation 1.1.

2 Main Result

In this section, we introduce some new definitions to prove the main results. We begin with the following definitions. Let (X, S) be a S-metric space. Let $\perp$ be an arbitrary relation on X.

Definition 2.1. The S-metric space (X, S) is $\perp$-complete if every Cauchy $\perp$-sequence is convergent.

Let (X, S) be a S-metric space. Let $\perp$ be an arbitrary relation on X. In the following, we denote this by $(X, S, \perp)$.

Definition 2.2.
- A mapping $f : (X, S, \perp) \to (X, S, \perp)$ is $\perp$-preserving if for $a \perp b$ we have $f(a) \perp f(b)$ for all $a, b \in X$.
- A map$f : (X, S, \perp) \to (X, S, \perp)$ is is said to be $(S, \perp)$-contraction if there exists a constant $0 \le L < 1$ such that

$$S(f(x), f(x), f(y)) \le LS(x, x, y),$$

  for all $x, y \in X$, $x \perp y$.
- A map $f : (X, S, \perp) \to (X, S, \perp)$ is $(S, \perp)$-continuous if for $\perp$-sequence $\{x_n\}$ in X such that $x_n \to x$, we have $f(x_n) \to f(x)$.

Now, we are ready to prove the main theorem of this paper which can be consider as a real extension of Theorem 1.11 (Theorem 3.11 of [2]).

Theorem 2.3. Let $(X, S, \perp)$ be a $\perp$-complete S-metric space such that there exists $x_0 \in X$ such that $x_0 \perp f(x)$ for all $x \in X$. Let $f : X \to X$ be $\perp$-preserving, $(S, \perp)$-continuous and $(S, \perp)$-contraction. Then f has a unique fixed point $x^* \in X$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} f^n(x) = x^*$ i.e. f is a Picard operator (P.O.).

Proof . By hypothesis, there exists $x_0 \in X$ such that $x_0 \perp f(x)$ for all $x \in X$. It follows that $x_0 \perp f(x_0)$. Let

$$x_1 := f(x_0), x_2 := f(x_1) = f^2(x_0), \cdots, x_{n+1} := f(x_n) = f^n(x_0).$$

Since f is $\perp$-preserving, $\{f^n(x_0)\}_{n=0}^{\infty}$ is an $\perp$-sequence. For $n = 0, 1, \cdots$, we get by induction

$$\begin{aligned} S(f^n(x_0), f^n(x_0), f^{n+1}(x_0)) &\le LS(f^{n-1}(x_0), f^{n-1}(x_0), f^n(x_0)) \\ &\le \cdots \\ &\le L^n S(x_0, x_0, f(x_0)). \end{aligned}$$

In order to show that the R-sequence $\{f^n(x_0)\}$ is Cauchy, consider $m, n \in \mathbb{N}$ such that $m > n$. From the definition of the $(S,\perp)$-metric space and by Lemma 1.5 we have

$$\begin{aligned}
S(f^n(x_0),f^n(x_0),f^m(x_0)) \leq & S(f^n(x_0),f^n(x_0),f^{n+1}(x_0))+S(f^n(x_0),f^n(x_0),f^{n+1}(x_0)) \\
& +S(f^m(x_0),f^m(x_0),f^{n+1}(x_0)) \\
= & 2S(f^n(x_0),f^n(x_0),f^{n+1}(x_0))+S(f^{n+1}(x_0),f^{n+1}(x_0),f^m(x_0)) \\
\leq & 2S(f^n(x_0),f^n(x_0),f^{n+1}(x_0))+S(f^{n+1}(x_0),f^{n+1}(x_0),f^{n+2}(x_0)) \\
& +S(f^{n+1}(x_0),f^{n+1}(x_0),f^{n+2}(x_0))+S(f^m(x_0),f^m(x_0),f^{n+2}(x_0)) \\
= & 2S(f^n(x_0),f^n(x_0),f^{n+1}(x_0))+2S(f^{n+1}(x_0),f^{n+1}(x_0),f^{n+2}(x_0)) \\
& +S(f^{n+2}(x_0),f^{n+2}(x_0),f^m(x_0)) \\
\leq & \cdots \\
\leq & 2\Sigma_{i=n}^{m-2}S(f^i(x_0),f^i(x_0),f^{i+1}(x_0))+S(f^{m-1}(x_0),f^{m-1}(x_0),f^m(x_0)) \\
\leq & 2L^nS(x_0,x_0,f(x_0))[1+L+L^2+\cdots] \\
\leq & \frac{2L^n}{1-L}S(x_0,x_0,f(x_0)).
\end{aligned}$$

Thus, for $m>n$ we have

$$S(f^n(x_0),f^n(x_0),f^m(x_0)) \leq \frac{2L^n}{1-L}S(x_0,x_0,f(x_0)). \tag{2.1}$$

From the above we find that $\{f^n(x_0)\}_{n=0}^{\infty}$ is Cauchy $\perp$-sequence. By $\perp$-completeness of X, there exists $x^* \in X$ such that $f^n(x_0) \to x^*$. On the other hand, f is $(S,\perp)$-continuous and hence $f(f^n(x_0)) \to f(x^*)$. As n tends to ∞ we have

$$f(x^*) = \lim_{n\to\infty} f(f^n(x_0)) = \lim_{n\to\infty} f^{n+1}(x_0) = x^*.$$

Therefore, x^* is a fixed point of f. To prove the uniqueness of the fixed point, let $y^* \in X$ be a fixed point of f. Then we have $f^n(y^*) = y^*$ for all $n \in \mathbb{N}$. By our choice of x_0 in the hypothesis we have $x_0 \perp y^*$ (because $y^* = f(y^*) \in f(X)$). Since f is $\perp$-preserving, we have

$$f^n(x_0) \perp f^n(y^*),$$

for all $n \in \mathbb{N}$. On the other hand, f is a $(S,\perp)$-contraction, then we have

$$\begin{aligned}
S(x^*,x^*,y^*) = S(f^n(x^*),f^n(x^*),f^n(y^*)) \leq & S(f^n(x^*),f^n(x^*),f^n(x_0))+S(f^n(x^*),f^n(x^*),f^n(x_0)) \\
& +S(f^n(y^*),f^n(y^*),f^n(x_0)) \\
\leq & L^nS(x^*,x^*,x_0)+L^n(x^*,x^*,x_0)+L^n(y^*,y^*,x_0) \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Then $S(x^*,x^*,y^*)=0$, hence $x^*=y^*$. Let $x \in X$ be arbitrary. By hypothesis we have $x_0 \perp f(x)$. Since f is $\perp$-preserving, then

$$f^n(x_0) \perp f^n(f(x)),$$

for all $n \in \mathbb{N}$. On the other hand, f is a $(S,\perp)$-contraction, then we get

$$\begin{aligned}
S(f^n(f(x)),f^n(f(x)),x^*) = & S(f^n(f(x)),f^n(f(x)),f^n(x^*)) \\
\leq & S(f^n(f(x)),f^n(f(x)),f^n(x_0))+S(f^n(f(x)),f^n(f(x)),f^n(x_0)) \\
& +S(f^n(x^*),f^n(x^*),f^n(x_0)) \\
\leq & 2L^nS(f(x),f(x),x_0)+L^n(x^*,x^*,x_0) \to 0 \quad \text{as} \quad n\to\infty.
\end{aligned}$$

So, $\lim_{n\to\infty} f^n(f(x)) = x^*$. Hence, $\lim_{n\to\infty} f^n(x) = x^*$. Therefore, f is a P.O. □

One can easily prove the following result.

Corollary 2.4. Let $(X,S,\perp)$ be an O-complete S-metric space. Let $f: X \to X$ be $\perp$-preserving, $(S,\perp)$-continuous and $(S,\perp)$-contraction. Then f has a unique fixed point $x^* \in X$. Furthermore, for any $x \in X$ we have $\lim_{n\to\infty} f^n(x) = x^*$ i.e. f is a Picard operator (P.O.).

3 Application in integral equation

Consider the integral equation

$$x(t) = \int_0^T K(t, s, x(s))ds + g(t), \quad t \in I = [0, T],$$

where $T > 0$. The aim of this section is to give an existence and uniqueness theorem for a solution of the above integral equation using results in the previous section. Let

$$X = \{u \in C(I, \mathbb{R}); u(t) > 1 \quad \text{for almost every} \quad t \in I\}.$$

Suppose the mapping

$$S : X \times X \times X \to \mathbb{R}^+$$

defined by

$$S(x, y, z) = \sup_{t \in I} |x(t) - y(t)| + \sup_{t \in I} |x(t) - z(t)| + \sup_{t \in I} |y(t) - z(t)|,$$

for $x, y, z \in X$. Define the following relation $\perp$ in X:

$$x \perp y \quad \text{if} \quad x(t)y(t) \geq y(t),$$

for almost every $t \in I$. Its easy to see that (X, S) is $\perp$-complete S-metric space.

Theorem 3.1. Suppose the following hypotheses hold:

1. $K : I \times I \times \mathbb{R} \to \mathbb{R}$ and $g : I \to [1, \infty)$ are continuous.
2. There exists a continuous function $G : I \times I \to [0, \infty)$ such that

$$|K(t, s, u) - K(t, s, v)| \leq G(t, s)|u - v|,$$

   for each $u, v \in \mathbb{R}$, $u \perp v$ and each $t, s \in I$.
3. $\sup_{t \in I} \int_0^T G(t, s)ds < r$ for each $r < 1$.

Then the integral equation 1.1 has a solution $u \in C(I, \mathbb{R})$.

Proof . In $(S, \perp)$-metric space $(X, S, \perp)$a mapping

$$A : (X, S, \perp) \to (X, S, \perp),$$

can be defined by

$$Ax(t) = \int_0^T K(t, s, x(s))ds + g(t),$$

for almost every $t \in I$. Note that if $x \in X$ is a fixed point of A, then x is a solution to the 1.1. First, we claim that for every $x \in X$, $Ax \in X$. To see this, for every $t \in I$, $x \in X$, we have

$$Ax(t) = \int_0^T K(t, s, x(s))ds + g(t) \geq 1.$$

One can conclude that $Ax(t) > 1$ and we have $Ax \in X$. Now, we check that the hypotheses in Theorem 2.3 is satisfied. to do this, we show that

1. There exists $x_0 \in x$ such that $x_0 \perp A(x)$ for all $x \in X$.
2. A is $\perp$-preserving.
3. A is $(S, \perp)$-contraction.
4. A is $(S, \perp)$-continuous.

Proof .

1. Put $x_0 = 2$ (the constant function $x_0 = 2$), we have $2 \perp A(x)$ for all $x \in X$.

2. We recall that A is $\perp$-preserving if for every $x, y \in X$, $x \perp y$, we have $Ax \perp Ay$. We have shown above that $Ax(t) > 1$ for every $t \in I$, which implies that $Ax(t)Ay(t) \geq Ay(t)$ for all $t \in I$. So $Ax \perp Ay$.
3. Let $x, y \in X$, $x \perp y$ and $t \in I$, we have

$$\begin{aligned} |Ax(t) - Ay(t)| &= |\int_0^T K(t,s,x(s))ds + g(t) - \int_0^T K(t,s,y(s))ds - g(t)| \\ &= |\int_0^T [K(t,s,x(s)) - K(t,s,y(s))]ds| \\ &\leq \int_0^T |K(t,s,x(s)) - K(t,s,y(s))|ds \\ &\leq \int_0^T G(t,s)|x(s) - y(s)|ds \\ &\leq \sup_{t\in I} |x(t) - y(t)| \sup_{t\in I} \int_0^T G(t,s)ds \\ &\leq r \sup_{t\in I} |x(t) - y(t)|. \end{aligned}$$

So,

$$\sup_{t\in I} |Ax(t) - Ay(t)| \leq r \sup_{t\in I} |x(t) - y(t)|.$$

Therefore, we have

$$S(Ax, Ax, Ay) = 2 \sup_{t\in I} |Ax(t) - Ay(t)| \leq 2r \sup_{t\in I} |x(t) - y(t)| = rS(x, x, y).$$

This proves that A is $(S, \perp)$-contraction with Lipchitz constant $\lambda = r < 1$.
4. Let $\{x_n\}$ be an $(S, \perp)$-sequence in X such that $\{x_n\}$ converges to some $x \in X$. Since A is $\perp$-preserving, $\{Ax_n\}$ is an $(S, \perp)$-sequence, too. For each $n \in \mathbb{N}$, by (2) we have

$$|Ax_n - Ax| \leq \lambda |x_n - x|.$$

As n goes to infinity, it follows that A is $\perp$-continuous.

□

The mapping A satisfies the hypotheses of the Theorem 2.3. Thus, existence and uniqueness of its fixed point $x^* \in X$ has been guaranteed by Theorem 2.3 . As noted above x^* is a unique solution to integral equation 1.1. □

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