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Strong convergence for α-nonexpansive mapping using a partial order induced by a function

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Abstract

In this work, we introduce a partial ordering on a Banach space induced by a real valued function and prove some convergence theorems for α-nonexpansive mapping in a ordered Banach space to a fixed point of mapping using this partial ordering. Moreover we give example to furnish the definition of the partial ordering induced by a real valued function.

Keywords: α-nonexpansive mapping, Strong Convergence, Partial order, Fixed point
2020 MSC: 47H05, 47H09, 47H10

1 Introduction

A very popular and interesting area of research is to investigate the convergence of the Nonexpansive mappings. Many authors [4, 7, 10, 5, 6, 9] have given the evidence of the existence of a fixed point for nonexpansive mappings defined on a uniformly convex Banach space. Later direction [12]- [13] in this research has been given for the Lipschitz condition to be satisfied for the pair of elements related by the partial ordering which seems to be a week assumption.Recently many authors [1, 2, 11, 15] have worked on the convergence result of monotone α-nonexpansive type mappings in ordered Banach space where the condition is restricted for the pair of comparable elements of the space.

In this work, we use more generalized approach where the condition to the non-expansive mappings can be used on any pair of point in the space and we show the convergence of the mapping using such condition over an ordered Banach space where the partial order is induced by a real-valued function. Moreover we give the example of the partial order is induced by a real-valued function. Recall that a mapping $U : D(U) \to R(U)$ is said to be monotone if $Uv \preceq Uw$ whenever $v \preceq w$ $\forall v, w \in D(U)$. Aoyama and Kohsaka [2], introduced the concept of a α-nonexpansive mapping as:

Let X be a Banach space and $U : C \to C$ be mapping. Then U is said to be α-nonexpansive for some $\alpha < 1$ if

$$||U(v) - U(w)||^2 \leq \alpha||U(v) - w||^2 + \alpha||v - U(w)||^2 + (1 - 2\alpha)||v - w||^2 \quad \forall x, y \in C \subseteq X \tag{1.1}$$

Obviously, nonexpansive mapping is 0-nonexpansive mapping. Recently, Muangchoo-In et. al. [11] extended the α-nonexpansive to $\alpha - \beta$-nonexpansive mappings as follows:

Let X be a Banach space and $U : C \to C$ be mapping. Then U is said to be α-nonexpansive for some $\alpha, \beta < 1$ if

$$||U(v) - U(w)||^2 \leq \alpha||U(v) - w||^2 + \beta||v - U(w)||^2 + (1 - (\alpha + \beta))||v - w||^2 \quad \forall x, y \in C \subseteq X \tag{1.2}$$

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The Mann iteration scheme [3] uses the concept of nonexpansivity of a mapping from linear space. $U : X \to X$ is a mapping and $\{\beta_n\}$ being a real non-negative sequence satisfying specified criteria. The generated sequence from an arbitrary $w_0 \in X$ is given as

$$w_n = (1-\beta_n)w_{n-1} + \beta_n U w_n \tag{1.3}$$

Ishikawa [8] introduced the Ishikawa iteration given as follows

$$w_{n+1} = (1-\beta_n)w_n + \beta_n U y_n \tag{1.4}$$

$$y_n = (1-\sigma_n)w_n + \sigma_n U w_n \tag{1.5}$$

For each $n \geq 1$, where $\beta_n \quad and \quad \sigma_n \in [0,1]$.
Now we state lemma which is useful in proving our main result.

Lemma 1.1. [14] Suppose that E is a uniformly convex Banach space and $0 < p \leq q < 1$ for all $n = 1, 2 \ldots$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequence of E such that $\lim_{n\to\infty}||x_n|| \leq r, \lim_{n\to\infty}||y_n|| \leq r$ and $\lim_{n\to\infty}||t_n x_n + (1-t_n)y_n|| \leq r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty}||x_n - y_n|| = 0$.

The order intervals over an ordered Banach Space X with the partial order "$\preceq$" are assumed to be closed and convex. Any of the two subsets are known to be an order interval

$$[c, \to) = \{w \in X : b \preceq w\} \quad or \quad (\leftarrow, b] = \{w \in X; w \preceq b\}$$

for any $b \in X$. And so the the subset

$$[c, d] = \{w \in X; c \leq w \leq d\} = [c, \to] \cap [\leftarrow, d]$$

is also closed and convex for any $c, d \in X$.

2 Main Results

First we define the following:

Definition 2.1. Let X be a Banach Space and ψ a function s.t. $\psi : X \to R$. Then a partial ordering on X defined by:

$$p \preceq q \iff \psi(p) - \psi(q) \geq \gamma||p-q|| \quad \forall p, q \in X \quad and \quad \gamma > 0 \tag{2.1}$$

is called a partial ordering induced by the function ψ and X is said to be ordered Banch space.

Lemma 2.2. Let X be a Banach Space and a function $\psi : X \to R$. We define a partial ordering on X as follows:

$$p \preceq q \iff \psi(p) - \psi(q) \geq \gamma||p-q|| \quad \forall p, q \in X \quad and \quad \gamma > 0 \tag{2.2}$$

Then " $\preceq$ " is a partial order on X and is called a partial order induced by ψ.

Proof . For all $p \in X, \quad \psi(p) - \psi(p) \geq \gamma||p-p||$ then $p \preceq p$ that is " $\preceq$ " is reflexive.
Now for $p, q \in X$ s.t. $p \preceq q$ and $q \preceq p$ then

$$\psi(p) - \psi(q) \geq \gamma||p-q|| \text{ for } \gamma > 0$$

And

$$\psi(q) - \psi(p) \geq \gamma||q-p|| \text{ for } \gamma > 0$$

combining we get $\psi(p) - \psi(q) = 0$ i.e.$||p-q|| = 0$ This shows i.e. $p = q$. Thus " $\preceq$ " is antisymmetric.
Again for $s, p, q \in X$ s.t.$s \preceq p$ and $p \preceq q$ then

$$\psi(s) - \psi(p) \geq \gamma||s-p|| \text{ for } \gamma > 0,$$

and

$$\psi(p) - \psi(q) \geq \gamma ||p - q|| \text{ for } \gamma > 0.$$

We get

$$\begin{aligned} \gamma||s-q|| &\leq \gamma||s-p|| + \gamma||p-q|| \\ &\leq \psi(s) - \psi(p) + \psi(p) - \psi(q) \end{aligned}$$

or

$$\psi(s) - \psi(q) \geq \gamma||s-q||.$$

This shows that $s \preceq q$.Thus " $\preceq$ " is transitive. And so the relation " $\preceq$ " is a partial order on X. □

Now we will give an example to furnish definition and lemma.

Example 2.3. Let $X = R^2$ and the norm on X is given by

$$||s|| = \sqrt{s_1^2 + s_2^2} \quad for \quad s = (s_1, s_2).$$

Let us define the function $psi : X \to R$ by

$$\psi(s) = 2\sqrt{s_1^2 + s_2^2} \quad for \quad s = (s_1, s_2).$$

It is obvious that s is a Banach space with the norm defined on it. again by using the equation (3), we write

$$s \preceq q \iff 2\sqrt{s_1^2 + s_2^2} - 2\sqrt{q_1^2 + q_2^2} \geq \gamma\sqrt{(s_1 - q_1)^2 + (s_2 - q_2)^2}$$

for $s = (s_1, s_2), q = (q_1, q_2) \in X \quad and \quad \gamma > 0$. For clarity, let us assume $\gamma = 2$ so that

$$s \preceq q \iff \sqrt{s_1^2 + s_2^2} - \sqrt{q_1^2 + q_2^2} \geq \sqrt{(q_1 - q_1)^2 + (s_2 - q_2)^2}$$

for $s = (s_1, s_2), q = (q_1, q_2) \in X$. It follows that $(2, 2) \preceq (1, 1), (2, 2) \preceq (1/2, 1/2)$ but $(2, 2) \npreceq (-1/4, -1/4), (3, 2) \npreceq (1, 1)$ etc. Therefore X is a partially ordered Banach space and the partial order is induced by the real valued function ψ.

Now we prove our convergence result over an ordered Banach space.

Theorem 2.4. Let $(X, " \preceq ")$ be an uniformly convex ordered Banach space endowed with the partial order " $\preceq$ " induced by the real valued bounded below function $\psi : X \to R$. Let $U : X \to X$ be a monotone and α-nonexpansive mapping. Let $\{v_n\}$ be a Mann iterative sequence defined by (2) with $v_1 \preceq Uv_1$ then the sequence $\{v_n\}$ is strongly converges to a unique fixed point of U.

Proof . Given $v_1 \preceq U(v_1)$. Since the order relation is closed and convex, we can write

$$v_1 \preceq (1 - \beta_1)v_1 + \beta_1 U(v_1) \preceq U(v_1)$$

so by Mann iteration (2) we will have $v_1 \preceq v_2 \preceq U(v_1)$. Since U is monotone, continuing in this way we get

$$v_1 \preceq v_2 \preceq v_3 \ldots v_n \preceq v_{n+1}.$$

Therefore by the condition of partial order induced by ψ on X, we get

$$\psi(v_1) \geq \psi(v_2) \geq \psi(v_3) \ldots \geq \psi(v_n) \geq \psi(v_{n+1}).$$

In other words, the sequence $\{\psi(v_n)\}$ is decreasing sequence of real numbers. And since ϕ is bounded from below, $\{\psi(v_n)\}$ is convergent sequence od real numbers and hence is Cauchy. So, for $\epsilon > 0$ there exist $n_0 \in N$ such that for all $m > n > n_0$, we have

$$|\psi(v_m) - \psi(v_n)| < \epsilon.$$

Since $x_n \preceq x_m$, by the definition of $\preceq$, we have

$$||v_m - v_n|| < \tfrac{\epsilon}{\lambda} = \epsilon_1(say).$$

This shows that the sequence $\{v_n\}$ is Cauchy in X and so converges to a point $v \in X$ i.e.

$$\lim_{n\to\infty} ||v_n - v|| = 0. \tag{2.3}$$

Therefore the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ also converges to the point $v \in X$. i.e.

$$\lim_{n\to\infty} ||v_{n_k} - v|| = 0. \tag{2.4}$$

Since the mapping $U : X \to X$ is α-nonexpansive, for all $v, w \in X$ and $\alpha < 1$, we have

$$||U(v) - U(w)||^2 \leq \alpha||U(v) - w||^2 + \alpha||v - U(w)||^2 + (1 - 2\alpha)||v - w||^2. \tag{2.5}$$

For $r \in F(U)$ and $u_n \in X$, using (4), we have

$$\begin{aligned}
||U(v_n) - r||^2 &= ||U(v_n) - U(r)||^2 \\
&\leq \alpha||U(v_n) - r||^2 + \alpha||v_n - U(r)||^2 + (1 - 2\alpha)||v_n - r||^2 \\
&\leq \alpha||U(v_n) - r||^2 + \alpha||v_n - r||^2 + (1 - 2\alpha)||v_n - r||^2 \\
&\leq \alpha||U(v_n) - r||^2 + (1 - \alpha)||v_n - r||^2
\end{aligned}$$

i.e

$$||U(v_n) - r|| \leq ||v_n - r||. \tag{2.6}$$

Again by using the iteration in (2), we have

$$\begin{aligned}
||v_{n+1} - r||^2 &= ||(1 - \beta_n)v_n + \beta_n U(v_n) - r||^2 \\
&\leq (1 - \beta_n)||v_n - r||^2 + \beta_n||U(v_n) - r||^2 \\
&\leq (1 - \beta_n)||v_n - r||^2 + \beta_n||v_n - r||^2 \\
&\leq ||v_n - r||^2
\end{aligned}$$

i.e

$$||v_{n+1} - r|| \leq ||v_n - r|| \tag{2.7}$$

for any $n \geq 1$. This means that $||v_n - r||$ is a decreasing and bounded sequence, which implies that $\lim_{n\to\infty}||v_n - r||$ exists. Let $\lim_{n\to\infty}||v_n - r|| = l$.

Now $\lim_{n\to\infty}||v_{n+1} - r|| = l$ means

$$\begin{aligned}
\lim_{n\to\infty} ||(1 - \beta_n)v_n + \beta_n U(v_n) - r|| &= l \\
\lim_{n\to\infty} ||(1 - \beta_n)(v_n - r) + \beta_n(U(v_n) - r)|| &= l
\end{aligned}$$

and we have $||U(v_n) - r|| \leq ||v_n - r||$ which will imply $\lim_{n\to\infty}||U(v_n) - r|| \leq l$. Therefore by using Lemma 1.1, we get

$$\lim_{n\to\infty} ||U(v_n) - v_n|| = 0$$

and we have $\lim_{n\to\infty} v_n = v$, therefore

$$\lim_{n\to\infty} U(v_n) = v. \tag{2.8}$$

Using (4) we can write

$$||U(v_n) - U(r)||^2 \leq \alpha||U(v_n) - r||^2 + \alpha||v_n - U(r)||^2 + (1 - 2\alpha)||v_n - r||^2.$$

Letting $n \to \infty$ and using (7) we get

$$\begin{aligned} \lim_{n\to\infty} ||U(v_n) - U(r)||^2 &\leq \alpha \lim_{n\to\infty} ||U(v_n) - r||^2 + \alpha \lim_{n\to\infty} ||v_n - U(r)||^2 + (1 - 2\alpha) \lim_{n\to\infty} ||v_n - r||^2 \\ ||v - r||^2 &\leq \alpha||v - r||^2 + \alpha||v - r||^2 + (1 - 2\alpha)||v - r||^2 \end{aligned}$$

or

$$||v - r||^2 \leq ||v - r||^2$$

which leads to $v = r$. This completes the proof. □

Similar results can be proved for $\alpha - \beta$-nonexpansive mappings as follows:

Theorem 2.5. Let $(X, " \preceq ")$ be an uniformly convex ordered Banach space endowed with the partial order $" \preceq "$ induced by the real valued bounded below function $\psi : X \to R$. Let $U : X \to X$ be a monotone and $\alpha - \beta$-nonexpansive mapping. Let $\{v_n\}$ be a Mann iterative sequence defined by (2) with $v_1 \preceq Uv_1$ then the sequence $\{v_n\}$ is strongly converges to a unique fixed point of U.

Proof . Given $v_1 \preceq U(v_1)$. Since the order relation is closed and convex, we can write

$$v_1 \preceq (1 - \beta_1)v_1 + \beta_1 U(v_1) \preceq U(v_1)$$

so by Mann iteration (2) we will have $v_1 \preceq v_2 \preceq U(v_1)$.Since U is monotone, continuing in this way we get

$$v_1 \preceq v_2 \preceq v_3 \ldots v_n \preceq v_{n+1}.$$

Therefore by the condition of partial order induced by ψ on X, we get

$$\psi(v_1) \geq \psi(v_2) \geq \psi(v_3) \ldots \geq \psi(v_n) \geq \psi(v_{n+1}).$$

In other words, the sequence $\{\psi(v_n)\}$ is decreasing sequence of real numbers. And since ϕ is bounded from below, $\{\psi(v_n)\}$ is convergent sequence od real numbers and hence is Cauchy. So, for $\epsilon > 0$ there exist $n_0 \in N$ such that for all $m > n > n_0$, we have

$$|\psi(v_m) - \psi(v_n)| < \epsilon.$$

Since $x_n \preceq x_m$, by the definition of $\preceq$, we have

$$||v_m - v_n|| < \tfrac{\epsilon}{\lambda} = \epsilon_1(say).$$

This shows that the sequence $\{v_n\}$ is Cauchy in X and so converges to a point $v \in X$ i.e.

$$\lim_{n\to\infty} ||v_n - v|| = 0. \tag{2.9}$$

Therefore the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ also converges to the point $v \in X$. i.e.

$$\lim_{n\to\infty} ||v_{n_k} - v|| = 0. \tag{2.10}$$

Since the mapping $U : X \to X$ is $\alpha - \beta$-nonexpansive, for all $v, w \in X$ and $\alpha < 1$, we have

$$||U(v) - U(w)||^2 \leq \alpha||U(v) - w||^2 + \beta||v - U(w)||^2 + (1 - (\alpha + \beta))||v - w||^2 \tag{2.11}$$

For $r \in F(U)$ and $v_n \in X$, using (4), we have

$$\begin{aligned} ||U(v_n) - r||^2 &= ||U(v_n) - U(r)||^2 \\ &\leq \alpha||U(v_n) - r||^2 + \beta||v_n - U(r)||^2 + (1 - (\alpha + \beta))||v_n - r||^2 \\ &\leq \alpha||U(v_n) - r||^2 + \beta||v_n - r||^2 + (1 - (\alpha + \beta))||v_n - r||^2 \\ &\leq \alpha||U(v_n) - r||^2 + (1 - (\alpha + \beta))||v_n - r||^2 \end{aligned}$$

i.e

$$||U(v_n) - r|| \leq ||v_n - r||. \tag{2.12}$$

Rest of the proof is directly followed from Theorem 2.3. □

3 conclusion

In the presented work, using a partial order which is induced by a real function on a uniformly convex Banach space we investigate the convergence results of a mapping to a fixed point in the space. Our approach of investigation is different from the previous approach of investigation which is offered by many authors i.e. to prove the convergence by a partial order defined using a Cone over the underlying space.

Remark 3.1. Similar kind of results to be proved using the scheme by Ishikawa [8]. Moreover if the condition of monotonicity can be removed for the investigation of convergence, would also be of interest.

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References

[1] B. Abdullatif and A. Khamsi, *Mann iteration process for monotone nonexpansive mappings*, Fixed Point Theory Appl. **2015** (2015), 177.

[2] K. Aoyama and F. Kohsaka, *Fixed point theorem for α-nonexpansive mappings in Banach spaces*, Nonlinear Anal. **74** (2011), no. 13, 4387–4391.

[3] J. Borwein, S. Reich and I. Shafrir, *Krasnoselskii Mann iterations in normed spaces*, Canad. Math. Bull. **35** (1992), 21–28.

[4] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Nat. Acad. Sci. USA **54** (1965), 1041–1044.

[5] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory*, Cambridge Stud. Adv. Math., vol. 28. Cambridge University Press, Cambridge, 1990.

[6] K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Dekker, New York, 1984.

[7] D. Göhde, *Zum prinzip der kontraktiven abbildung.* Math. Nachr. **30** (1965), 251–258.

[8] S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.

[9] M.A. Khamsi and W.A. Kirk, *An introduction to metric spaces and fixed point theory*, Wiley, New York, 2001.

[10] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Month. **72** (1965), 1004–1006.

[11] K. Muangchoo, D. Thongtha, P. Kuman and Y. Je, *Fixed point theorems and convergence theorems for monotone (α, β)-nonexpansive mappings in ordered Banach spaces*, Creat. Math. Inf. **26** (2017), no. 2, 163–180.

[12] J.J. Nieto and R. Rodríguez-López, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order **22** (2005), 223–239.

[13] A.C.M. Ran and M.C.B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Am. Math. Soc. **132** (2004), 1435–1443.

[14] J. Schu, *Weak and strong convergence to fixed points of asymptotically nonexpansive mappings*, Bull. Aust. Math. Soc. **43** (1991), 153–159.

[15] R. Shukla, R. Pant and M. Sen, *Generalized α-nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl. **2017** (2017), 4.