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Approximating common fixed points of mean nonexpansive mappings in hyperbolic spaces

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Abstract

In this paper, we prove some fixed points properties and demiclosedness principle for mean nonexpansive mapping in uniformly convex hyperbolic spaces. We further propose an iterative scheme for approximating a common fixed point of two mean nonexpansive mappings and establish some strong and $\triangle$-convergence theorems for these mappings in uniformly convex hyperbolic spaces. The results obtained in this paper extend and generalize corresponding results in uniformly convex Banach spaces, CAT(0) spaces and other related results in literature.

Keywords: Mean nonexpansive mappings, uniformly convex hyperbolic spaces, strong and $\triangle$-convergence theorem, three step iteration
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1 Introduction

Let (X, d) be a metric space and C be a nonempty closed and convex subset of X. A mapping $T : C \to C$ is said to be

(i) *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y), \ \forall x, y \in C,$$

(ii) *Suzuki-generalized nonexpanpansive* (or said to satisfy condition (C)) if

$$\frac{1}{2}d(x, Tx) \leq d(x, y) \implies d(Tx, Ty) \leq d(x, y), \ \forall x, y \in C,$$

(iii) *mean nonexpansive* if

$$d(Tx, Ty) \leq ad(x, y) + bd(x, Ty), \ \forall x, y \in C, \ a, b \geq 0, \ a + b \leq 1.$$

For more details on Suzuki-type and contraction-type mappings, see [26, 27, 28, 29].

Remarks:

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(1) It is worth mentioning that nonexpansive mappings are Suzuki-generalized nonexpansive mappings. However, Suzuki [38] gave an example of a Suzuki-generalized nonexpansive mapping which is not nonexpansive.
(2) We also mention that every nonexpansive mapping is a mean nonexpansive mapping. However, we give the following example to show that there exists a mean nonexpansive mapping which is not a nonexpansive mapping.

Example 1.1. Let $T : [0,1] \to [0,1]$ be defined by

$$Tx = \begin{cases} \frac{1}{4}, & \text{if } x \in [0, \frac{1}{2}], \\ \frac{1}{8}, & \text{if } x \in (\frac{1}{2}, 1), \\ 0, & \text{if } x = 1. \end{cases}$$

Then, T is a mean nonexpansive mapping with $a = b = \frac{1}{2}$.

Proof . Case 1: Let $x \in [0,1]$ and $y \in (\frac{1}{2}, 1)$, then we have that $\frac{1}{2} - Ty < y - Ty$, which implies that $\frac{1}{2}\left(\frac{1}{2} - \frac{1}{8}\right) < \frac{1}{2}|y - Ty|$.
So that

$$|Tx - Ty| \leq \frac{1}{8} < \frac{1}{2}\left(\frac{1}{2} - \frac{1}{8}\right) < \frac{1}{2}|y - Ty| \leq \frac{1}{2}|x - y| + \frac{1}{2}|x - Ty|.$$

Case 2: Let $x \in [0,1]$ and $y = 1$, then by similar argument as in **Case 1**, we obtain

$$|Tx - Ty| \leq \frac{1}{4} < \frac{1}{2}|y - Ty| \leq \frac{1}{2}|x - y| + \frac{1}{2}|x - Ty|.$$

Case 3: Let $x \in (\frac{1}{2}, 1)$ and $y \in [0, \frac{1}{2}]$, then we have that $1 - (\frac{1}{2} + \frac{1}{4}) < 2x - (y + Ty)$, which implies $\frac{1}{8} < \frac{1}{2}|2x - (y - Ty)|$.
So that

$$|Tx - Ty| = \frac{1}{8} < \frac{1}{2}|2x - (y + Ty)| \leq \frac{1}{2}|x - y| + \frac{1}{2}|x - Ty|.$$

Case 4: Let $x = 1$ and $y \in [0, \frac{1}{2}]$, then by similar argument as in **Case 3**, we obtain

$$|Tx - Ty| = \frac{1}{4} < \frac{1}{2}|2x - (y + Ty)| \leq \frac{1}{2}|x - y| + \frac{1}{2}|x - Ty|.$$

For the cases where $x, y \in [0, \frac{1}{2}]$, $x, y \in (\frac{1}{2}, 1)$ and $x = y = 1$, we have that $|Tx - Ty| = 0$. Thus, we conclude that T is mean nonexpansive with $a = b = \frac{1}{2}$. However, we see clearly that T is not continuous. Therefore, T cannot be nonexpansive. For more examples of mean nonexpansive mappings, see [30, 49, 48]. □

Although it was shown in [30] that increasing mean nonexpansive mappings are Suzuki-generalized nonexpansive mappings. Nakprasit [30] gave an example of a mean nonexpansive mapping which is not a Suzuki-generalized nonexpansive mapping. The class of mean nonexpansive mappings was first introduced by Zhang [47], who proved that a mean nonexpansive mapping has a fixed point in a weakly compact convex subset C (with normal structure) of a Banach space. Since then, several authors have studied mean nonexpansive mappings in Banach spaces. For example, Zuo [49] studied some fixed point theorems for mean nonexpansive mappings in Banach spaces and proved that under certain conditions, a mean nonexpansive mapping has a fixed point in C, where C is a nonempty, closed and convex subset of a Banach space. Furthermore, he proved that if T is a mean nonexpansive mapping and $\{x_n\}$ is a sequence in C, then the sequence $\{x_n - Tx_n\}$ converges strongly to 0. For other extensive studies on mean nonexpansive mappings in Banach spaces, see [16, 35, 45, 46] and the references therein. Recently, mean nonexpansive mappings was introduced and studied in CAT(0) by Zhou and Cui [48] using the following Ishikawa iteration: For $x_1 \in C$, $\{t_n\}, \{s_n\} \subset [0,1]$, define $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1 - s_n)x_n \oplus s_n T x_n, \\ x_{n+1} = (1 - t_n)x_n \oplus t_n T y_n, \ n = 1, 2, \ldots. \end{cases} \tag{1.1}$$

They proved both strong and Δ-convergence theorems for the sequence $\{x_n\}$ generated by the above algorithm. For recent results on approximating fixed points of nonlinear mappings in CAT(0) space, see [6, 8, 31, 32].

Beside the nonlinear mappings involved in the study of fixed point theory, the role played by the spaces involved is also very important. Several fixed point results and iterative algorithms for approximating the fixed points of nonlinear mappings in Hilbert and Banach spaces have been obtained in literature, for example, see [2, 3, 4, 5, 12, 13, 18, 19, 33, 34, 39, 40, 41, 42, 43]. It is easier working with Banach space due to its convex structures. However, metric space do not naturally enjoy this structure. Therefore the need to introduce convex structures to it arises. The concept of convex metric space was first introduced by Takahashi [44] who studied the fixed points for nonexpansive mappings in the setting of convex metric spaces. Since then, several attempts have been made to introduce different convex structures on metric spaces. An example of a metric space with a convex structure is the hyperbolic space. Different convex structures have been introduced on hyperbolic spaces resulting to different definitions of hyperbolic spaces (see [14, 22, 36]). Although the class of hyperbolic spaces defined by Kohlenbach [22] is slightly restrictive than the class of hyperbolic spaces introduced in [14], it is however, more general than the class of hyperbolic spaces introduced in [36]. Moreover, it is well-known that Banach spaces and CAT(0) spaces are examples of hyperbolic spaces introduced in [22]. Some other examples of this class of hyperbolic spaces includes Hadamard manifords, Hilbert ball with the hyperbolic metric, Catesian products of Hilbert balls and $\mathbb{R}$-trees, see [7, 14, 11, 15, 22, 36].

It is worth mentioning that, as far as we know, no work has been done on fixed point problems for mean nonexpansive mappings in hyperbolic spaces. Therefore, it is necessary to extend results on fixed point problems for mean nonexpansive mappings from uniformly convex Banach spaces and CAT(0) spaces to uniformly convex hyperbolic spaces, since the class of uniformly convex hyperbolic spaces generalizes the class of uniformly convex Banach spaces as well as CAT(0) spaces.

Motivated by all these facts, we study some fixed points properties and demiclosedness principle for mean nonexpansive mappings in uniformly convex hyperbolic space introduced in [22], and establish both strong and Δ-convergence theorems for approximating a common fixed point of two mean nonexpansive mappings using the iterative scheme introduced by Abbas and Nazir [1].

2 Preliminaries

Throughout this paper, our study is in hyperbolic space introduced by Kohlenbach [22].

Definition 2.1. A hyperbolic space (X, d, W) is a metric space (X, d) together with a convex mapping $W : X^2 \times [0, 1] \to X$ satisfying

1. $d(u, W(x, y, \alpha)) \leq (1 - \alpha)d(u, x) + \alpha d(u, y)$,
2. $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$,
3. $W(x, y, \alpha) = W(y, x, 1 - \alpha)$,
4. $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha)d(x, y) + \alpha d(z, w)$, for all $w, x, y, z \in X$ and $\alpha, \beta \in [0, 1]$.

Example 2.2. [37] Let X be a real Banach space which is equipped with norm $||.||$. Define the function $d : X^2 \to [0, \infty)$ by

$$d(x, y) = ||x - y||.$$

Then, we have that (X, d, W) is a hyperbolic space with mapping $W : X^2 \times [0, 1] \to X$ defined by $W(x, y, \alpha) = (1 - \alpha)x + \alpha y$.

Definition 2.3. [37] Let X be a hyperbolic space with a mapping $W : X^2 \times [0, 1] \to X$.

(i) A nonempty subset C of X is said to be convex if $W(x, y, \alpha) \in C$ for all $x, y \in C$ and $\alpha \in [0, 1]$.

(ii) X is said to be uniformly convex if for any $r > 0$ and $\epsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$

$$d(W(x, y, \frac{1}{2}), z) \leq (1 - \delta)r,$$

provided $d(x, z) \leq r, d(y, z) \leq r$ and $d(x, y) \geq \epsilon r$.

(iii) A mapping $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \epsilon)$ for a given $r > 0$ and $\epsilon \in (0, 2]$, is known as a modulus of uniform convexity of X. The mapping η is said to be monotone, if it decreases with r (for a fixed ϵ).

Definition 2.4. Let C be a nonempty subset of a metric space X and $\{x_n\}$ be any bounded sequence in C. For $x \in X$, consider a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x_n, x).$$

The asymptotic radius $r(C, \{x_n\})$ of $\{x_n\}$ with respect to C is given by

$$r(C, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in C\}.$$

A point $x \in C$ is said to be an asymptotic center of the sequence $\{x_n\}$ with respect to $C \subseteq X$ if

$$r(x, \{x_n\}) = \inf\{r(y, \{x_n\}) : y \in C\}.$$

The set of all asymptotic centers of $\{x_n\}$ with respect to C is denoted by $A(C, \{x_n\})$. If the asymptotic radius and the asymptotic center are taken with respect to X, then they are simply denoted by $r(\{x_n\})$ and $A(\{x_n\})$ respectively. It is well-known that in uniformly convex Banach spaces and CAT(0) spaces, bounded sequences have unique asymptotic center with respect to closed and convex subsets.

Definition 2.5. [23]. A sequence $\{x_n\}$ in X is said to be $\triangle$-converge to $x \in X$, if x is the unique asymptotic center of $\{x_{nk}\}$ for every subsequence $\{x_{nk}\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$.

Remark 2.6. [24]. We note that $\triangle$-convergence coincides with the usually weak convergence known in Banach spaces with the usual Opial property.

Lemma 2.7 ([25]). Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset C of X.

Lemma 2.8 ([9]). Let X be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η and let $\{x_n\}$ be a bounded sequence in X with $A(\{x_n\}) = \{x\}$. Suppose $\{x_{nk}\}$ is any subsequence of $\{x_n\}$ with $A(\{x_{nk}\}) = \{x_1\}$ and $\{d(x_n, x_1)\}$ converges, then $x = x_1$.

Lemma 2.9 ([20]). Let (X, d, W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x^* \in X$ and $\{t_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x^*) \le c$, $\limsup_{n\to\infty} d(y_n, x^*) \le c$ and $\lim_{n\to\infty} d(W(x_n, y_n, t_n), x^*) = c$, for some $c > 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Definition 2.10. Let C be a nonempty subset of a hyperbolic space X and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is called a Fejér monotone sequence with respect to C if for all $x \in C$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \le d(x_n, x).$$

Proposition 2.11. [17] Let $\{x_n\}$ be a sequence in X and C be a nonempty subset of X. Suppose that $T : C \to C$ is any nonlinear mapping and the sequence $\{x_n\}$ is Fejer monotone with respect to C, then we have the following:

(i) $\{x_n\}$ is bounded.
(ii) The sequence $\{d(x_n, x^*)\}$ is decreasing and converges for all $x^* \in F(T)$.
(iii) $\lim_{n\to\infty} d(x_n, F(T))$ exists.

3 Main Results

3.1 Fixed Points Properties and Demiclosedness Principle

Theorem 3.1. Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $T : C \to C$ be a mean nonexpansive mapping with $b < 1$ and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

Proof . We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to some $y \in C$. We show that $y \in F(T)$ as follows:

Observe that

$$\begin{aligned} d(x_n, Ty) = d(Tx_n, Ty) &\le ad(x_n, y) + bd(x_n, Ty) \\ \implies d(x_n, Ty) - bd(x_n, Ty) &\le ad(x_n, y) \\ \implies d(x_n, Ty) &\le \frac{a}{1-b} d(x_n, y) \\ \implies d(x_n, Ty) &\le d(x_n, y). \end{aligned}$$

Since $\lim_{n\to\infty} d(x_n, y) = 0$, then by sandwich theorem, we obtain

$$\lim_{n\to\infty} d(x_n, Ty) = 0.$$

By the uniqueness of limit, we have that

$$Ty = y.$$

Hence, $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $x, y \in F(T)$ and $\alpha \in [0, 1]$. Then, we have

$$\begin{aligned} d(x, T(W(x, y, \alpha))) = d(Tx, T(W(x, y, \alpha))) &\le ad(x, W(x, y, \alpha)) + bd(x, T(W(x, y, \alpha))), \\ \text{which implies } d(x, T(W(x, y, \alpha))) &\le \frac{a}{1-b} d(x, W(x, y, \alpha)) \\ &\le d(x, W(x, y, \alpha)) \end{aligned} \tag{3.1}$$

Using similar argument, we have

$$d(y, T(W(x, y, \alpha))) \le d(y, W(x, y, \alpha)). \tag{3.2}$$

Using (3.1) and (3.2), we have

$$\begin{aligned} d(x, y) &\le d(x, T(W(x, y, \alpha))) + d(T(W(x, y, \alpha)), y) \\ &\le d(x, W(x, y, \alpha)) + d(W(x, y, \alpha), y) \\ &\le (1-\alpha)d(x, x) + \alpha d(x, y) + (1-\alpha)d(x, y) + \alpha d(y, y) \\ &\le d(x, y). \end{aligned} \tag{3.3}$$

Hence, we conclude that (3.1) and (3.2) are $d(x, T(W(x, y, \alpha))) = d(x, W(x, y, \alpha))$ and $d(y, T(W(x, y, \alpha))) = d(y, W(x, y, \alpha))$ respectively. Because if $d(x, T(W(x, y, \alpha))) < d(x, W(x, y, \alpha))$ or $d(y, T(W(x, y, \alpha))) < d(y, W(x, y, \alpha))$, then the inequality in (3.3) becomes strictly less than, which therefore gives us a contradiction, that is, $d(x, y) < d(x, y)$. Hence, we have that

$$T(W(x, y, \alpha)) = W(x, y, \alpha) \ \forall x, y \in F(T) \text{ and } \alpha \in [0, 1].$$

Thus, $W(x, y, \alpha) \in F(T)$, which implies that $F(T)$ is convex. □

Corollary 3.2. Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $T : C \to C$ be a nonexpansive mapping and $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.

We now establish the demiclosedness principle for mean nonexpansive mappings in hyperbolic spaces.

Theorem 3.3. Let C be a nonempty closed and convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T : C \to C$ be mean nonexpansive mapping with $b < 1$ and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x^*$. Then $x^* \in F(T)$.

Proof . Since $\{x_n\}$ is a bounded sequence in C, we have from Lemma 2.7 that $\{x_n\}$ has a unique asymptotic center in C. Also, since $\Delta - \lim_{n\to\infty} x_n = x^*$, we have that $A(\{x_n\}) = \{x^*\}$.

Now,

$$d(x_n, Tx^*) \le d(x_n, Tx_n) + d(Tx_n, Tx^*)$$
$$\le d(x_n, Tx_n) + ad(x_n, x^*) + bd(x_n, Tx^*)$$
$$\text{which implies } d(x_n, Tx^*) \le \frac{1}{1-b}[d(x_n, Tx_n) + ad(x_n, x^*)].$$

Taking $\limsup_{n\to\infty}$ of both sides, we have

$$r(Tx^*, \{x_n\}) = \limsup_{n\to\infty} d(x_n, Tx^*) \le \frac{1}{1-b} \limsup_{n\to\infty}[d(x_n, Tx_n) + ad(x_n, x^*)] \le \limsup_{n\to\infty} d(x_n, x^*) = r(x^*, \{x_n\}).$$

By the uniqueness of the asymptotic center of $\{x_n\}$, we have $Tx^* = x^*$. Hence, $x^* \in F(T)$. □

Corollary 3.4. Let C be a nonempty closed and convex subset of complete uniformly convex hyperbolic space X with monotone modulus of convexity η. Let $T : C \to C$ be nonexpansive mapping with $b < 1$ and $\{x_n\}$ be a bounded sequence in C such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and $\Delta - \lim_{n\to\infty} x_n = x^*$. Then $x^* \in F(T)$.

3.2 Strong and Δ-Convergence Theorems

In [1], Abbas and Nazir introduced a three step iteration and showed that its rate of convergence is comparatively faster than some existing iteration processes (see [1] for more details). We now propose this iteration process in the frame work of hyperbolic spaces for approximating a common fixed point of two mean nonexpansive mappings.

Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X and $T, S : C \to C$ be two mean nonexpansive mappings. For $x_1 \in C$, we construct the sequence $\{x_n\}$ as follows:

$$\begin{cases} z_n = W(x_n, Sx_n, \gamma_n), \\ y_n = W(Sx_n, Tz_n, \beta_n), \\ x_{n+1} = W(Ty_n, Tz_n, \alpha_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

For the rest of this paper, we denote the set of common fixed points of T and S by Γ, that is, $\Gamma := F(T) \cap F(S)$. Thus, using Algorithm 3.4, we state and prove strong and Δ-convergence theorems for approximating an element in Γ. We begin with the following lemmas.

Lemma 3.5. Let C be a nonempty closed and convex subset of a hyperbolic space X. Let $S, T : C \to C$ be two mean nonexpansive mappings. Suppose $\Gamma \neq \emptyset$ and the sequence $\{x_n\}$ is defined by (3.4), then

(i) $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in \Gamma$.
(ii) $\lim_{n\to\infty} d(x_n, \Gamma)$ exists.

Proof . Let $x^* \in \Gamma$, then from (3.4), we have

$$\begin{aligned} d(z_n, x^*) &= d(W(x_n, Sx_n, \gamma_n), x^*) \\ &\le (1-\gamma_n)d(x_n, x^*) + \gamma_n(Sx_n, x^*) \\ &\le (1-\gamma_n)d(x_n, x^*) + \gamma_n[ad(x_n, x^*) + bd(x_n, x^*)] \\ &= [1 - \gamma_n + a\gamma_n + b\gamma_n]d(x_n, x^*) \\ &\le d(x_n, x^*). \end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\begin{aligned} d(y_n, x^*) &= d(W(Sx_n, Tz_n, \beta_n), x^*) \\ &\le (1-\beta_n)d(Sx_n, x^*) + \beta_n(Tz_n, x^*) \\ &\le (1-\beta_n)[ad(x_n, x^*) + bd(x_n, x^*)] + \beta_n[ad(z_n, x^*) + bd(z_n, x^*)] \\ &\le (1-\beta_n)[ad(x_n, x^*) + bd(x_n, x^*)] + \beta_n[ad(x_n, x^*) + bd(x_n, x^*)] \\ &= [a + b]d(x_n, x^*) \\ &\le d(x_n, x^*). \end{aligned} \tag{3.6}$$

Also, using (3.4), (3.5) and (3.6), we have

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(Ty_n, Tz_n, \alpha_n), x^*) \\ &\leq (1-\alpha_n)d((Ty_n, x^*) + \alpha_n d(Tz_n, x^*) \\ &\leq (1-\alpha_n)[ad(y_n, x^*) + bd(y_n, x^*)] + \alpha_n[ad(z_n, x^*) + bd(z_n, x^*)] \\ &\leq (1-\alpha_n)[ad(x_n, x^*) + bd(x_n, x^*)] + \alpha_n[ad(x_n, x^*) + bd(x_n, x^*)] \\ &= [a+b]d(x_n, x^*) \\ &\leq d(x_n, x^*). \end{aligned} \tag{3.7}$$

Inequality (3.7) implies that $\{x_n\}$ is Fejér monotone with respect to Γ. Thus, by Proposition 2.11, we have that $\{x_n\}$ is bounded, $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in \Gamma$ and $\lim_{n\to\infty} d(x_n, \Gamma)$ exists. □

Lemma 3.6. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T, S : C \to C$ be two mean nonexpansive mappings. Suppose $\Gamma \neq \emptyset$ and the sequence $\{x_n\}$ is defined by (3.4), then $\lim_{n\to\infty} d(x_n, Sx_n) = \lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, Tz_n) = \lim_{n\to\infty} d(z_n, Tz_n) = 0$.

Proof . From Lemma 3.5, we have that $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in \Gamma$. Suppose that $\lim_{n\to\infty} d(x_n, x^*) = c$. If we take $c = 0$, then we are done. So, we consider the case where $c > 0$. It is clear from (3.5) that

$$d(z_n, x^*) \leq d(x_n, x^*),\ n \in \mathbb{N}. \tag{3.8}$$

Thus, taking $\limsup_{n\to\infty}$ of both sides of (3.8), we have

$$\limsup_{n\to\infty} d(z_n, x^*) \leq c. \tag{3.9}$$

By the definition of T, we get

$$\begin{aligned} d(Tz_n, x^*) &\leq ad(z_n, x^*) + bd(z_n, x^*) \\ &= [a+b]d(z_n, x^*), \end{aligned}$$

taking $\limsup_{n\to\infty}$ of both sides, we have

$$\limsup_{n\to\infty} d(Tz_n, x^*) \leq c. \tag{3.10}$$

Also from (3.6), we have

$$d(y_n, x^*) \leq d(x_n, x^*),$$

taking $\limsup_{n\to\infty}$ of both sides, we have

$$\limsup_{n\to\infty} d(y_n, x^*) \leq c. \tag{3.11}$$

More so,

$$\begin{aligned} d(Ty_n, x^*) &\leq ad(y_n, x^*) + bd(y_n, Tx^*) \\ &\leq d(y_n, x^*), \end{aligned}$$

taking $\limsup_{n\to\infty}$ of both sides and using (3.11), we have

$$\limsup_{n\to\infty} d(Ty_n, x^*) \leq c. \tag{3.12}$$

From (3.4), we have that

$$d(x_{n+1}, x^*) = d(W(Ty_n, Tz_n, \alpha_n), x^*),$$

which implies

$$\lim_{n\to\infty} d(W(Ty_n, Tz_n, \alpha_n), x^*) = c. \tag{3.13}$$

Then by Lemma 2.9, we have

$$\lim_{n\to\infty} d(Ty_n, Tz_n) = 0. \tag{3.14}$$

Again, from the definition of S, we have

$$\begin{aligned} d(Sx_n, x^*) &\le ad(x_n, x^*) + bd(x_n, x^*) \\ &= [a+b]d(x_n, x^*) \\ &\le d(x_n, x^*), \end{aligned}$$

taking $\limsup_{n\to\infty}$ of both sides, we have

$$\limsup_{n\to\infty} d(Sx_n, x^*) \le c.$$

Now,

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(Ty_n, Tz_n, \alpha_n), x^*) \\ &\le (1-\alpha_n)d(Ty_n, x^*) + \alpha_n d(Tz_n, x^*) \\ &\le (1-\alpha_n)d(Ty_n, x^*) + \alpha_n[d(Tz_n, Ty_n) + d(Ty_n, x^*)] \\ &= d(Ty_n, x^*) + \alpha_n d(Tz_n, Ty_n) \\ &\le ad(y_n, x^*) + bd(y_n, x^*) + \alpha_n d(Tz_n, Ty_n) \\ &= [a+b]d(y_n, x^*) + \alpha_n d(Tz_n, Ty_n) \\ &\le d(y_n, x^*) + \alpha_n d(Tz_n, Ty_n), \end{aligned}$$

taking $\liminf_{n\to\infty}$ of both sides and using (3.14), we have

$$c \le \liminf_{n\to\infty} d(y_n, x^*). \tag{3.15}$$

It then follows from (3.11) and (3.15) that

$$\lim_{n\to\infty} d(y_n, x^*) = c. \tag{3.16}$$

So that

$$\lim_{n\to\infty} d(W(Sx_n, Tz_n, \beta_n), x^*) = c.$$

Thus, using Lemma 2.9, we have

$$\lim_{n\to\infty} d(Sx_n, Tz_n) = 0. \tag{3.17}$$

Furthermore, using (3.17) and (3.14), we have

$$\lim_{n\to\infty} d(Sx_n, Ty_n) \le \lim_{n\to\infty} d(Sx_n, Tz_n) + \lim_{n\to\infty} d(Tz_n, Ty_n) = 0. \tag{3.18}$$

Also,

$$\begin{aligned} d(y_n, x^*) &= d(W(Sx_n, Tz_n, \beta_n), x^*) \\ &\le (1-\beta_n)d(Sx_n, x^*) + \beta_n d(Tz_n, x^*) \\ &\le (1-\beta_n)[d(Sx_n, Tz_n) + d(Tz_n, x^*)] + \beta_n d(Tz_n, x^*) \\ &= (1-\beta_n)d(Sx_n, Tz_n) + d(Tz_n, x^*) \\ &\le (1-\beta_n)d(Sx_n, Tz_n) + ad(z_n, x^*) + b(z_n, x^*) \\ &= (1-\beta_n)d(Sx_n, Tz_n) + (a+b)d(z_n, x^*) \\ &\le (1-\beta_n)d(Sx_n, Tz_n) + d(z_n, x^*), \end{aligned}$$

taking $\liminf_{n\to\infty}$ of both sides and using (3.16) and (3.17), we have

$$c \le \liminf_{n\to\infty} d(z_n, x^*). \tag{3.19}$$

From (3.9) and (3.19), we have

$$\lim_{n\to\infty} d(z_n, x^*) = c$$

That is,

$$\lim_{n\to\infty} d(W(x_n, Sx_n, \gamma_n), x^*) = c.$$

Then using Lemma 2.9, we have that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{3.20}$$

Also, we have

$$d(x_n, Tz_n) \leq d(x_n, Sx_n) + d(Sx_n, Tz_n),$$

which implies from (3.20) and (3.17) that

$$\lim_{n\to\infty} d(x_n, Tz_n) = 0. \tag{3.21}$$

Again,

$$d(x_n, Ty_n) \leq d(x_n, Tz_n) + d(Tz_n, Ty_n),$$

which implies from (3.14) and (3.21) that

$$\lim_{n\to\infty} d(x_n, Ty_n) = 0. \tag{3.22}$$

From (3.20), we obtain

$$\begin{aligned} d(z_n, x_n) &= d(W(x_n, Sx_n, \gamma_n), x_n) \\ &\leq \gamma_n d(Sx_n, x_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.23}$$

Also, from (3.21) and (3.23), we obtain

$$\lim_{n\to\infty} d(z_n, Tz_n) \leq \lim_{n\to\infty} d(z_n, x_n) + \lim_{n\to\infty} d(x_n, Tz_n) \to 0, \text{ as } n \to \infty. \tag{3.24}$$

That is

$$\lim_{n\to\infty} d(z_n, Tz_n) = 0.$$

Hence, the proof is complete. □

Theorem 3.7. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S, T : C \to C$ be two mean nonexpansive mappings such that $b < 1$. Suppose that $\Gamma \neq \emptyset$ and the sequence $\{x_n\}$ is defined by (3.4), then $\{x_n\}$ $\triangle$-converges to a common fixed point of T and S.

Proof . Let $W_\Delta(x_n) := \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We now show that $W_\Delta(x_n) \subset \Gamma$ and that $W_\Delta(x_n)$ contains exactly one point.

Let $u \in W_\Delta(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$, since $\{u_n\}$ is bounded by Lemma 3.5. This implies from Lemma 2.7 that we can find a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$, for some $v \in C$. By Lemma 3.6, we have that $\lim_{n\to\infty} d(v_n, Tv_n) = \lim_{n\to\infty} d(v_n, Sv_n) = 0$, which together with Theorem 3.3 gives that $v \in \Gamma$. Therefore, $d(u_n, v)$ converges and by Lemma 2.8, we have that $v = u \in \Gamma$. Hence, $W_\Delta(x_n) \subset \Gamma$.

Next, we show that $W_\Delta(x_n)$ contains exactly one point. Let $A(\{x_n\}) = \{x\}$ and $\{u_n\}$ be arbitrary subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 3.5, we have that $d(x_n, u)$ converges, since $u \in \Gamma$. Thus, by Lemma 2.8, we have that $u = x \in \Gamma$. Hence, $W_\Delta(x_n) = \{x\}$. Therefore, $\{x_n\}$ $\triangle$-converges to a common fixed point of T and S. □

Theorem 3.8. Suppose that the assumptions in Theorem 3.7 holds, then the sequence $\{x_n\}$ defined by (3.4) converges strongly to $x^* \in \Gamma$ if and only if $\liminf_{n\to\infty} d(x_n, \Gamma) = 0$, where $d(x_n, \Gamma) = \inf\{d(x_n, x^*) : x^* \in \Gamma\}$.

Proof . Suppose that the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$ and since $0 \leq d(x_n, \Gamma) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, \Gamma) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, \Gamma) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, \Gamma) = 0$. Then, from Lemma 3.5, we obtain that $\lim_{n\to\infty} d(x_n, \Gamma) = 0$. Suppose that $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{p_k\}$ is a sequence in Γ such that for all $n \geq 1$,

$$d(x_{n_k}, p_k) \leq \frac{1}{2^k}.$$

From (3.7), it obtain that

$$d(x_{n_k+1}, p_k) < \frac{1}{2^k},$$

which implies

$$\begin{aligned} d(p_{k+1}, p_k) &\leq d(p_{k+1}, x_{n_k+1}) + d(x_{n_k+1}, p_k) \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ &< \frac{1}{2^{k-1}}. \end{aligned}$$

This shows that $\{p_k\}$ is a Cauchy sequence in Γ. Also, by Theorem 3.1, we have that Γ is a closed subset of X. Thus, $\{p_k\}$ is a convergent sequence in Γ. Let $\lim_{n\to\infty} p_k = x^*$, then $x^* \in \Gamma$, and we have

$$d(x_{nk}, x^*) \leq d(x_{nk}, p_k) + d(p_k, x^*) \to 0 \text{ as } n \to \infty,$$

which implies $\lim_{n\to\infty} d(x_{n_k}, x^*) = 0$. Since $\lim_{n\to\infty} d(x_n, x^*)$ exists, then we conclude that the sequence $\{x_n\}$ converges strongly to $x^* \in \Gamma$. Hence, the proof is complete. □

By setting $S = T$ in Algorithm 3.4, we obtain the following corollaries.

Corollary 3.9. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $T : C \to C$ be a mean nonexpansive mapping such that $b < 1$. Suppose that $F(T) \neq \emptyset$ and the sequence $\{x_n\}$ is defined by

$$\begin{cases} z_n = W(x_n, Sx_n, \gamma_n), \\ y_n = W(Sx_n, Tz_n, \beta_n), \\ x_{n+1} = W(Ty_n, Tz_n, \alpha_n), \quad n \in \mathbb{N}, \end{cases} \tag{3.25}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$.

Then $\{x_n\}$ $\triangle$-converges to a fixed point of T.

Corollary 3.10. Suppose that the assumptions in Corollary 3.9 holds, then the sequence $\{x_n\}$ defined by (3.25) converges strongly to $x^* \in F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x_n, F(T)) = \inf\{d(x_n, x^*) : x^* \in F(T)\}$.

In view of Remark (2), we have the following corollaries.

Corollary 3.11. Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $S, T : C \to C$ be two nonexpansive mappings such that $b < 1$. Suppose that $\Gamma \neq \emptyset$ and the sequence $\{x_n\}$ is defined by (3.4), then $\{x_n\}$ $\triangle$-converges to a common fixed point of T and S.

Corollary 3.12. Suppose that the assumptions in corollary 3.11 holds, then the sequence $\{x_n\}$ defined by (3.4) converges strongly to $x^* \in \Gamma$ if and only if $\liminf_{n\to\infty} d(x_n, \Gamma) = 0$, where $d(x_n, \Gamma) = \inf\{d(x_n, x^*) : x^* \in \Gamma\}$.

Remarks:

(i) It follows from Corollaries 3.11 and 3.12, that the results presented in this paper extend corresponding results from the class of nonexpansive mappings to the more general class of mean nonexpansive mappings in hyperbolic spaces.

(ii) Our results extend and complement corresponding results in Banach vector spaces and CAT(0) spaces, since both spaces are examples of the hyperbolic space considered in this paper.

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